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Normality of monomial ideals in two sets of variables

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Abstract

We study the normality of the monomial ideals in two sets of variables $L = I_kJ_r + I_sJ_t \subset K[X_1, \dots, X_m; Y_1, \dots, Y_n]$, K is a field, $k+r = s+t$, where I_k (resp.J_r) is the ideal of R generated by all the monomials of degree k (resp.r) in the variables $X_1, \dots, X_m$(resp.$Y_1, \dots, Y_n$). If L is not normal, we determine one element of the integral closure of all non complete powers of L.

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Introduction

In a recent work [4] G.Restuccia and R.Villarreal introduce the class of square-free ideals of mixed products in a polynomial ring over a field k in two sets of variables. They are square-free monomial ideals generated in the same degree that are integrally closed ([5], §7.5). In [4] the authors studied when each power of a mixed product ideal is complete. In this case the ideal is said normal. This property is linked to properties of graded algebras arising from I. The most important of such algebras is the Rees algebra $Rees(I) = \bigoplus_{i\geq 0} I^it^i$ ([1], §1.5, §4.5).An important result says that if I is normal, then $Rees(I)$ is normal ([5], 3.3.18).

It is possible to introduce the same class of mixed product ideals in a polynomial ring in two sets of variables in the not square-free case. More precisely, if $R = K[X_1, \dots, X_m; Y_1, \dots, Y_n]$ is the polynomial ring in two sets

Key Words: Monomial ideals; Graded rings; Rees algebras.

of variables over a field K , given the non negative integers k,r,s,t such that $k + r = s + t$, we can define the monomial ideals of R:

$$L = I_k J_r + I_s J_t,$$

where I_k (resp.J_r) is the ideal of R generated by all the monomials of degree k (resp.r) in the variables $X_1, \dots, X_m$(resp.$Y_1, \dots, Y_n$).

The aim of this work is to study the normality of these monomial ideals as in the square-free case. We obtain again a complete classification of the ideals of this class. If the ideal L is not normal, we determine the powers of L that result complete and for all powers that are not complete we find a monomial that lies in the integral closure of the power but it does not lie in the power. The technics used are similar to those used in [4] and in [3]. The results obtained about the normality coincide with those obtained in [4] in all cases, except for the ideals $L = J_r + I_m$ and $L = J_r + I_m J_t$ that are normal if they are square-free monomial ideals, contrary they are not normal in the not square-free case.

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1

Let $R = K[X_1, \dots, X_m; Y_1, \dots, Y_n]$ be a polynomial ring over a field K in two sets of variables. Given the non negative integers k,r,s,t such that $k+r = s+t$, we define the monomial ideals of R:

$$L = I_k J_r + I_s J_t,$$

where I_k (resp.J_r) is the ideal of R generated by all the monomials of degree k (resp.r) in the variables $X_1, \dots, X_m$ (resp.$Y_1, \dots, Y_n$).

It is easy to see that we have the following classes of monomial ideals of R arising from the definition of L:

1) $L = J_r + I_r$, with $r > 1$

2) $L = J_r + I_m J_t$, with $r = m + t$

3) $L = J_r + I_s J_t$, with $r = s + t$ and $s \neq m$

4) $L = I_k J_r + I_s J_t$, with $k + r = s + t$

5) $L = I_k J_r$, with $k, r > 1$

6) $L = I_k J_r + I_{k+1} J_{r-1}$, with $k, r > 0$.

Definition 1.1 *The integral closure of L is the set of all elements of R which are integral over L. We denote this set by $\overline{L}$.*
If $L = \overline{L}$, L is said to be integrally closed or complete. If all the powers of L , L^p, $p \geq 1$, are complete, the ideal L is said to be normal.

Remark 1.1 *The monomial ideal I_k (resp. J_r) is normal because $I_k = (I_1)^k$ (resp. $J_r = (J_1)^r$) (see [5], 3.3.18).*

As the integral closure of a monomial ideal is again a monomial ideal, one has the following description for the integral closure of L:

$$\overline{L} = (f \mid f \text{ is monomial in R and } f^i \in L^i, \text{for some i} \geq 1),$$

(see [5],7.3.3).

Now, we study the classes $1), 2), 3), 4)$. We will prove that they are not normal ideals. In fact, we have the following:

Proposition 1.1 *Let $R = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n]$ be a polynomial ring over a field K. Let L be one of the following ideals:*

a) $L = J_r + I_r,\ r > 1$.

b) $L = J_r + I_m J_t$, *with* $r = m + t$.

c) $L = J_r + I_s J_t$, *with* $r = s + t$ *and* $s \neq m$.

Then L^i is not integrally closed for all $i \geq 1$.

1. *If i is odd, there exists $f = (X_1 Y_1 Y_2^{r-2})^i \in \overline{L^i}/L^i$.*
2. *If i is even, there exists $f = (X_1 Y_1^r Y_2^{r-1})^{\frac{i}{2}} \in \overline{L^i}/L^i$.*

Proof.
a) $L = J_r + I_r,\ r > 1$.
From the equalities
1. $f^r = X_1^{ri} Y_1^{ri} Y_2^{ri(r-2)} = (X_1^r)^i (Y_1^r)^i (Y_2^r)^{i(r-2)}$,

2. $f^r = X_1^{r\frac{i}{2}} Y_1^{r^2\frac{i}{2}} Y_2^{\frac{ir}{2}(r-1)} = (X_1^r)^{\frac{i}{2}} (Y_1^r)^{\frac{ir}{2}} (Y_2^r)^{\frac{i}{2}(r-2)}$,
it follows that f^r is in L^{ri}. By a counting degree argument it follows that f is not in L^i.
b)$L = J_r + I_m J_t$.

From the equalities

1. $f^m = X_1^{mi} Y_1^{mi} Y_2^{mi(r-2)}$,

it is possible to write f^m as the product of an element of $I_m J_t$ and $m-1$ elements of J_r, that is
$f^m = (X_1^m Y_2^t)^i \prod_{s=1}^{m-1} (Y_1^{h_s} Y_2^{k_s})^i$,
with $h_s + k_s = r$ for all $s = 1, \ldots, m-1$, $\sum_{s=1}^{m-1} h_s = m$ and $\sum_{s=1}^{m-1} k_s = m(r-2) - t$, it follows that f^m is in L^{mi}.

2. $f^m = X_1^{\frac{mi}{2}} Y_1^{\frac{mir}{2}} Y_2^{\frac{i}{2}m(r-1)} = (X_1^m Y_2^t)^{\frac{i}{2}} \prod_{s=1}^{\frac{i}{2}(2m-1)} (Y_1^{h_s} Y_2^{k_s})^i$

with $h_s + k_s = r$ for all $s = 1, \ldots, m-1$, $\sum_{s=1}^{\frac{i}{2}(2m-1)} h_s = rm\frac{i}{2} - t$ and $\sum_{s=1}^{\frac{i}{2}(2m-1)} k_s = m(r-1)\frac{i}{2}$, it follows that f^m is in L^{mi}.

c)$L = J_r + I_s J_t$

We prove that $f^s \in L^{si}$ in the same way of the previous case choosing $m = s$.

Remark 1.2 *In the squarefree case the ideals $J_r + I_r$ and $L = J_r + I_s J_t$ are not normal ideal, while the ideal $L = J_r + I_m J_t$ is normal (see [4]).*

Remark 1.3 *A general case of $L = J_r + I_r$ is the ideal $L = J_r + I_m$, with $r \neq m$. This ideal isn't normal too. In fact we have that L^i is not integrally closed for all $i \geq 1$.*

There are the following cases:

a) If r,m are even, then there exists

$$f = \begin{cases} (X_1^{\frac{m}{2}} Y_1^{(\frac{r}{2}-1)} Y_2)^i \in \overline{L^i} \setminus L^i, & \text{if } i \text{ is odd} \\ (X_1^{\frac{m}{2}} Y_1^r Y_2^{\frac{r}{2}})^{\frac{i}{2}} \in \overline{L^i} \setminus L^i & \text{if } i \text{ is even} \end{cases}$$

To show that f lies in the integral closure of L^i, it suffices to observe the equalities

1. $f^2 = (X_1^{\frac{m}{2}} Y_1^{(\frac{r}{2}-1)} Y_2)^{2i} = (X_1^m)^i (Y_1^{r-2} Y_2^2)^i$,
it follows that f^2 is in L^{2i}. As $deg_X(f) = \frac{m}{2}i$ and $deg_Y(f) = \frac{r}{2}i$, by a counting degree argument it follows that $deg(f) = \frac{(m+r)}{2}i$ and f is not in L^i.

2. $f^2 = (X_1^{\frac{m}{2}} Y_1^r Y_2^{\frac{r}{2}})^i = (X_1^m)^{\frac{i}{2}} (Y_1^r)^i (Y_2^r)^{\frac{i}{2}}$,
it follows that f^2 is in L^{2i}. As $deg_X(f) = \frac{mi}{4}$ and $deg_Y(f) = \frac{3ri}{4}$, by a counting degree argument it follows that $deg(f) = \frac{(m+3r)}{4}i$ and f is not in L^i.

b) If $(m, r) = m \neq 1$ odd, then there exists

$$f = \begin{cases} (X_1 X_2 Y_1^{(m-2)\frac{r}{m}})^i \in \overline{L^i} \setminus L^i & \text{if } i \text{ is odd} \\ (X_1 Y_1^{r-\frac{r}{m}} Y_2^r)^{\frac{i}{2}} \in \overline{L^i} \setminus L^i & \text{if } i \text{ is even} \end{cases}$$

c) If $(m,r) = r \neq 1$ odd, then there exists

$$f = \begin{cases} (X_1^{\frac{m}{r}} Y_1^{\frac{m}{r}} Y_2^{r-2})^i \in \overline{L^i} \setminus L^i & \text{if } i \text{ is odd} \\ (X_1^{\frac{m}{r}} Y_1^r Y_2^{r-1})^{\frac{i}{2}} \in \overline{L^i} \setminus L^i & \text{if } i \text{ is even} \end{cases}$$

We prove the cases $b)$ and $c)$ in the similar way as the case $a)$.

Proposition 1.2 *Let $R = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n]$ be a polynomial ring over a field K. Let $L = I_k J_r + I_s J_t$ be an ideal of R , with $k > 1, s = k+2, t \geq 1, k+r = s+t$. Then :*

1. *If i is odd, there exists $f = (X_1 X_2^k Y_1^{r-1})^i \in \overline{L^i}/L^i$.*

2. *If i is even, there exists $f = (X_1^k X_2^{k+1} Y_1^{2r-1})^{\frac{i}{2}} \in \overline{L^i}/L^i$.*

Proof.
1. Let $f = (X_1 X_2^k Y_1^{r-1})^i$ be a monomial of R. To show that f lies in the integral closure of L^i, it suffices to observe the equality

$$f^2 = X_1^{2i} X_2^{2ki} Y_1^{2i(r-1)} = (X_2^k Y_1^r)^i (X_1^2 X_2^k Y_1^{r-1})^i,$$

it follows that $f^2 \in L^{2i}$.
2. Let $f = (X_1^k X_2^{k+1} Y_1^{2r-1})^{\frac{i}{2}}$ be a monomial of R. Since

$$f^2 = X_1^{ik} X_2^{i(k+1)} Y_1^{i(2r-1)} = (X_1 X_2^{k-1} Y_1^r)^{\frac{3i}{2}} (X_1^{2k-3} X_2^{5-k} Y_1^{r-2})^{\frac{i}{2}},$$

it follows that $f^2 \in L^{2i}$.
By counting degree argument it follows that f is not in L^i.

Remark 1.4 *In the square-free case, the powers of the ideal $L = I_k J_r + I_s J_t$ are not complete* (see [4]).

2

Let $R = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n]$ be the polynomial ring of Section 1. We consider the remaining two classes of ideals of R examined before.

i) $L = I_k J_r + I_{k+1} J_{r-1}$,

ii) $L = I_k J_r$.

We will be able to prove that they are both normal.
A crucial result for obtaining the normality of $i)$ is the following:

Lemma 2.1 *Let $L = I_kJ_r + I_{k+1}J_{r-1}$ and $L' = I_{k-1}J_r + I_kJ_{r-1}$ (resp. $L' = I_kJ_{r-1} + I_{k+1}J_{r-2}$) $\subset R = K[X_1, \dots, X_m; Y_1, \dots, Y_n]$. If $\wp \subset R$ is a face ideal, such that $X_i \notin \wp$, for some i (resp. $Y_j \notin \wp$, for some j), then*

$$(L)_\wp = (L')_\wp = J_{r-1}.$$

Proof. If we localize L and L' at $\wp$, the variable X_i is invertible in $(L)_\wp$ and in $(L')_\wp$. Since $X_i^{k-1} \in I_{k-1}$ and $X_i^k \in I_k$, we have $(I_{k-1})_\wp = R$ and $(I_k)_\wp = R$, and it follows $(I_{k-1}J_r)_\wp = (I_{k-1})_\wp(J_r)_\wp = (J_r)_\wp$ and $(I_kJ_{r-1})_\wp = (I_k)_\wp(J_{r-1})_\wp = (J_{r-1})_\wp$. Hence

$$(L')_\wp = (J_r)_\wp + (J_{r-1})_\wp = (J_{r-1})_\wp.$$

In the same way we have

$$(L)_\wp = (J_r)_\wp + (J_{r-1})_\wp = (J_{r-1})_\wp.$$

Then $(L)_\wp = (L')_\wp$.

Remark 2.1 *In the square-free case, we have $(I_k)_\wp = (I'_{k-1})_\wp$, where I'_{k-1} is a square-free ideal of R generated by monomials of degree $k-1$ in the variables $X_1, \dots, X_{i-1}, X_{i+1}, \dots X_m$ and $\wp \subset R$ a face ideal, with the variable $X_i \notin \wp$. The same result is obtained for J_r (see [5], 7.5.1). Hence for all mixed product ideals we have $(L)_\wp = (L')_\wp$.*
In the not square-free case, the result is true only for the ideals $L = I_kJ_r + I_{k+1}J_{r-1}$, $L = I_kJ_r$,$L = I_kJ_r + I_sJ_t$, and L' is the ideal generated in the degree $k+r-1$ by all the variables. For other ideals the localization produces the ring R. For example, if $L = I_r + J_r$ and $X_i \notin \wp$, we have $(L)_\wp = (I_r)_\wp + (J_r)_\wp = R + (J_r)_\wp = R$.

Proposition 2.1 *Let $L = I_kJ_r + I_{k+1}J_{r-1}$, with $k \geq 0$ and $r \geq 1$. Then L is complete.*

Proof. By induction on $k+r$. If $k+r = 1$, then $k = 0$, $r = 1$ and $L = J_1 + I_1$ is integrally closed.

Assume $k+r > 1$. By induction hypothesis the ideal $L' = I_{k-1}J_r + I_kJ_{r-1}$, generated in the degree $k+r-1$, is complete. We set $M = \overline{L}/L$. If $M \neq (0)$, take an associated prime ideal $\wp$ of M. Since $M \hookrightarrow R/L$, an associated prime ideal of M is an associated prime of R/L, this implies that $\wp$ is a face ideal, since the monomial ideal L has a primary decomposition into monomial ideals and every associated prime is a face ideal (see [5], 5.1.3). Suppose that $\wp \neq \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal, then there exists a variable $X_i \notin \wp$. From Lemma 2.1, we have:

$$(L')_\wp = (L)_\wp$$

and

$$M_\wp = (\overline{L}/L)_\wp = (\overline{L})_\wp/(L)_\wp = (\overline{L'})_\wp/(L')_\wp = 0,$$

because L' is complete. Contradiction, because $\wp$ is in the support of M. Hence the maximal ideal $\mathcal{M}$ is the only associated prime of M and there exists a monomial $f \in (\overline{L}/L)$ such that $(L : f) = \mathcal{M}$. The support of f contains one of the variables Y_i: if $f = \underline{X}^a$, then $f \in \overline{L} \Rightarrow f^i \in L^i$ for some $i \geq 1$. Hence we must have $r = 1$ and $f^i \in (I_{k+1})^i$. As I_{k+1} is normal then $f \in (I_{k+1}) \subset L$. Contradiction, because $f \notin L$. Let $Y_1 \in supp(f)$ such that $deg_{Y_1}(f) \geq deg_{Y_i}(f)$ for $i = 2, \ldots, n$. Then we can write

$$Y_1 f = g\omega,$$

where ω is a monomial of L (of degree $k + r$) and g is a monomial of R. (We observe that $deg(g) > 0$ because $f^i \in L^i$ and $Y_1 \notin supp(g)$ because $\notin L$.)

We assume that Y_j divides g for $j \neq 1$. Let $c = deg_{Y_1}(f)$, as Y_1^{c+1} divides $Y_1 f$ then Y_1^{c+1} divides ω. Assume that $Y_1 \in supp(\omega)$ and note that $Y_j \in supp(\omega)$; if $Y_j \notin supp(\omega)$ the equality

$$Y_1 f = (\omega Y_j/Y_1)(Y_1 g/Y_j),$$

implies that $f \in L$.

Theorem 2.1 *Let $L = I_k J_r + I_{k+1} J_{r-1}$, with $k \geq 0$ and $r \geq 1$. Then L is normal.*

Proof. By induction on $k + r$. If $k + r = 1$, $L = I_1 + J_1$ is normal. Now we assume $k + r \geq 2$ and we use induction on p, for all $p \geq 1$.
$p = 1$: $L = \overline{L}$ by lemma 2.1.
$p > 1$: we assume L^i complete for $1 \leq i < p$. We set $M = \overline{L^p}/L^p$. If $M \neq (0)$, take an associated prime ideal $\wp$ of M. Since $M \hookrightarrow R/L^p$, an associated prime ideal of M is an associated of R/L^p, this implies that $\wp$ is a face ideal (since the monomial ideal $L^p = q_1 \cap \cdots \cap q_s$ is a primary decomposition into monomial ideals and every associated prime is a face ideal (see [5] 5.1.3)).We suppose that $\wp \neq \mathcal{M}$, $\mathcal{M}$ is a maximal ideal. If a variable $X_i \notin \wp$ then (by lemma 2.1):

$$({L'}^p)_\wp = (L^p)_\wp,$$

where $L' = I_{k-1} J_r + I_k J_{r-1}$ generated in the degree $k + r - 1$.
We have

$$M_\wp = (\overline{L^p}/L^p)_\wp = (\overline{L^p})_\wp/(L^p)_\wp = (\overline{{L'}^p})_\wp/({L'}^p)_\wp \subseteq (\overline{{L'}^{p-1}})_\wp/({L'}^{p-1})_\wp = 0,$$

because $(L')^{p-1}$ is complete by induction hypothesis (on $k + r$ and p). This is a contradiction, because $\wp$ is in the support of M. Hence the maximal ideal

$\mathcal{M}$ is the only associated prime of M and there exists a monomial $f \in (\overline{L^p}/L^p)$ such that $(L^p : f) = \mathcal{M}$. The support of f contains one of the variables Y_i: if $f = \underline{X}^a$, then $f \in \overline{L^p} \Rightarrow f^i \in L^{pi}$ for some $i \geq 1$. Hence we must have $r = 1$ and $f^i \in (I_{k+1})^{ip}$. As I_{k+1} is normal then $f \in (I_{k+1})^p \subset L^p$. This is a contradiction because $f \notin L^p$. Let $Y_1 \in supp(f)$ such that $deg_{Y_1}(f) \geq deg_{Y_i}(f)$ for $i = 2, \ldots, n$. Then we can write

$$Y_1 f = g\omega_1 \cdots \omega_p,$$

where $\omega_1 \ldots \omega_p$ are monomials of L (of degree $k + r$) and g is a monomial of R. (We observe that $deg(g) > 0$ because $f^i \in L^{ip}$ and $Y_1 \notin supp(g)$ because $f \notin L^p$.)

Case I) We assume that Y_j divides g for $j \neq 1$. Let $c = deg_{Y_1}(f)$. As Y_1^{c+1} divides $Y_1 f$ then Y_1^{c+1} divides $\omega_1 \cdots \omega_p$. Assume that $Y_1 \in supp(\omega_i)$ for $i = 1, \ldots, c+1$ and note that $Y_j \in supp(\omega_i)$ for $i = 1, \ldots, c+1$; if $Y_j \notin supp(\omega_i)$ the equality

$$Y_1 f = \omega_1 \cdots (\omega_i Y_j/Y_1) \cdots \omega_{c+1} \cdots \omega_p (Y_1 g/Y_j),$$

implies that $f \in L^p$.

Case II) Assume that $g = \underline{X}^a$ and X_j divides g.

a) First suppose that there exists a monomial ω_l of the form

$$\omega_l = (X_{i_1} \cdots X_{i_k})(Y_1 Y_{j_2} \cdots Y_{j_r}),$$

with $1 \leq i_1 \leq \ldots \leq i_k \leq m,\ 1 \leq j_2 \leq \ldots \leq j_r \leq n$ and $Y_1 \in supp(\omega_l)$. If $Y_1 \notin supp(\omega_l)$ and $X_j \in supp(\omega_l)$, then we can write

$$Y_1 f = \omega_1 \cdots \omega_{l-1}(X_{i_1} \cdots X_{i_k} X_j)(Y_{j_2} \cdots Y_{j_r})\omega_{l+1} \cdots \omega_p (Y_1 g/X_j),$$

it follows $f \in L^p$.

Then there exists a monomial ω_q of the form:

$$\omega_q = \begin{cases} (1)\ (X_{s_1} \cdots X_{s_{k+1}})(Y_{t_1} \cdots Y_{t_{r-1}}) \\ (2)\ (X_{s_1} \cdots X_{s_k})(Y_{t_1} \cdots Y_{t_r}) \end{cases},$$

with $1 \leq s_1 \leq \ldots \leq s_{k+1} \leq m,\ 1 \leq t_1 \leq \ldots \leq t_r \leq n$ and $X_j \notin supp(\omega_q)$. In the case (1): $X_{s_1}, \ldots, X_{s_{k+1}} \not\subseteq X_{i_1}, \ldots, X_{i_k}$ and let $X_{s_1} \notin \{X_{i_1}, \ldots, X_{i_k}\}$. From the equality

$$Y_1 f = g\omega_l\omega_q \prod_{i \neq l,q} \omega_i = (Y_1 g/X_j)(X_{s_1}\omega_l/Y_1)(X_j\omega_q/X_{s_1}) \prod_{i \neq l,q} \omega_i,$$

it follows $f \in L^p$.

In the case (2): $\{Y_{t_1}, \ldots, X_{t_r}\} \nsubseteq \{Y_{j_2}, \ldots, Y_{j_r}\}$ and let $Y_{t_1} \notin \{Y_{j_2}, \ldots, Y_{j_r}\}$. From the equality

$$Y_1 f = g\omega_l\omega_q \prod_{i\neq l,q} \omega_i = (Y_1 g/X_j)(Y_{t_1}\omega_l/Y_1)(X_j\omega_q/Y_{t_1}) \prod_{i\neq l,q} \omega_i,$$

it follows $f \in L^p$.

b) Suppose that all monomials ω_l that contain Y_1 in their support are

$$\omega_l = (X_{i_1} \cdots X_{i_{k+1}})(Y_1 Y_{j_2} \cdots Y_{j_{r-1}}).$$

There exists

$$\omega_q = \begin{cases} (1)\ (X_{s_1} \cdots X_{s_k})(Y_{t_1} \cdots Y_{t_r}) \\ (2)\ (X_{s_1} \cdots X_{s_{k+1}})(Y_{t_1} \cdots Y_{t_{r-1}}) \end{cases}.$$

From now on, by using the same technic used in [5](Prop 7.5.8), we obtain the proof.

Remark 2.2 *In the square-free case, the ideal $L = I_k J_r + I_{k+1} J_{r-1}$ is normal too* (see [4]).

Theorem 2.2 *Let $L = I_k J_r$, with $k, r > 1$. Then L is normal .*

Proof. First we prove that L is complete. It is enough to prove that $I_k \cap J_r$ is integrally closed and $I_k J_r = I_k \cap J_r$.

To prove that $\overline{I_k \cap J_r} = \overline{I_k} \cap \overline{J_r}$, it is enough to prove that $\overline{I_k \cap J_r} \subseteq \overline{I_k} \cap \overline{J_r}$, since $\overline{I_k} \cap \overline{J_r} = I_k \cap J_r \subseteq \overline{I_k \cap J_r}$. For all $z \in \overline{I_k \cap J_r}$ there exists an equation $z^l + a_1 z^{l-1} + \cdots + a_{l-1} z + a_l = 0$, with $a_i \in (I_k \cap J_r)^i$ for all $i = 1, \ldots, l$. It follows that $a_i \in (I_k)^i$ and $a_i \in (J_r)^i$. Hence $z \in \overline{I_k} \cap \overline{J_r}$.

Now, let $f \in I_k$ and $g \in J_r$, $G.C.D.(f, g) = 1$, it follows that fg is a $l.c.m(f, g)$, hence $fg \in I_k \cap J_r$.

Then L is complete.

For all $i > 0$, it results

$$L^i = (I_k)^i (J_r)^i = (I_k)^i \cap (J_r)^i,$$

hence L^i is integrally closed, because $(I_k)^i$ and $(J_r)^i$ are integrally closed.

Remark 2.3 *In the square-free case, the ideal as $L = I_{k+1}$, $L = I_k J_r$ is normal too* (see [4]).

For computing examples we used the computer algebra program [2], that was able to find the monomials of the integral closure of L^i in the simplest cases.

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