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Almost and quasi-Leinster groups

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Abstract. In this paper, we extend the parallelism between perfect numbers and Leinster groups, by introducing the new concepts of almost and quasi-Leinster groups which parallel almost and quasi-perfect numbers. These are small deviations from perfect numbers; very few results and/or examples are known about them.

We investigate nilpotent almost/quasi-/Leinster groups and find some examples and conditions for the existence of such groups for classes of non-nilpotent groups: ZM (Zassenhaus metacyclic) groups, affine groups, dihedral groups and dicyclic groups.

1 Introduction

Throughout this paper, all the integers are positive and all the groups are finite. We denote the set of positive integers by $\mathbb{N}^*$. By *proper divisor* of an integer, we understand a divisor of the number that is different from the integer itself. Similarly, by a *proper subgroup* of a group, we understand a subgroup of the group except the group itself. For a group G, we denote by $L(G)$ the lattice of its subgroups and by $N(G)$ the sublattice of normal subgroups.

In 2001, Tom Leinster introduced in [9] a group theoretical analogue to perfect numbers: finite groups where the order of the group is equal to the sum of the orders of its proper normal subgroups. He called these groups perfect, but since this name was already used, these groups were later called Leinster groups [11].

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Finding examples of Leinster groups of odd order proved to be difficult, taking ten years. An example was produced by François Brunault in reply to Tom Leinsters post on MathOverflow [20]. The use of computational programs brought thousands of examples of Leinster groups, see [13]. Additionally, the existence of Leinster groups among different classes of groups (e.g. generalized quaternion groups, semisimple groups) was studied, see [1]. Also, Leinster groups of specific orders, products of primes, such as p^2q^2, $pqrs$, have been studied, see [1, 2].

Our paper aims to further study Leinster groups and introduce two new related concepts: almost and quasi-Leinster groups, group-theoretical analogues to almost and quasi-perfect numbers.

In the Preliminaries section, we introduce the necessary theoretical background and the new concepts. We also prove a result characterizing powers of even perfect numbers.

In the second section, we characterize almost/quasi-/Leinster nilpotent groups. In the subsequent sections, we find examples of quasi-Leinster groups among ZM-groups, and completely characterize almost/quasi-/Leinster affine, dihedral and dicyclic groups.

2 Preliminaries

We begin with the number theoretic concepts and then present their group-theoretic equivalents.

We start with two classical integer-valued functions ([11], Definition 2.1.):

$$D(n) = \sum_{d|n} d \tag{1}$$

and

$$\delta(n) = \frac{D(n)}{n}, \tag{2}$$

where $n \in \mathbb{N}^*$.

Now, we can classify integers based on the values of D and δ:

Definition 2.1. *([6], Pages 71, 74)*

(1) An integer n is called perfect if it is equal to the sum of its proper divisors, i.e.:

$$\delta(n) = 2 \tag{3}$$

(2) An integer is called abundant if it is smaller than the sum of its proper divisors, i.e.:

$$\delta(n) > 2. \tag{4}$$

(3) An integer is called deficient if it is greater than the sum of its proper divisors, i.e.:

$$\delta(n) < 2. \tag{5}$$

(4) An abundant integer is called quasi-perfect if the sum of its proper divisors is greater than itself by one, i.e.:

$$D(n) = 2n + 1. \tag{6}$$

(5) A deficient integer is called almost perfect if the sum of its proper divisors is smaller than itself by one, i.e.:

$$D(n) = 2n - 1. \tag{7}$$

Example 2.1 ([6], Pages 71, 74)**.** *(1)* $6 = 1+2+3$ *and* $28 = 1+2+4+7+14$ *are perfect numbers.*

(2) The only known almost perfect numbers are of type 2^s *with* $s > 0$*.*

(3) There are no known quasi-perfect numbers.

Proposition 2.1 ([6], Page 71)**.** *All even perfect numbers are of the form* $f(r) = 2^{r-1}(2^r - 1)$*, where* $r \geq 2$ *and* $2^r - 1$ *is prime.*

Example 2.2 ([19])**.** *There are 52 known perfect even numbers:*

$$P_i, \quad i \in \{1, \ldots, 52\}.$$

Using notation in Proposition 2.1, the first seven are: $P_1 = 6 = f(2)$*,* $P_2 = 28 = f(3)$*,* $P_3 = 496 = f(5)$*,* $P_4 = 8128 = f(6)$*,* $P_5 = 33550336 = f(13)$*,* $P_6 = 8589869056 = f(17)$ *and* $P_7 = 137438691328 = f(19)$*.*

Using Proposition 2.1, we prove a small corollary that will be useful in our study.

Corollary 2.2. *Let* p *be a prime,* P_i *an even perfect number and* k *a positive integer such that:*

$$P_i + 1 = p^k. \tag{8}$$

Then $k = 1$*.*

Proof. Since P_i is an even perfect number, using 2.1, Equation (8) becomes:

$$p^k - 1 = 2^{r-1}(2^r - 1), \tag{9}$$

where $2^r - 1$ is a Mersenne prime $r \geq 2$.
We assume, by contradiction, that k is even, i.e. $k = 2\ell, \ell \in \mathbb{N}^*$. Equation (9) becomes:

$$(p^\ell - 1)(p^\ell + 1) = 2^{r-1}(2^r - 1).$$

Numbers $p^\ell - 1$ and $p^\ell + 1$ are even consecutive numbers. It follows that $r - 1 \geq 2 \iff r \geq 3$.
Since $2^r - 1$ is prime, it divides only one of the factors in the left-hand side (either $p^\ell - 1$, or $p^\ell + 1$). Since they are both even, it follows that $2^{r+1} - 2$ is a divisor of one of the factors. Because $2^{r+1} - 2 > 2^{r-2}$, and $p^\ell + 1 > p^\ell - 1$, we get $p^\ell - 1 = 2^{r-2}$ and $p^\ell + 1 = 2^{r+1} - 2$. Subtracting these, it follows that $2^{r+1} - 4 = 2^{r-2}$, which yields a contradiction, hence k is odd.
Now, we prove that k can actually be just 1.
Assume by contradiction, that $k > 1$. In this case, Equation (9) becomes:

$$2^{r-1}(2^r - 1) = (p - 1)(p^{k-1} + \cdots + p + 1).$$

Since p^k is odd, it follows that p is odd. Together with the fact that k is odd, we get that $p^{k-1} + \cdots + p + 1$ is odd. Since $p - 1$ is odd, it follows that $2^{r-1} \mid (p - 1)$. Also, because $2^r - 1$ is prime, it divides just one of the factors in the right-hand side. Both the factors in the right hand side are greater than 1; it follows that $2^{r-1} = p - 1$ and $2^r - 1 = p^{k-1} + \cdots + p + 1$. We get that: $2p - 3 = p^{k-1} + \cdots + p + 1$, which is equivalent to $p^{k-1} + \cdots + p - 2p = -4$. Since p is odd, this has no solutions.
This concludes the proof. □

Example 2.3. *Using GAP [18], we found* 4 *solutions for* (8)*:*

$$P_i \in \{P_1, P_2, P_5, P_7\}.$$

They were obtained by checking the first 39 *even perfect numbers.*

While even perfect numbers are characterized, no odd perfect numbers are known:

Conjecture 1 ([6], Page 71)**.** *There are no odd perfect numbers.*

Now, we can present the analogues concepts for groups. We start with the definitions of the functions D and δ.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7] and [8].

Definition 2.2 ([11], Definition 2.1.)**.** *Given a group G, we define the following:*

$$D(G) = \sum_{N \lhd G} |N|, \tag{10}$$

and

$$\delta(G) = \frac{D(G)}{|G|}. \tag{11}$$

The easiest computations of the functions D and δ are those for cyclic groups.

Example 2.4 ([9], Example 2.1.)**.** *For a cyclic finite group C_n, the following hold:*

$$D(C_n) = D(n), \tag{12}$$

and

$$\delta(C_n) = \delta(n). \tag{13}$$

We recall some basic properties of the two functions given in Definition 2.2.

Remark 2.1 ([11], Observation 3.1.)**.** *The following hold:*

(1) $\delta(G) > 1$, for all $G \neq \{1\}$.

(2) $\delta(G/N) \leq \delta(G)$, for all $N \lhd G$.

(3) D, δ are multiplicative functions:

If G_1, G_2 groups such that $\gcd(|G_1|, |G_2|) = 1$, then

$$D(G_1 \times G_2) = D(G_1)D(G_2) \tag{14}$$

$$\delta(G_1 \times G_2) = \delta(G_1)\delta(G_2) \tag{15}$$

Now, we can state the definitions for Leinster, abundant and deficient groups. The first ones have been studied in [9, 11, 1, 2, 13].

Definition 2.3 ([11], Definition 2.1.)**.** *Let G be a group.*

(1) G is called Leinster if $\delta(G) = 2$.

(2) G is called abundant if $\delta(G) > 2$.

(3) G is called deficient if $\delta(G) < 2$.

We can extend the analogy with integers, by introducing quasi-Leinster and almost Leinster groups.

Definition 2.4. *Let G be a group.*

(1) G is called quasi-Leinster if

$$D(G) = 2|G| + 1. \tag{16}$$

(2) G is called almost Leinster if

$$D(G) = 2|G| - 1. \tag{17}$$

Example 2.5. *The non-abelian group $G_{p,q}$ of order pq, $p < q$ primes such that $p \mid (q-1)$, is not almost/quasi-Leinster. To see this, we compute the value of $D(G_{p,q})$. From Sylow's theorems, it follows that $G_{p,q}$ has as proper normal subgroups the trivial one and the Sylow one of order q. Therefore*

$$D(G_{p,q}) = 1 + q + pq.$$

Since $|G_{p,q}| = pq$, if we assume that $G_{p,q}$ is almost/quasi-Leinster, it follows that: $2pq \pm 1 = 1 + q + pq$ which is equivalent to

$$pq \pm 1 = 1 + q.$$

Since $p \geq 2$ and $q \geq 3$, this equation has no solution.

Now, we catalog the almost/quasi-/Leinster groups in different classes of groups.

3 Nilpotent groups

The first result characterizes nilpotent groups woth $\delta(G) \leq 2$, in particular nilpotent almost Leinster groups.

Proposition 3.1 ([11])**.** *Let G be a nilpotent group. The following are equivalent:*

(1) $\delta(G) \leq 2$

(2) The group G is cyclic and $|G|$ is perfect or deficient.

Proof. The implication (1)⇒(2) is Proposition 3.2 from [11]. The converse follows from (12) and (13). □

We can restate Proposition 3.1, using the new terminology.

Corollary 3.2. *Let G be a nilpotent group. The following are equivalent:*

(1) The group G is almost/Leinster.

(2) The group G is cyclic and $|G|$ is almost/perfect.

The following proposition characterizes quasi-Leinster nilpotent groups.

Proposition 3.3. *Let G be a nilpotent group. The following are equivalent:*

(1) The group G is quasi-Leinster.

(2) The group G is cyclic and $|G|$ is quasi-perfect.

Proof. The converse statement follows from (12). We prove the direct.

Since G is nilpotent, it can be written as the direct product of its Sylow p-subgroups:

$$G = G_1 \times G_2 \times \cdots \times G_k,$$

where $|G_i| = p_i^{n_i}$, for all $i \in \{1, 2, \ldots, k\}$. Since the function D is multiplicative, it follows that:

$$D(G) = \prod_{i=1}^{k} D(G_i).$$

Thus, we get

$$\prod_{i=1}^{k} D(G_i) = 2 \prod_{i=1}^{k} |G_i| + 1.$$

We assume, by contradiction, that G_1 is not cyclic. It follows that $n_1 \geq 2$. In addition, the final theorem from [12] states that the number of subgroups of index p_1 is congruent to $1 + p_1 \pmod{p_1^2}$; in particular, it is greater or equal to $1 + p_1$. These subgroups are normal, so $D(G) \geq 1 + p_1^{\alpha_1} + (p_1 + 1)p_1^{\alpha_1 - 1}$. It follows that:

$$\delta(G_1) \geq \frac{1 + p_1^{n_1} + (p_1 + 1)p_1^{n_1 - 1}}{p_1^{n_1}} = 2 + \frac{1}{p_1} + \frac{1}{p_1^{n_1}}\,.$$

Since G is quasi-Leinster and δ is multiplicative, it follows that

$$\begin{aligned} 2 + \frac{1}{p_1^{n_1} \ldots p_k^{n_k}} = \delta(G) &= \delta(G_1) \ldots \delta(G_k) \\ &\geq \left(2 + \frac{1}{p_1} + \frac{1}{p_1^{n_1}}\right)\left(1 + \frac{1}{p_2^{n_2}}\right) \ldots \left(1 + \frac{1}{p_k^{n_k}}\right) \\ &> 2 + \frac{1}{p_1^{n_1} \ldots p_k^{n_k}}\,, \end{aligned}$$

a contradiction. Thus G_1 is cyclic. Analogously, all G_i, $i = 2, \ldots, k$, are cyclic and therefore G is cyclic. □

Now, we study different classes of finite non-nilpotent groups.

4 ZM-groups

Definition 4.1 ([17], Pages 144, 145)**.** *A ZM-group (Zassenhaus metacyclic) is a finite group with all Sylow subgroups cyclic.*

We recall some properties from literature for these groups.

Proposition 4.1 ([7])**.** *The presentation of ZM-groups is given by:*

$$ZM(m,n,r) = \langle a, b : a^m = b^n = 1, b^{-1}ab = a^r \rangle,$$

where the triplet $(m,n,r) \in (\mathbb{N}^*)^3$ *satisfies the following conditions:*

$$\begin{cases} \gcd(m,n) = \gcd(m,r-1) = 1, \\ r^n \equiv 1 \pmod{m}. \end{cases} \tag{18}$$

Lemma 4.2 ([16])**.** *The order of the group* $ZM(m,n,r)$ *is* mn*.*

Proposition 4.3. *There exist quasi-Leinster ZM-groups.*

Proof. According to [11, Theorem 5.7], the value of $D(ZM(m,n,r))$ is given by:

$$D(ZM(m,n,r)) = mn \sum_{n_1 | n} \frac{\delta(\gcd(m, r^{n_1} - 1))}{n_1}. \tag{19}$$

Assume that m is prime, and let d be the order of r modulo m. According to [11, Corollary 5.9.], Equation (19) becomes:

$$D(ZM(m,n,r)) = mD(n) + D\left(\frac{n}{d}\right). \tag{20}$$

Suppose further that $n = d$. It follows that:

$$D(ZM(m,n,r)) = mD(n) + 1. \tag{21}$$

If we assume that $ZM(m,n,r)$ is quasi-Leinster, it follows from Lemma 4.2 that:

$$D(ZM(m,n,r)) = 2mn + 1,$$

which, using (21), implies $D(n) = 2n$, i.e. n is perfect. On the other hand, it follows from (18), that

$$r \not\equiv 1 \pmod{m} \text{ and } r \not\equiv 0 \pmod{m}. \tag{22}$$

Since m is prime, it follows that the order of r modulo m is $m-1$. Since we have assumed that the order of r is n, it follows that: $m - 1 = n$, or, equivalently $m = n + 1$, which has solutions, according to Example 2.3. Any $r \in \{2, \ldots, m-1\}$ satisfies the conditions from (22). □

Example 4.1. *Under the assumptions from the proof of Proposition 4.3, from 2.3, we get* 4 *values for* n*, and the corresponding values for* m*. For each of these, any* $r \in \{2, \dots, m-1\}$ *give an example of quasi-Leinster ZM group. For each* m*, we write just one, for* $r = 3$*:*

- $(m, n, r) = (7, 6, 3)$,
- $(m, n, r) = (29, 28, 3)$,
- $(m, n, r) = (33550337, 33550336, 3)$,
- $(m, n, r) = (137438691329, 137438691328, 3)$.

5 Affine groups

Definition 5.1 ([10], Chapter 2)**.** *Given* $q = p^k$ *with* p *prime and* $k \in \mathbb{N}^*$*, we consider the affine group*

$$\mathrm{Aff}(\mathbb{F}_q) = \mathbb{F}_q \rtimes \mathbb{F}_q^* = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\},$$

with matrix multiplication as the group operation.

The normal subgroups of $\mathrm{Aff}(\mathbb{F}_q)$ are described in the following proposition.

Proposition 5.1 ([4])**.** *There is a bijection between the subgroups of* $\mathbb{F}_q^*$ *and the non-trivial normal subgroups of* $\mathrm{Aff}(\mathbb{F}_q)$*, namely*

$$H \leq \mathbb{F}_q^* \mapsto N_H = \begin{pmatrix} H & \mathbb{F}_q \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in H, b \in \mathbb{F}_q \right\} \lhd \mathrm{Aff}(\mathbb{F}_q).$$

Proposition 5.2. *(1) There are no Leinster affine groups.*

(2) The cyclic group of order 2 *is the only almost Leinster affine group.*

(3) An affine group $\mathrm{Aff}(F_q)$ *is quasi-Leinster if and only if* q *is prime and* $q - 1$ *is perfect.*

Proof. Since $\mathbb{F}_q^*$ is cyclic, it follows from Proposition 5.1 that:

$$N(\mathrm{Aff}(\mathbb{F}_q)) = \{1\} \cup \{N_H : H \leq \mathbb{F}_q^*\}.$$

Then

$$D(\mathrm{Aff}(\mathbb{F}_q)) = 1 + \sum_{H \leq \mathbb{F}_q^*} |N_H| = 1 + q \sum_{H \leq \mathbb{F}_q^*} |H| = 1 + qD(q-1).$$

We obtain that:

(1) $\mathrm{Aff}(\mathbb{F}_q)$ is Leinster if and only if $1 + qD(q-1) = 2q(q-1)$. This yields $q \mid 1$, which is absurd.

(2) $\mathrm{Aff}(\mathbb{F}_q)$ is almost Leinster if and only if $1 + qD(q-1) = 2q(q-1) - 1$. It follows that $q \mid 2$, thus $q = 2$, i.e. $\mathrm{Aff}(\mathbb{F}_q) \cong C_2$.

(3) $\mathrm{Aff}(\mathbb{F}_q)$ is quasi-Leinster if and only if $1 + qD(q-1) = 2q(q-1) + 1$. This implies $D(q-1) = 2(q-1)$, i.e. $q-1$ is perfect. According to Corollary 2.2, $q = p$.

□

Example 5.1. *Using Example 2.3, we get* 4 *examples of quasi-Leinster affine groups:* $\mathrm{Aff}(\mathbb{F}_7)$, $\mathrm{Aff}(\mathbb{F}_{29})$, $\mathrm{Aff}(\mathbb{F}_{33550337})$, *and* $\mathrm{Aff}(\mathbb{F}_{137438691329})$.

6 Dihedral groups

Definition 6.1 ([15], Page 188)**.** *Given $n \in \mathbb{N}^* \setminus \{1\}$, the corresponding dihedral group is given by*

$$D_{2n} = \langle r, s : r^n = s^2 = e, srs^{-1} = r^{-1} \rangle.$$

The normal subgroups of D_{2n} are described in the following result.

Lemma 6.1 ([5])**.** *The normal subgroups of D_{2n} are given by:*

$$N(D_{2n}) = \begin{cases} \{D_{2n}\} \cup \{\langle r^d \rangle : d \mid n\}, & n \text{ odd} \quad (23) \\ \{D_{2n}\} \cup \{\langle r^d \rangle : d \mid n\} \cup \{\langle r^2, s \rangle, \langle r^2, rs \rangle\}, & n \text{ even}. \quad (24) \end{cases}$$

In particular, for any $t \mid (2n)$, $t \neq 2$, there exists at most one normal subgroup of index t of D_{2n}. For $t = 2$, $\langle r \rangle \lhd D_{2n}$ is a normal subgroup of index 2. If n is odd, this is the only one. If n is even, D_{2n} has three normal subgroups of index 2: $\langle r \rangle$, $\langle r^2, s \rangle$, $\langle r^2, rs \rangle$ of index 2.

We are now able to study when D_{2n} is a quasi-/almost/Leinster group.

Proposition 6.2. *The dihedral group D_{2n} is quasi-/almost/Leinster if and only if n is odd and quasi-/almost/Leinster.*

Proof. If we assume that D_{2n} is almost/quasi-/Leinster, we get

$$D(D_{2n}) = 4n + a, a \in \{-1, 0, 1\}. \quad (25)$$

We compute the value of $D(D_{2n})$, distinguishing two cases based on the parity of n:

n **odd.** Using (23) from Lemma 6.1, together with (25), we get

$$4n + a = 2n + D(n), a \in \{-1, 0, 1\},$$

which is equivalent to $D(n) = 2n \pm a$, i.e. n is almost/quasi-/perfect.

n **even.** Again, using (25) and (24) from Lemma 6.1, we get:

$$4n + D(n) = 4n + a, a \in \{-1, 0, 1\},$$

which has no solution.

□

Remark 6.1. *Proposition 6.2 recovers the result for Leinster dihedral groups from [9].*

7 Dicyclic groups

Definition 7.1 ([14], Page 128)**.** *Given $n \in \mathbb{N}^* \backslash \{1\}$, the corresponding dicyclic group is given by:*

$$\mathrm{Dic}_{4n} = \langle a, x : a^{2n} = 1, x^2 = a^n, ax = xa^{-1} \rangle.$$

The normal subgroups of the dicyclic groups can be described in a similar way to those of the dihedral groups.

Lemma 7.1 ([14], Lemma 3.3)**.** *The normal subgroups of* Dic_{4n} *are given by:*

$$N(\mathrm{Dic}_{4n}) = \begin{cases} \mathrm{Dic}_{4n} \cup \{\langle a^d \rangle : d \mid 2n\}, & n \text{ odd} \quad (26) \\ \mathrm{Dic}_{4n} \cup \{\langle a^d \rangle : d \mid 2n\} \cup \{\langle a^2, ax \rangle, \langle a^2, a^2 x \rangle, & n \text{ even }. \quad (27) \end{cases}$$

This allows us to prove the following proposition.

Proposition 7.2. *The dicyclic group* Dic_{4n} *is quasi-/almost/Leinster if and only if n is odd, and $2n$ quasi-/almost/Leinster.*

Proof. If Dic_{4n} is almost/quasi-/Leinster, we get

$$D(\mathrm{Dic}_{4n}) = 8n + a, a \in \{-1, 0, 1\}. \quad (28)$$

Using (26) from Lemma 7.1, we compute $D(\mathrm{Dic}_{4n})$, taking into account the parity of n.

n **even.** In this case, it follows from (27), that $D(\mathrm{Dic}_{4n}) = D(2n) + 8n$. Together with (28), we get $D(2n) + 8n = 8n + a, a \in \{-1, 0, 1\}$, which has no solution.

n **odd.** In this case, $D(\mathrm{Dic}_{4n}) = D(2n) + 4n$. It follows that $D(2n) + 4n = 8n + a, a \in \{-1, 0, 1\}$, i.e.:

$$D(2n) = 4n + a, a \in \{-1, 0, 1\}. \tag{29}$$

(1) If $D(2n) = 4n - 1$, then $2n$ is almost perfect.

(2) If $D(2n) = 4n$, then $2n$ is perfect.

(3) If $D(2n) = 4n + 1$, then $2n$ is quasi-perfect.

□

Remark 7.1. *In the context of the previous proof, when n is odd, Equation (29) has no known solutions for $a \in \{-1, 1\}$.*
Indeed, there are no known almost perfect numbers of the form $4k+2$, $k \in \mathbb{N}^*$, so there are no known solutions for $a = -1$.
Similarly, there are no known solution for $a = 1$, since there are no known quasi-perfect numbers.
For $a = 0$, or equivalently, $2n$ perfect, there is just one solution, $n = 3$, which yields the unique Leinster group Dic_{12}.

Remark 7.2. *Sometimes, the dicyclic groups are called generalized quaternion groups, see [15, Page 252].*
Our results above recover the result for Leinster generalized quaternion groups from [1, Proposition 2.1.].

8 Conclusions

We introduced quasi-/almost Leinster groups, which are analogues to quasi-/almost perfect numbers.
We proved that the only nilpotent almost and quasi-Leinster groups are the cyclic groups of almost and quasi-perfect order.
In addition, we studied a few well-known classes of non-nilpotent groups (ZM-groups, affine groups, dihedral groups and dicyclic groups).
We found examples of almost/quasi-Leinster groups for the first two. We proved that the existence of almost/quasi-/Leinster dihedral groups D_{2n} groups is equivalent to n being odd, and $2n$ being almost/quasi-/perfect.
Similarly, for dicyclic groups, we proved that the existence of an almost/quasi-/Leinster group Dic_{4n} is equivalent to n being odd and $2n$ being almost/quasi-/perfect.
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