



Extension of Paul Erdős graph on hypergroups

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Abstract

This paper introduces ultra non-commutative graphs derived from ultra non-commutative hypergroups. Vertices represent ultra non-commutative elements of the hypergroup U , defined as $A_U = \{x \in U \mid \exists y \in U, y \circ x \cap x \circ y = \emptyset\}$. An edge connects vertices x and y if and only if $y \circ x \cap x \circ y = \emptyset$. We investigate properties of the associated graph $\Gamma(U)$, including connectivity, Hamiltonian conditions, and planarity.

1 Introduction

Graph theory has emerged as a powerful tool across diverse fields in recent years. Studying algebraic (hyper)structures through the lens of graph properties has become an exciting area of research, yielding numerous intriguing results and open questions. Researchers have extensively explored the connection between (hyper)groups and graphs by associating graphs with these algebraic structures and investigating their algebraic properties.

For instance, the idea of non-commuting graphs, initially proposed for finite groups to analyze the structure of non-central elements [1], has been effectively adapted to hypergroups, where it illustrates the algebraic characteristics of non-commuting elements through the connectivity and structure of graphs [17]. This adaptation highlights the effectiveness of graph-theoretic methods in investigating the intricate interactions within hyperoperations. Fundamental research on equivalence relations in hyperstructures has established crucial frameworks for defining quotient hyperstructures, contrasting

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traditional and contemporary methods for forming equivalence classes in hyperalgebra [4]. Concurrently, the concept of geometric polygroups has provided a fresh algebraic viewpoint on geometric transformations, linking symmetry and reflection attributes in geometry with polygroup operations [5]. Graph vertex colorings have also been employed to define color hypergroups and related join spaces, which represent combinatorial groupings and algebraic interactions among color classes [12]. In a different research avenue, path hypergroupoids have been created by establishing set-valued operations on graph paths, with investigations uncovering their connections to commutativity and connectivity [18]. Likewise, path hypercompositions in automata have been suggested to characterize nondeterministic transitions through hyperoperations, connecting automata theory with hyperstructure analysis [21]. Recently, Cayley digraphs over polygroups have been introduced as a generalization of classical Cayley graphs, demonstrating how the algebraic features of polygroups such as their non-deterministic, set-valued operations can be represented in directed graph frameworks [24]. Graphs and hypergraphs were studied in [13] and [19]. Together, these studies illustrate a dynamic research trajectory at the convergence of hyperalgebra, combinatorics, and geometry. In this context, one can also refer to the subsequent papers: [14], [16], and [20].

A comprehensive overview of graphs defined on groups, encompassing non-commuting, deep commuting, nilpotency, solvability, and Engel graphs, is available in [7]. A recent collection of theoretical and applied research delving into the relationship between algebraic structures and graphs can be found in [8]. In this note we propose a construction of a graph starting from an ultra non-commutative hypergroup, i.e. a finite hypergroup $(H, \circ)$, having elements x and y such that $y \circ x \cap x \circ y = \emptyset$. Paul Erdős (as mentioned in Neumann [22]), utilized a similar approach in 1975. Let G be a non-abelian group and $Z(G)$ be the set of elements of G commuting with all elements in G , i.e. the center of G . Take the set of vertices $V = \{x \in G \mid x \notin Z(G)\}$. Two vertices x and y are joined whenever they are not commute, i.e. $xy \neq yx$. Abdollahi et al. [1], coined the term "non-commuting graph" for Erds graph $\Gamma(G)$. Their work delved into the influence of the graph-theoretic attributes of $\Gamma(G)$ on the group-theoretic characteristics of G . Iranmanesh et al. in [17] extending this notion to the hypergroups, instead of a non-abelian group, they considered a *non-central hypergroup*, i.e., the associated fundamental group $\overline{H} = H/\beta$ is non-abelian. Two vertices x and y whenever $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$, where ω_H is the core (heart) of the hypergroup $(H, \circ)$. The properties of an associated graph of non-central hypergroup, H are related to the fundamental group H/β . Now let $(U, \circ)$ be a finite hypergroup and set $A_U = \{x \in U \mid \exists y \in U : y \circ x \cap x \circ y = \emptyset\}$. If $A_U \neq \emptyset$, then $(U, \circ)$ is called an *ultra non-commutative hypergroup*.

In this paper we extend Erdős construction to ultra non-commutative hypergroups as follows: A_U is the set of vertices and join two distinct vertices x, y if $y \circ x \cap x \circ y = \emptyset$. Building on the concept of non-commuting graphs, our paper explores the following: After reviewing fundamental graph theory concepts and establishing notation, Section 3 introduces essential hypergroup theory definitions used throughout the paper. We define ultra non-commutative hypergroups and ultra non-commuting graphs, providing examples of such graphs over ultra non-commutative hypergroups. We delve into properties of the ultra non-commuting graph $\Gamma(U)$, including its connectivity, Hamiltonian conditions, and planarity.

2 Preliminaries

A simple graph G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes), and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges. We call $V(G)$ the vertex set and $E(G)$ the edge set of G . An edge $\{v, w\}$ is said to join the vertices v and w , and is usually abbreviated to vw . For the following basic notions we refer the readers to the fundamental book in graph theory by Bondy and Murty [6].

Definition 2.1. [17] The degree of a vertex v of G is the number of edges incident with v , and is written $\deg(v)$. The minimum degree of G is denoted by $\delta(G)$.

Definition 2.2. [25] A subgraph of a graph G is a graph, each of whose vertices belongs to $V(G)$ and each of whose edges belongs to $E(G)$.

Definition 2.3. [25] Two graphs G and H are isomorphic, written $G \cong H$, if there is a one-one correspondence between the vertices of G and those of H such that the number of edges joining any two vertices of G is equal to the number of edges joining the corresponding vertices of H .

Definition 2.4. [15, 25]

- (1) A *walk* in a graph G is a sequence of vertices $v_0, v_1, \dots, v_k$, in which any two consecutive edges are adjacent. A walk is a *closed* if $v_0 = v_k$. A *path* is a walk without repeated vertices. The *length* of a walk is the number of edges in it. A *connected graph* is a graph having a path between every pair of vertices.
- (2) A *cycle* in a graph G is a closed path of length at least 3. We use the notation C_n for a cycle of n . A *Hamiltonian cycle* is a closed path that visits every vertex in the graph exactly once. A graph is Hamiltonian if it has a Hamiltonian cycle.

Let $x = v_1, v_2, \dots, v_n = y$ be a path between two vertices x and y in a given graph G . By $d(x, y)$, we mean the minimum length of all paths from x to y . If there are no walks between x and y , let $d(x, y) = \infty$ by convention. It is clear that a graph G is connected if $d(x, y) < \infty$ for all $x, y \in V$. For a connected simple finite graph G define the *diameter* of G as $\text{diam}(G) = \max\{d(x, y) \mid x, y \in V(G), x \neq y\}$.

Theorem 2.5. (Dirac's Theorem) [25] If G is a simple graph with $n \geq 3$ vertices and if $\delta(G) \geq \frac{n}{2}$, then G has a Hamiltonian cycle.

A simple graph in which each pair of distinct vertices are adjacent is a complete graph. We denote the complete graph on n vertices by K_n . If the vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B , then G is a bipartite graph. Alternatively, a bipartite graph is one whose vertices can be colored black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B). A complete bipartite graph is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the bipartite graph with n black vertices and m white vertices by $K_{n,m}$. It is obvious that each K_n is a Hamiltonian graph whenever $n \geq 3$, while $K_{n,m}$ is a Hamiltonian graph if and only if $n = m \geq 2$. A planar graph is a graph that can be drawn in the plane without crossings that is, so that no two edges intersect geometrically except at a vertex to which both are incident. For example, K_5 and $K_{3,3}$ are not planar graphs, while K_4 is a planar graph.

Theorem 2.6. [25] If G is a connected simple planar graph with $v(> 3)$ vertices and e edges, then $e \leq 3v - 6$.

As a result we get the following.

Proposition 2.7. [17] Every simple planar graph contains a vertex of degree at most 5.

Theorem 2.8. [17] If a maximal planar graph has vertices with $v > 2$, then it has $3v - 6$ edges.

Theorem 2.9. (Kuratowski's Theorem) [25] A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

3 On ultra non-commuting graphs

This section focuses on constructing and analyzing the properties of a specific graph associated with an ultra non-commutative hypergroup, that is, a hypergroup $(U, \circ)$ where not all products commute ($\exists x, y : x \circ y \cap y \circ x = \emptyset$).

For a comprehensive understanding of hyperstructure theory, refer to the foundational works by Corsini [9], Davvaz and Leoreanu-Fotea [11], Corsini and Leoreanu [10], and Vougiouklis [26]. We begin by defining a hypergroup. A hypergroup is a non-empty set H equipped with a hyperoperation $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ that satisfies the associative property: $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$, and the reproduction axiom: $x \in H$, $x \circ H = H \circ x = H$ for all $x \in H$. A crucial concept in hypergroup theory is the "heart" of a hypergroup, which, in some ways, parallels the center of a group. To define the heart of a hypergroup, we introduce the β -relation, also known as the fundamental relation due to its significant properties. Further details regarding its meaning and applications can be found in Al Tahan et al. [3], and Novák et al. [23]. Define first, for all $n \geq 1$, on a hypergroup $(H, \circ)$ the relation β_n as follows:

$a\beta_nb \Leftrightarrow \exists(z_1, \dots, z_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n z_i$ and take $\beta = \bigcup_{i=1}^n \beta_i$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H . Denote by β^* the transitive closure of β , so β^* is an equivalence relation on H , see Corsini [9].

It is well known that β^* is the smallest strongly regular relation on a hypergroup $(H, \circ)$, such that the quotient H/β^* is a group with respect to the following operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z), \forall z \in x \circ y$. H/β^* is called the *fundamental group* associated with H . The *heart* ω_H of the hypergroup H is the set of all elements x of H , for which the equivalence class $\beta^*(x)$ is the identity element of the fundamental group H/β^* .

Definition 3.1. [9] Suppose $(U, \cdot)$ and $(K, \circ)$ are hypergroups and α corresponds one to one from U to K . α is called an *isomorphism* whenever $\alpha(x \cdot y) = \alpha(x) \circ \alpha(y)$, for every x and y in H .

Definition 3.2. A finite hypergroup $(U, \circ)$ is called *ultra non-commutative* if there exist elements $x, y \in U$ such that the hyperproducts $y \circ x$ and $x \circ y$ have no elements in common, i.e., $y \circ x \cap x \circ y = \emptyset$.

Example 3.3. If a group G has an order less than or equal to 5, then G is abelian. Consequently, the set of elements in G that do not commute with any other element is empty.

Definition 3.4. Given an ultra non-commutative hypergroup $(U, \circ)$, we construct an associated graph $\Gamma(U)$ as follows: The set of vertices of $\Gamma(U)$ is denoted by A_U . Two distinct vertices x and y are connected by an edge if and only if the hyperproducts $x \circ y$ and $y \circ x$ have no elements in common.

Example 3.5. Consider the hypergroup defined by Table 1.

Here we have the set of vertices $A_U = \{x, y\}$ and the associated graph is depicted in the Figure 1.

$(U, \circ)$	x	y	z
x	x	x	U
y	y	y	U
z	U	U	U

Table 1: Cayley Table of the hypergroup $(U, \circ)$ Figure 1: Ultra non-commuting graph of U

Example 3.6. Let us consider the hypergroup represented by the Table 2:

$(K, \circ)$	x	y	z	r	t
x	x	y	z	r	t
y	y	y	y	K	y
z	z	z	z	K	z
r	r	K	K	K	K
t	t	z	z	K	z

Table 2: Caley Table of the hypergroup $(K, \circ)$

In this example the set of vertices is $A_K = \{y, z, t\}$ and the associated graph is depicted in the Figure 2.

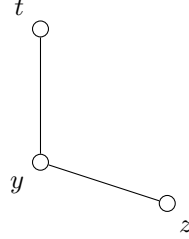


Figure 2: Ultra non-commuting graph of K .

Example 3.7. Consider the hypergroup defined by the Table 3.

$(L, \circ)$	x	y	z	t
x	x	x	L	L
y	y	y	L	L
z	L	L	z	z
t	L	L	t	t

Table 3: Cayley Table of the hypergroup $(L, \circ)$

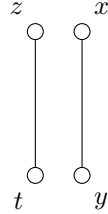


Figure 3: Ultra non-commuting graph of L .

Proposition 3.8. *Let U and K be two ultra non-commutative hypergroups, such that $U \cong K$. Then $\Gamma(U) \cong \Gamma(K)$.*

Proof. As ultra non-commutative hypergroups U and K are isomorphic there exists α one to one correspondence from A_U to A_K . Let $(x, y) \in E_U$, we prove

$(\alpha(x), \alpha(y)) \in E_K$. Since

$$\alpha(x)\alpha(y) \bigcap \alpha(y)\alpha(x) = \alpha(xy) \bigcap \alpha(yx) = \alpha(xy) \bigcap yx = \emptyset.$$

Hence $(\alpha(x), \alpha(y)) \in E_K$. On the other hand, let $(a, b) \in E_K$, in the following we prove that $(\alpha^{-1}(a), \alpha^{-1}(b)) \in E_U$. Since $(\alpha^{-1}(a)\alpha^{-1}(b)) \bigcap (\alpha^{-1}(b)\alpha^{-1}(a)) = \alpha^{-1}(ab \bigcap ba) = \emptyset$. Then $(\alpha^{-1}(a), \alpha^{-1}(b)) \in E_U$. Hence $(x, y) \in E_U$, if and only if $(\alpha(x), \alpha(y)) \in E_K$. Therefore we have $\Gamma(U) \cong \Gamma(K)$. $\square$

Remark 3.9. The following example shows that the converse of Proposition 3.8 is not true.

Example 3.10. Let us consider the hypergroup represented by the Table 4. Then the associated graph is the same with Example 3.5 while $U \not\cong L$.

$(L, \circ)$	x	y	z	t
x	x	x	L	L
y	y	y	L	L
z	L	L	L	L
t	L	L	L	L

Table 4: Cayley Table of the hypergroup $(L, \circ)$

Let $(U, \circ)$ be a hypergroup (or more generally a hyperstructure). The relation β^* is defined as the transitive closure of the relation β , where $x\beta^*y$ if and only if there exists a finite sequence $x = x_0, x_1, \dots, x_n = y$ such that $x_i\beta x_{i+1}$, for all i and x, y in U . The class $\bar{x}$ can be thought of as: $\bar{x} = \{y \in U : x\beta^*y\}$. Lets define the equivalence class of a nonempty subset D of U under the relation β^* as bellow: $D = \{\bar{x} : x \in D\}$.

Proposition 3.11. *Let $(U, \circ)$ be a an ultra non-commutative hypergroup and $A_U = \{x_1, x_2, \dots, x_n\}$. Then $A_{\bar{U}} \subseteq \bar{A}_U$.*

Proof. Suppose that $A_{\bar{U}} = \{\bar{x} \in U/\beta^* | \bar{y} \in U/\beta^* : \bar{x}\bar{y} \neq \bar{y}\bar{x}\}$. Since $\bar{x} \in A_{\bar{U}}$, there exists $\bar{y} \in A_{\bar{U}}$, such that $\bar{x}\bar{y} \neq \bar{y}\bar{x}$. As a result $xy \bigcap yx = \emptyset$. Thus $x \in A_U$. Therefore $\bar{x} \in \bar{A}_U$. So we have $A_{\bar{U}} \subseteq \bar{A}_U$. $\square$

Corollary 3.12. *If U is an ultra non-commutative hypergroup, then $\Gamma(\bar{U})$ is subgraph of $\Gamma(U)$.*

Definition 3.13. Suppose $(U, \circ)$ is an ultra non-commutative hypergroup, we set: $B_U = \{x \in A_U | \bar{x} \notin A_{\bar{U}}\}$ and for all $x \in A_U$ we have $A_U(x) = \{y \in A_U | x \circ y \bigcap y \circ x = \emptyset\}$.

Example 3.14. Let us consider the hypergroup $U = \{e, x, y, z\}$ represented by the Table 5.

$(U, \circ)$	e	x	y	z
e	e	x	y	z
x	x	e	y	z
y	y	z	$\{e, x, z\}$	$\{e, x, y\}$
z	z	b	$\{e, x, z\}$	$\{e, x, y\}$

Table 5: Cayley Table of the hypergroup $(U, \circ)$

In this case have: $A_U = \{x, y, z\}$, $A_{\overline{U}} = \emptyset$, $B_U = \{x, y, z\}$, $\overline{A_U} = \{\bar{e}\}$ and $A_U(x) = \{y, z\}$, $A_U(y) = \{x\}$, $A_U(z) = \{x\}$.

Remark 3.15. If H is a group, then $B_U = \emptyset$.

Theorem 3.16. For every ultra non-commutative hypergroup U , we have:

- (i) $A_{\overline{U}} = \overline{A_U - B_U}$.
- (ii) $\overline{A_U - B_U} = \overline{A_U} - \overline{B_U}$.

Proof. (i) Let $\bar{x} \in \overline{A_U - B_U}$. Then there exists $y \in A_U - B_U$, such that $\bar{x} = \bar{y}$. Because $y \notin B_U$, then $\bar{y} \in A_{\overline{U}}$ so $\bar{x} \in A_{\overline{U}}$. Therefore $\overline{A_U - B_U} \subseteq A_{\overline{U}}$. Now let $\bar{x} \in A_{\overline{U}}$. Because $A_{\overline{U}} \subseteq \overline{A_U}$, then $\bar{x} \in \overline{A_U}$. So there exists $y \in A_U$, such that $\bar{x} = \bar{y}$. By the definition of B_U as a result we conclude that $y \notin B_U$. Thus $y \in A_U - B_U$. Therefore $\bar{y} \in \overline{A_U - B_U}$, and so $\bar{x} \in \overline{A_U - B_U}$. Hence $A_{\overline{U}} \subseteq \overline{A_U - B_U}$.

(ii) Now let $\bar{x} \in \overline{A_U - B_U}$. Then there exists $y \in A_U - B_U$ such that $\bar{x} = \bar{y}$. Since $\bar{y} \in A_{\overline{U}} \subseteq \overline{A_U}$, then $\bar{y} \in \overline{A_U}$ and $\bar{y} \notin \overline{B_U}$. Therefore $\bar{y} \in \overline{A_U - B_U}$ and so $\bar{x} \in \overline{A_U - B_U}$. Hence $\overline{A_U - B_U} \subseteq \overline{A_U} - \overline{B_U}$. Let $\bar{x} \in \overline{A_U} - \overline{B_U}$, then $\bar{x} \in \overline{A_U}$. So there exists $y \in A_U$ such that $\bar{x} = \bar{y}$. Since $\bar{y} \notin \overline{B_U}$. We have $y \notin B_U$. Thus $y \in A_U - B_U$. Consequently $\bar{y} \in \overline{A_U - B_U}$, and so $\bar{x} \in \overline{A_U - B_U}$. Finally, as a result we get $\overline{A_U} - \overline{B_U} \subseteq \overline{A_U - B_U}$. $\square$

Proposition 3.17. [1] For any non-abelian group G , the graph $\Gamma(G)$ is connected.

Lemma 3.18. If $x, y \in A_U$ and $\bar{x}, \bar{y} \in A_{\overline{U}}$, then there is a path between x, y .

Proof. If $\bar{x}, \bar{y} \in A_{\overline{U}}$. Then according the Proposition 3.17, there exists $\bar{a}_0 = \bar{x}, \bar{a}_1, \dots, \bar{a}_n = \bar{y}$ such that $\bar{x}\bar{a}_1 \neq \bar{a}_1\bar{x}, \bar{a}_1\bar{a}_2 \neq \bar{a}_2\bar{a}_1, \dots, \bar{a}_{n-1}\bar{y} \neq \bar{y}\bar{a}_{n-1}$. Consequently $xa_1 \cap a_1x = \emptyset, a_1a_2 \cap a_2a_1 = \emptyset, \dots, a_{n-1}y \cap ya_{n-1} = \emptyset$, so there exists a path from x to y . $\square$

Theorem 3.19. *If for all $x \in A_U$, $A_U(x) \cap (A_U - B_U) \neq \emptyset$, then $\Gamma(U)$ is connected.*

Proof. Suppose that $x, z \in A_U$. Because for all $x \in A_U$, $A_U(x) \cap (A_U - B_U) \neq \emptyset$, then there exists $y \in A_U(x)$ and $y \in A_U - B_U$. Thus $x \circ y \cap y \circ x = \emptyset$ and $y \notin B_U$. Therefore $\bar{y} \in A_{\bar{U}}$. On the other hand if $z \in A_U$, then $A_U(z) \cap (A_U - B_U) \neq \emptyset$. Hence there exists $y' \in A_U(z)$, such that $z \circ y' \cap y' \circ z = \emptyset$ and $y' \notin B_U$, therefore $\bar{y'} \in A_{\bar{U}}$. Because $\bar{y}$ and $\bar{y'}$ are in $A_{\bar{U}}$, by Lemma 3.18, there exists a path between y and y' . Thus there exists a path from x to z . $\square$

Recall that for any element x in a hypergroup $(U, \circ)$ we denote by $\bar{x} = \beta(x)$ the equivalence class of x modulo the relation β .

Proposition 3.20. *Let U be an ultra non-commutative hypergroup, and $A_{\bar{U}} = \overline{A_U}$. Then,*

(1) *if $x \in A_U$, then $\bar{x} = \beta(x) \subseteq A_U$.*

(2) $A_U = \bigcup_{\bar{x} \in A_{\bar{U}}/\beta} \bar{x}$.

Proof. (1) If $x \in A_U$. Then $\bar{x} \in A_{\bar{U}}$, therefore there exists $\bar{y} \in A_{\bar{U}}$, such that $\bar{y} \otimes \bar{x} \neq \bar{x} \otimes \bar{y}$. Now let $z \in \bar{x}$, it follows that $\bar{x} = \bar{z}$ and so $\bar{y} \otimes \bar{z} \neq \bar{z} \otimes \bar{y}$. As a result $z \circ y \cap y \circ z = \emptyset$. Hence $z \in A_U$, consequently $\beta(x) \subseteq A_U$.

(2) It follows immediately from part (1) and equivalence $x \in A_U \iff \bar{x} \in A_{\bar{U}}/\beta$. $\square$

Proposition 3.21. *Consider $a, b \in A_U$, $a, b \in \bar{x}$, and $\bar{x} \in A_{\bar{U}}$, then there exists a path between a, b .*

Proof. Suppose there is no path between a, b . Since $\bar{x} \in A_{\bar{U}}$, thus there exists $\bar{c} \in A_{\bar{U}}$, such that $\bar{c} \otimes \bar{x} \neq \bar{x} \otimes \bar{c}$. As a result $\bar{c} \otimes \bar{a} \neq \bar{a} \otimes \bar{c}$, therefore $c \circ a \cap a \circ c = \emptyset$ and $\bar{c} \otimes \bar{b} \neq \bar{b} \otimes \bar{c}$. As a result $c \circ b \cap b \circ c = \emptyset$. We conclude that there is a path between a, b , which is a contradiction. $\square$

Corollary 3.22. *Let $(U, \circ)$ be an ultra non-commutative hypergroup and $B_U = \emptyset$. Then the graph $\Gamma(A_U)$ is connected.*

Proof. Because $B_U = \emptyset$, $A_U(x) \cap (A_U - B_U) \neq \emptyset$, for all $x \in A_U$. By Theorem 3.19, we conclude that $\Gamma(U)$ is connected. $\square$

Proposition 3.23. *Let $(U, \circ)$ be an ultra non-commutative hypergroup and $A_{\bar{U}} = \overline{A_U}$. Then $\Gamma(U)$ is connected.*

Proof. Since $A_{\bar{U}} = \overline{A_U}$, it results that $B_U = \emptyset$. By corollary 3.22, it follows that $\Gamma(U)$ is connected. $\square$

Corollary 3.24. *Let $(G, \circ)$ be a non-commutative group. Because $B_U = \emptyset$, we have $\Gamma(G)$ is connected.*

Lemma 3.25. *Let $(G, \circ)$ be a non-commutative group. Then $\Gamma(G)$ is not a complete graph.*

Proof. Assume, for the sake of contradiction, that $\Gamma(G)$ is a complete graph. This implies that the diameter of $\Gamma(G)$, denoted by $\text{diam}(\Gamma(G))$, is 1. If $x \in A_G$: Since $\text{diam}(\Gamma(G)) = 1$, x must be its own inverse ($x = x^{-1}$). Otherwise, x and x^{-1} would be connected by an edge in the complete graph, implying that $x \circ x^{-1} x^{-1} \circ x$, which is a contradiction. If $x \notin A_G$: There must exist an element $y \in A_G$ such that $x \circ y \in A_G$. If not, $x \circ y$ would commute with every element in A_G , leading to a contradiction. Specifically, if $x \circ y$ commutes with all elements in A_G , then y would also commute with all elements in A_G , contradicting the assumption that $y \in A_G$. Since $x \circ y \in A_G$, it follows that $(x \circ y)^{-1} = x \circ y$. This implies that $y^{-1} \circ x^{-1} = x \circ y = y \circ x$ (as $x \in A_G$). Consequently, $x = x^{-1}$. Therefore, we have shown that $x = x^{-1}$ for all $x \in G$. This implies that G is an abelian group, which contradicts our initial assumption. Hence, $\Gamma(G)$ cannot be a complete graph. $\square$

Proposition 3.26. *If $(U, \circ)$ is an ultra non-commutative hypergroup and $A_{\overline{U}} = \overline{A_U}$, then $\Gamma(U)$ is not a complete graph.*

Proof. Suppose that $\Gamma(U)$ be a complete graph. We have $xy \cap yx = \emptyset$ and $\overline{x}, \overline{y} \in \overline{A_U}$, for all $x, y \in A_U$. As a result $\overline{x}, \overline{y} \in A_{\overline{U}}$ and $\overline{y} \otimes \overline{x} \neq \overline{x} \otimes \overline{y}$. Therefore $A_{\overline{U}}$ is a complete graph, which is a contradiction. $\square$

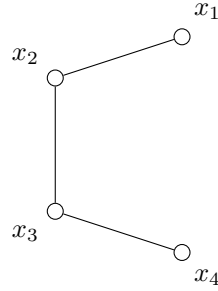
Proposition 3.27. *If $(U, \circ)$ is an ultra non-commutative graph and $A_{\overline{U}} = \overline{A_U}$, then $\text{diam}(\Gamma(U)) = 2$.*

Proof. Using Proposition 3.26, we know that the associated graph $\Gamma(U)$ is not complete, so there exist $x, y \in A_U$, such that $d(x, y) \neq 1$. Since $x \circ y \cap y \circ x \neq \emptyset$ and $\overline{x}, \overline{y} \in A_{\overline{U}}$. Hence there is $\overline{z} \in A_{\overline{U}}$, such that $\overline{z} \otimes \overline{x} \neq \overline{x} \otimes \overline{z}$ or $\overline{y} \otimes \overline{z} \neq \overline{z} \otimes \overline{y}$. It results that $x \circ z \cap z \circ x = \emptyset$ or $y \circ z \cap z \circ y = \emptyset$. Consequently $d(x, y) = 2$. $\square$

Example 3.28. Consider the hypergroup defined by the Table 6.

Then $\text{diam}(\Gamma(H_4)) = 3$.

$(H_4, \circ)$	x_1	x_2	x_3	x_4
x_1	H_4	$\{x_1, x_2\}$	H_4	H_4
x_2	$\{x_3, x_4\}$	H_4	$\{x_1, x_2\}$	H_4
x_3	H_4	$\{x_3, x_4\}$	H_4	$\{x_1, x_2\}$
x_4	H_4	H_4	$\{x_3, x_4\}$	H_4

Table 6: Cayley Table of the hypergroup $(H_4, \circ)$ Figure 4: Ultra non-commuting graph of H_4 .

Definition 3.29. [11] We say that a hypergroup H is complete, if for all $(n, m) \in \mathbb{N}^2$, such that $n, m \geq 2$, and for all $(x_1, \dots, x_n, y_1, \dots, y_m) \in H^{n+m}$, the following implication is valid:

$$\prod_{i=1}^n x_i \cap \prod_{j=1}^m y_j \neq \emptyset \Rightarrow \prod_{i=1}^n x_i = \prod_{j=1}^m y_j.$$

Theorem 3.30. [11] A hypergroup $(H, \circ)$ is complete if and only if $H = \bigcup_{i \in G} X_i$, where G and X_i satisfy the conditions:

- (1) G is a group. (Indeed $G = H/\beta^*$).
- (2) $i \neq j$ implies $X_i \cap X_j = \emptyset$.
- (3) If $(a, b) \in X_i \times X_j$, then $a \circ b = X_{ij}$.

Theorem 3.31. If $(U, \circ)$ is an ultra non-commutative complete hypergroup, then $\text{diam}(\Gamma(U)) = 2$.

Proof. According Theorem 3.16, we have $A_{\overline{U}} \subseteq \overline{A_U}$. By Proposition 3.27, we have to prove that $\overline{A_U} \subseteq A_{\overline{U}}$. Let $\bar{x} \in \overline{A_U}$. Then there exists $y \in A_U$,

such that $\bar{x} = \bar{y}$. Therefore there exists $z \in A_U$, such that $y \circ z \cap z \circ y = \emptyset$. Since $y \in X_i, z \in X_j$, then $X_i \circ X_j \neq X_j \circ X_i$, such $X_{ij} \neq X_{ji}$, as a result $ij \neq ji$. Because U is complete hypergroup, according Theorem 3.30, $U = \bigcup_{i \in G} X_i$, where $G = U/\beta^*$ and X_i satisfy the conditions (1), (2) and (3). Thus $i = \bar{y}, j = \bar{z}$. Hence $\bar{y} \otimes \bar{z} \neq \bar{z} \otimes \bar{y}$, as a result $\bar{y} \in A_{\bar{U}}$. Therefore $\bar{x} \in A_{\bar{U}}$. $\square$

Corollary 3.32. *Let $(G, \circ)$ be a non-commutative group. Then $\text{diam}(\Gamma(G)) = 2$.*

The graph associated to hypergroup has a different property from the graph of groups, for example, for every natural number n , there is a hypergroup of diameter n .

Theorem 3.33. *Let $n \geq 4$, $H_n = \{x_1, x_2, \dots, x_n\}$ and hyperoperation $*$ on it defined as follows:*

$$x_i * x_j = \begin{cases} \{x_1, x_2\} & j=i+1 \\ \{x_3, x_4\} & j=i-1 \\ H_n & \text{otherwise} \end{cases}$$

*Then $(H_n, *)$ is a hypergroup.*

Proof. (1) For all $i \leq n$, we have $x_i * H_n = H_n * x_i = H_n$. Because $n \geq 4$, there exist $j \neq i+1, i-1$.

(2) For all $x_i, x_j, x_k \in H_n$, we have $(x_i * x_j) * x_k = H_n = x_i * (x_j * x_k)$. Indeed, $(x_i * x_j) * x_k = \{x_1, x_2\} * x_k$ or $\{x_3, x_4\} * x_k$ or $H_n * x_k$. Therefore $(x_i * x_j) * x_k = H_n$. On the other hand, $x_i * (x_j * x_k) = x_i * \{x_1, x_2\}$ or $x_i * \{x_3, x_4\}$ or $H_n * x_k$. Thus $x_i * (x_j * x_k) = H_n$. Consequently the associativity condition is valid. Thus $(H_n, *)$ is a hypergroup. $\square$

Theorem 3.34. *For all $n \geq 4$, $\text{diam}(\Gamma(H_n)) = n - 1$.*

Proof. By Definition of H_n , for all $x_i, x_j \in H_n$, x_i and x_j are adjacent if and only if i and j are consecutive numbers (i.e. $j = i - 1$ or $j = i + 1$). Therefore the minimum length of all paths from x_i to x_j ; $d(x_i, x_j) = n - 1$ and so $\text{diam}(\Gamma(H_n)) = n - 1$. The associated graph H_n , is a graph of order n as follows:

$$x_1 \text{ --- } x_2 \text{ --- } x_3 \text{ --- } \dots \text{ --- } x_n$$

$\square$

Figure 5: Ultra non-commuting graph of H_n .

Definition 3.35. An ultra non-commuting hypergroup U is called *Hamiltonian* if the associated graph $\Gamma(U)$ is a Hamiltonian graph.

Example 3.36. Let $S_3 = \{(1), (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\}$ be the symmetric group on three elements and assume $X_{(1)} = \{e\}$, $X_{(1,2)} = \{x\}$, $X_{(1,3)} = \{y\}$, $X_{(2,3)} = \{z\}$, $X_{(1,3,2)} = \{t\}$, $X_{(1,2,3)} = \{r, p, q\}$ and $U = \cup_{\sigma \in S_3} X_{\sigma}$. Then we define on U the hyperoperation $\circ$ as follows: $a \circ b = X_{\sigma_a \cdot \sigma_b}$, where $a \in X_{\sigma_a}$ and $b \in X_{\sigma_b}$. This is the method to obtain a complete hypergroup, starting from a group [2]. Besides, U is an ultra non-commutative, which is not a Hamiltonian hypergroup. The associated graph A_U is as below:

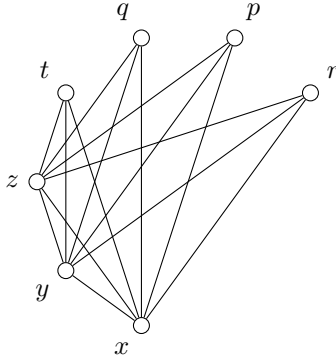


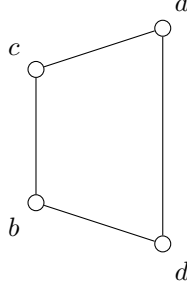
Figure 6: The associated graph A_U

Example 3.37. Consider the hypergroup $(U_1, \circ)$ defined by the Table 7.

$(U_1, \circ)$	e	a	b	c	d
e	e	a	b	c	d
a	a	$\{e, a, b\}$	$\{c, d\}$	$\{e, a, b\}$	$\{c, d\}$
b	b	$\{c, d\}$	$\{e, a, b\}$	$\{c, d\}$	$\{e, a, b\}$
c	c	$\{c, d\}$	$\{e, a, b\}$	$\{c, d\}$	$\{e, a, b\}$
d	d	$\{e, a, b\}$	$\{c, d\}$	$\{e, a, b\}$	$\{c, d\}$

Table 7: Hypergroup $(U_1, \circ)$

In this case the associated graph $\Gamma(U_1)$ is a Hamiltonian graph.

Figure 7: Associated graph $\Gamma(U_1)$

Theorem 3.38. *If U is an ultra non-commutative hypergroup such that $A_{\overline{U}} = \overline{A_U}$, and there exists $n \geq 1$ such that, $|\beta^*(x)| = n$, for all $x \in U$, then U is a Hamiltonian hypergroup.*

Proof. Because $A_{\overline{U}} = \overline{A_U}$, we have $A_{\overline{U}} \neq \emptyset$. Let $\bar{x} \in A_{\overline{U}} = \overline{U} - Z(\overline{U})$ and $C(\bar{x})$ be the set of elements that commute with $\bar{x}$ in $\overline{U}$. We have $|C(\bar{x})| \leq \frac{|\overline{U}|}{2}$. It follows that:

$$\deg(\bar{x}) = |\overline{U} - C(\bar{x})| > \frac{|\overline{U} - Z(\overline{U})|}{2} = \frac{|A_{\overline{U}}|}{2}.$$

Indeed, if $\deg(\bar{x}) \leq \frac{|\overline{U} - Z(\overline{U})|}{2}$, then more than half elements of $A_{\overline{U}}$ can commute with $\bar{x}$, therefore $|C(\bar{x})| \geq \frac{|\overline{U} - Z(\overline{U})|}{2} + |Z(\overline{U})|$, following that $|C(\bar{x})| > \frac{|\overline{U}|}{2}$, which is a contradiction. Moreover, we have

$$\deg(x) = \deg(\bar{x}) \cdot n = \deg(\bar{x}) \cdot \frac{|A_U|}{|A_{\overline{U}}|} > \frac{|A_{\overline{U}}|}{2} \cdot \frac{|A_U|}{|A_{\overline{U}}|} = \frac{|A_U|}{2}.$$

Hence, by Dirac's Theorem, it follows that $\Gamma(U)$ is a Hamiltonian graph. $\square$

Proposition 3.39. *If U is a complete ultra non-commutative hypergroup, such that $U = \bigcup_{i \in G} X_i$ and $|X_i| = n$, for all $i \in G$, then U is Hamiltonian.*

Proof. Because for all $x \in U$, $|\beta^*(x)| = |X_i| = n$. Hence, by Theorem 3.38, it follows that $\Gamma(U)$ is a Hamiltonian graph. $\square$

Corollary 3.40. *Every non-abelian group is a Hamiltonian hypergroup.*

Definition 3.41. A *planar graph* is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.

Example 3.42. Let us consider the hypergroup represented by the Table 8: In this example the associated graph is at Figure 8:

$(S_3, \circ)$	e	x	y	z	r	t
e	e	x	y	z	r	t
x	x	e	t	r	z	y
y	y	r	e	t	x	z
z	z	t	r	e	y	x
r	r	y	z	x	t	e
t	t	z	x	y	e	r

Table 8: Caley Table of the hypergroup $(S_3, \circ)$

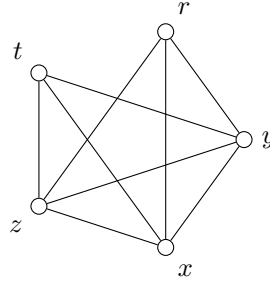


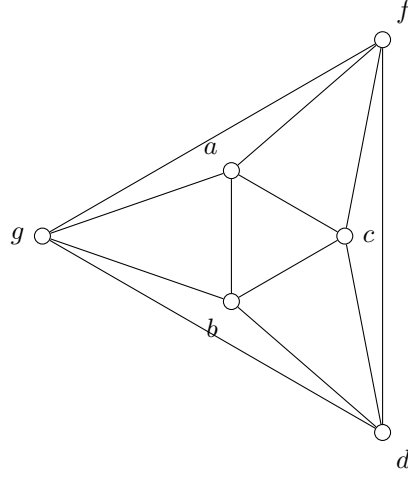
Figure 8: Ultra non-commuting graph of S_3 .

Proposition 3.43. [19] *A non-abelian group G has a planar Cayley graph $\Gamma(G)$ if and only if G is isomorphic to D_6 , D_8 , or Q_8 .*

Theorem 3.44. *Let $(U, \circ)$ be an ultra non-commutative hypergroup and $A_{\overline{U}} = \overline{A_U}$. If $\text{Card}(\beta^*(x) = \overline{x}) \geq 2$, for all $x \in U$, then U is non-planar.*

Example 3.45. The dihedral group D_8 and the quaternion group Q_8 , both of order 8, have isomorphic associated graphs as given on the Figure 9.

Proof. If $\Gamma(\overline{U})$ is not planar then $\Gamma(U)$ is not planar. Otherwise suppose $\Gamma(\overline{U})$ is planar. In this case if there exist $\overline{x}, \overline{y} \in A_{\overline{U}}$, such that $\text{Card}(\overline{x}) \geq 3$ and $\text{Card}(\overline{y}) \geq 3$, then we have $K_{3,3}$ is a subgraph of $\Gamma(A_U)$. If do not exists $\overline{x}, \overline{y} \in A_{\overline{U}}$, such that $\text{Card}(\overline{x}) \geq 3$ and $\text{Card}(\overline{y}) \geq 3$, then suppose that n_e be the number of edges and n_v be the number of vertices of A_U . As the graph $A_{\overline{U}}$, is planar, by Proposition 3.44, $\overline{U}$ is isomorphic to one of the groups D_6 ,

Figure 9: Ultra non-commuting graph of D_8 or Q_8 .

D_8 or Q_8 . Let $\bar{U} \cong D_6$, thus $\Gamma(\bar{U}) \cong \Gamma(S_3)$. By Example 3.42 we get the set of vertices $A_{\bar{U}} = \{x, y, z, t, r\}$.

Without loss of generality, suppose that:

Case (1) $Card(\bar{x}) = k \geq 2$ and $Card(\bar{y}) = Card(\bar{z}) = Card(\bar{t}) = Card(\bar{r}) = 2$, then the number of vertices of $\Gamma(A_U)$ is $n_v = 8 + k$ and the number of edges of $\Gamma(U)$ is $n_e = 20 + 8k$.

Case (2) If $Card(\bar{r}) = k \geq 2$ and $Card(\bar{y}) = Card(\bar{z}) = Card(\bar{t}) = Card(\bar{x}) = 2$, then the number of vertices of A_U is $n_v = 8 + k$ and the number of edges of $\Gamma(U)$ is $n_e = 24 + 6k$. With respect of cases (1) and (2) we have $3n_v - 6 < n_e$, and using Theorem 2.8, it follows that $\Gamma(U)$ is not planar.

Now let $\bar{U} \cong Q_8$. By Example 3.43 we get the set of vertices $A_{\bar{U}} = \{a, b, c, d, f, g\}$. Without loss of the generality, suppose that $Card(\bar{a}) = k \geq 2$ and $Card(\bar{b}) = Card(\bar{c}) = Card(\bar{d}) = Card(\bar{f}) = Card(\bar{g}) = 2$, then the number of vertices of $\Gamma(U)$ is $n_v = 10 + k$ and the number of edges of $\Gamma(U)$ is $n_e = 32 + 8k$. Because $3n_v - 6 < n_e$, it follows that $\Gamma(U)$ is not planar. $\square$

4 Conclusion

Paul Erdős defined a graph having the set of vertices $V = G - Z(G)$, where $Z(G)$ is the set of the elements of a non-abelian group $(G, \cdot)$ commuting with all elements in G , and joining two vertices x and y whenever $x \cdot y \neq y \cdot x$. This

construction has been extended in this note to the hypergroups framework, considering an ultra non-commutative hypergroup $(U, \circ)$, and considering the set of the vertices as $A_U = \{x \in U \mid \exists y \in U : x \circ y \cap y \circ x = \emptyset\}$ and join two vertices x and y whenever $x \circ y \cap y \circ x = \emptyset$. We have established the connectivity and necessary or sufficient conditions for the associated graph to be Hamiltonian or planar. In a future work other similar constructions will be investigated, in the sense that we will construct new graphs or hypergraphs associated with hypergroups, and vice-versa we will study the properties of the hyperstructures associated with some particular graphs. Moreover, we can explore how the graph theoretical properties of the associated graph can effect on the hypergroup theoretical properties of $(U, \circ)$.

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