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# On pseudomonotone variational inequalities

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**Abstract**

**Abstract**. There are mainly two definitions of pseudomonotone mappings. First, introduced by S. Karamardian in 1976, refers to the maps on finite-dimensional spaces while the second one, introduced by H. Brézis in 1968, is for operators on topological vector spaces in duality. Both represent extensions of the concept of a monotone mapping. We investigate variational inequalities on Banach spaces involving pseudomonotone operators in the sense of Karamardian. We emphasize the link between the two concepts in interconnection with the existence and uniqueness of the solutions of variational inequalities.

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## 1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space with the topological dual $(X^*, \|\cdot\|_*)$, $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbf{R}$ the pairing of elements from $X^*$ and $X$.

We denote by $2^{X^*}$ the totality of all nonempty subsets of $X^*$ and consider the multivalued or set-valued mapping $A : X \to 2^{X^*}$. Let $D(A) = \{x \in X : A(x) \neq \emptyset\}$ be its *effective domain*, $R(A) = \{f \in A(x) : x \in D(A)\}$ be its *range* and $G(A) = R(A) \times D(A)$ be its *graph*. We do not distinguish between a set-valued mapping $A$ and its graph $G(A)$. So that, $A$ or $G(A)$ is *monotone* if

$\langle f_1 - f_2, x_1 - x_2\rangle \geq 0$ for all $f_1 \in A(x_1)$ and $f_2 \in A(x_2)$ or for all $(x_1, f_1), (x_2, f_2) \in G(A)$.

To prove the existence of a solution of the operator equation (inclusion) involving a monotone mapping $A(x) \ni f$ it is necessary to assume that $A$ is

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maximal. The set-valued mapping $A$ is *maximal monotone* if its graph $G(A)$ has no monotone extension in $X^* \times X$. The maximality ensures some required topological properties.

The variational inequalities can be regarded as generalizations of these equations.

In a more general framework (we give up the finite dimesion of $X$) let $C$ be a nonempty closed convex set in $X$ and $A$ be a set-valued mapping from $X$ into $X^*$. Then, for a given $f \in X^*$, the problem of finding an element $u \in C$ such that

$$\langle Au - f, x - u \rangle \geq 0, \, \forall x \in C,$$

is called a *variational inequality* $(VI)$ *of the first kind*. More precisely, sometimes we denote it by $VI\ (A, C)$ and the set of solutions by $SOL\ (A, C)$.

Clearly, when $C = X$ or $u$ is an interior point of $C$, then we range over a neighborhood of $u$ and the variational inequality $VI\ (A, C)$ reduces to the equation $A(u) \ni f$.

For the existence of variational inequalities, H. Brézis [1] introduced the concept of pseudomonotone operators between two topological vector spaces in duality, using the filter-convergence. In the case of Banach spaces, there is a countable system of neighborhoods and we can bound ourselves to ordinary sequences. A theory of the pseudomonotone-like mappings in the last framework was elaborated by F.E. Browder and P. Hess [2] and it will be used below. Because the pseudomonotonicity appears as an extension of the maximal monotonicity, we emphasize first that a hemicontinuous maximal monotone operator $A$ with $D(A) = X$ is pseudomonotone.

For the sake of simplicity, we consider in the sequel, if other is not stated, that $f = 0$.

In section 2 we recall the definition of pseudomonotonicity in the sense of Karamardian. We consider some variational inequalities involving pseudomonotone operators and their properties.

## 2. Variational inequalities and algebraic pseudomonotonicity in the sense Karamardian.

Let $C$ be a nonempty closed convex subset of a real Banach space $X$. An operator $T : C \subset X \to X^*$ is *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0, \, \forall x, y \in C.$$

$T : C \subset X \to X^*$ is *pseudomonotone* if $\langle Ty, x - y \rangle \geq 0$ implies $\langle Tx, x - y \rangle \geq 0$ for every pair of points $x, y \in C$ , and $T$ is *strictly pseudomonotone* if $\langle Ty, x - y \rangle \geq 0$ implies $\langle Tx, x - y \rangle > 0$ for every pair of distinct points $x, y \in$

$C$. Other classes of algebraic monotone-like operators on finite-dimensional subspaces were introduced in [5].

The symbols "$\rightarrow$","$\rightharpoonup$" and "$\overset{*}{\rightleftharpoons}$ " denote the norm-convergence, the weak convergence and the weak$^*$ convergence, respectively. Then $T$ is *hemicontinuous* if the function $t \mapsto \langle T(tx + (1-t)y), x - y\rangle$ is continuous on $[0,1]$, and $T$ is *demicontinuous* if $x_n \to x$ in $X$ implies $Tx_n \overset{*}{\rightleftharpoons} Tx$ in $X^*$. Obviously, if $T$ is demicontinuous then the restrictions of to any finite-dimensional subspaces of $X$ are continuous.

Now we list some existence results for solutions of the $VI(T,C)$. We start with the following routine assertion:

**Lemma 2.1.** *Let $T : C \to X^*$ be a finite-dimensional continuous pseudomonotone operator. Then $u \in C$ is a solution of the inequality*

$$\langle Tu, x-u\rangle \geq 0,\ \forall x \in C, \tag{2.1}$$

*if and only if*

$$\langle Tx, x-u\rangle \geq 0,\ \forall x \in C. \tag{2.2}$$

*Moreover, the set of solutions $SOL\ (T,C)$ of the variational inequality* (2.1) *is closed and convex.*

In the sequel, $VI(T,C)$ and $SOL\ (T,C)$ are referred to (2.1). First we state an uniqueness result:

**Proposition 2.2.** *If $T : C \to X^*$ is strictly pseudomonotone, then of $VI(T,C)$ has at most one solution.*

Finally, we can establish the existence result:

**Theorem 2.3.** *Let $X$ be a real Banach space, that $C$ be a weakly compact subset of $X$ and $T : C \to X^*$ be a pseudomonotone operator. Then $VI(T,C)$ admits a solution $SOL\ (T,C)$ and is nonempty, weakly compact, and convex.*

**Corollary 2.4.** *Let $X$ be a real reflexive Banach space, that $C$ be a weakly compact, convex subset of $X$, and $T : C \to X^*$ a monotone operator. If for any pair of points $y, z \in X$, the condition*

$$\lim_{t\to o^+} \inf \langle T(y+tz), z\rangle \leq \langle Ty, z\rangle \tag{2.3}$$

*holds, then $VI(T,C)$ admits a solution and $SOL\ (T,C)$ is nonempty, weakly compact, and convex.*

We remark that condition (2.3) emphasized by J.-C. Yao [8] is weaker than the hemicontinuity assumption and Corollary 2.4 extends the standard Stampacchia's existence result.

As a consequence of Proposition 2.2 and Theorem 2.3, an uniqueness result holds:

**Theorem 2.5.** *Let $C$ be a weakly compact, convex subset of real Banach space $X$, and $T : C \to 2^{X^*}$ be a finite-dimensional continuous strictly pseudomonotone operator. Then $VI\ (T, C)$ admits a unique solution.*

An excellent survey of developments of the variational inequalities in finite-dimensional spaces has been performed in [3].

### 3. Set-valued mappings.

Now, we consider set-valued (multivalued) mappings $A : X \to 2^{X^*}$ and denote by $Conv\ (X^*)$ the totality of all convex closed subset of $X^*$. We introduce the *upper* and *lower support* functions for $A$ by the formulas

$$[A(x), y]_+ = \sup_{x^* \in A(x)} \langle x^*, y \rangle \text{ and } [A(x), y]_- = \inf_{x^* \in A(x)} \langle x^*, y \rangle,$$

with the *upper norm* on $Conv\ (X^*)$ defined by $\|A(x)\|_+ = \sup_{x^* \in A(x)} \|x^*\|_{X^*}$.

Our operators could be non-convex and non-closed set-valued, i.e., we distinguish $A(x)$ and $\bar{co}\ A(x)$ (the minimal closed convex set containing $A(x)$.

Denote $gr\ \bar{co}\ A = \{(x, g) \in D(A) \times X^* : g \in \bar{co}\ A(x)\}$. In addition, the following relations hold:

$$[A(x), y]_+ = [co\, A(x), y]_+ = [\bar{co}\ A(x), y]_+, \forall x, y \in X,$$

$$[A(x), y]_- = [co\, A(x), y]_- = [\bar{co}\ A(x), y]_-, \forall x, y \in X,$$

$$[A(x), y_1 + y_2]_+ \geq [A(x), y_1]_+ + [A(x), y_2]_-, \forall x, y_1, y_2 \in X,$$

$$[A(x), y_1 + y_2]_- \leq [A(x), y_1]_- + [A(x), y_2]_+, \forall x, y_1, y_2 \in X,$$

$$\|\bar{co}\ A(x)\|_+ = \|A(x)\|_+, \forall x, \in X.$$

We know the following definitions:

1) A mapping $A : X \to Conv\ (X^*)$ is called *upper semicontinuous* at $x \in D(A)$ if for each neighborhood $V$ of $A(x)$ in $X^*$ there is a neighborhood $U$ of $x$ in $X$ such that $A(U) \subset V$ and $A$ is *upper semicontinuous* if it is upper semicontinuous at each point $x \in D(A)$. The upper semicontinuity plays an important role in the fixed-point theory for multivalued maps.

2) The mapping $A$ is called *locally bounded* if for any $x \in \overline{D(A)}$ there are positive numbers $\varepsilon$ and $M$ such that $\|A(y)\|_{+} \leq M$ for $y \in X$ with $\|y - x\| < \varepsilon$.

We generalize these definitions by introducing the notion of generalized pseudomonotonicity, in the frame of Banach spaces which are not necesarily finite-dimensional.

**Definition 3.1.** The mapping $A : D(A) \subset X \to 2^{X^*}$ is *generalized pseudomonotone* if for any sequence $\{(x_n, x_n^*)\} \subset G(\bar{co}\ A)$ such that $(x_n, x_n^*) \rightharpoonup (x, x^*)$ in $X \times X^*$ and $\limsup \langle x_n^*, x_n - x\rangle \leq 0$, it follows that $x^* \in \bar{co}\ A(x)$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

Our study of variational inequalities involving set-valued mappings is based on the following Brouwer fixed-point extension ([9] see Zeidler, pp.453):

**Proposition 3.2.** *Let $C$ be a nonempty, convex, compact set in a locally convex space $X$ and $S : C \to 2^C$ a mapping such that the set $S(x)$ is nonempty and convex for all $x \in C$, and the preimages $S^{-1}(y)$ are relatively open with respect to $C$ for all $y \in C$. Then $S$ has a fixed point.*

We can now extend an existence result ([9],pp.453) to variational inequalities with set-valued mappings, i.e., the following problem:

*Find a pair $(u, g) \in C \times T(u)$ such that this satisfies the inequality*

$$\langle g, v - u\rangle \geq 0 \text{ for all } v \in C. \tag{3.1}$$

We give sufficient conditions for this problem to have solutions.

**Theorem 3.3.** *Let $T : C \subset X \to 2^{X^*}$ be a set-valued mapping defined on a nonempty subset $C \subset X$.*

*If the following conditions are satisfied:*

*(i) the mapping $T : C \subset X \to 2^{X^*}$ is locally bounded, upper semicontinuous, and generalized pseudomonotone where $X$ is a locally convex space and $X^*$ is the dual space of $X$ under the strong topology;*

*(ii) the set $C$ is nonempty, convex, and compact;*

*then the variational inequality* (3.1) *has a solution $(u, g) \in C \times T(u)$.*

**Proof.** In the contrary case, to each $h \in T(u)$ there corresponds an element $w \in C$ such that

$$\langle h, w - u\rangle < 0. \tag{3.2}$$

Define the multivalued mapping $S : C \to 2^C$ by

$$S(u) := \{w \in K : \langle h, w - u \rangle < 0\}.$$

Condition (3.2) implies that the set $S(u)$ is nonempty for all $u \in C$. In addition, $S(u)$ is convex.

We show that the set $S^{-1}(w)$ is relatively open in $C$. First of all, specify

$$S^{-1}(w) := \{u \in K : \langle h, w - u \rangle < 0\}.$$

Let $\{u_n\}$ be a (generalized) sequence in $C \backslash S^{-1}(u)$ with $u_n \to z$ and $h_n \in T(u_n)$, so that $\langle h_n, w - u_n \rangle \geq 0$ or $\langle h_n, u_n - w \rangle \leq 0$ for all $n$. By the local boundedness, we also have $h_n \rightharpoonup g$ in $X^*$. From the generalized pseudomonotonicity of $T$, we derive that $h \in T(z)$. Thus, $C \backslash S^{-1}(u)$ is relatively closed and $S^{-1}(u)$ is relatively open in $C$.

By the previous proposition, there exists a fixed point $u \in S(u)$. This lead to the contradiction $\langle h, u - u \rangle < 0$. Hence there is a $g \in T(u)$ and $u \in C$, satisfying (3.1).

## 4. Variational inequalities with set-valued mappings.

Let $C \subset D(A)$ be a convex closed set. O.A. Solonoukha [7] investigated the solvability for the multivariational inequality

$$[A(u), v - u]_+ \geq \langle f, v - u \rangle, \; v \in C. \tag{4.1}$$

involving the set-valued mapping $A : C \to 2^{X*}$, called briefly VISM.

We start giving an equivalence of VISM(A,C) with a usual multivalued mapping in the form (3.1).

**Lemma 4.1.** Let $u_o$ *be a solution of* VISM (4.1) *with* $\bar{\bar{co}}\ A(y)$*a bounded set. Assume that* $C$ *is also compact set and* $A$ *is locally bounded, upper semicontinuous and a generalized pseudomonotone mapping. Then there exists an element* $g \in \bar{\bar{co}}\ A(u_o)$ *such that*

$$\langle g, v - u_o \rangle \geq \langle f, v - u_o \rangle, \; \forall v \in C.$$

**Proof.** If the claim is not true, to each $g \in \bar{\bar{co}}\ A(u_o)$ there corresponds an element $w \in C$ such that $\langle h, w - u_o \rangle \geq \langle h, w - u_o \rangle$. We define a similar multivalued mapping $S$ and we follow the proof of Theorem 3.3.

This lemma allows us to approach the previous VISM(A,C) by a simpler and regular form. In this setting, the mapping $A$ is called *coercive* if

$$\frac{[A(x),x]_-}{\|x\|}\to\infty \text{ as } \|x\|\to\infty.$$

In the standard way [6], we can establish the following existence result:

**Theorem 4.2.** *Let $C\subset D(A)$ be a closed convex and compact set in a real reflexive Banach space and $A:C\to 2^{X^*}$ be a locally bounded, generalized pseudomonotone mapping.*

*Assume, further, that $A$ is coercive. Then VISM* (4.1) *has a nonempty weakly compact set of solutions for any $f\in X^*$.*

In the case $C=X$, if the mapping $A$ satisfies the assumptions of Theorem 4.2, then, for any $f\in X^*$, the operator inclusion

$$\overline{co}\ A(u)\ni f$$

has at least one solution $u\in X$. In other words, $\overline{co}\ A$ is surjective, i.e., $R(\overline{co}\ A)=X^*$ .

More generally, let $\varphi: Dom(\varphi)\to\mathbf{R}$ be a convex lower semicontinuous function with the domain $Dom\,\varphi=\{x\in X:\varphi(x)<\infty\}$. Consider the *variational inequality of second kind*, that is, for a given $f\in X^*$, find $u\in Dom(\varphi)$ such that

$$[A(u),v-u]_+ +\varphi(v)-\varphi(u)\geq\langle f,v-u\rangle,\ \forall v\in Dom(\varphi). \tag{4.2}$$

The corresponding coerciveness condition has the form

$$\frac{[A(x),x]_- +\varphi(x)}{\|x\|}\to\infty \text{ as } \|x\|\to\infty, \tag{4.3}$$

and we can prove a similar existence result:

**Theorem 4.3.** *Let $\varphi: Dom(\varphi)\to\mathbf{R}$ be a convex lower semicontinuous function on a real reflexive Banach space $X$ and $A: Dom(\varphi)\to 2^{X^*}$ be a locally bounded generalized pseudomonotone mapping. Assume, further, that $A$ satisfies the coerciveness condition* (4.3). *Then the* VISM (4.2) *has a nonempty weakly compact set of solutions for any $f\in X^*$.*

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