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The characterization of Nelson algebras by Sheffer stroke

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Abstract

In this study, Sheffer stroke Nelson algebras (briefly, s-Nelson algebras), (ultra) ideals, quasi-subalgebras, quotient sets, and fuzzy structures on these algebraic structures are introduced. The relationships between s-Nelson and Nelson algebras are analyzed. It is also shown that an s-Nelson algebra is a bounded distributive modular lattice, and the family of all ideals forms a complete distributive modular lattice. A congruence relation on an s-Nelson algebra is determined by an ideal and quotient s-Nelson algebras are constructed by this congruence relation. Finally, it is indicated that a quotient s-Nelson algebra constructed by the ultra ideal is totally ordered and that the cardinality of the quotient is less than or equal to 2.

1 Introduction

H. M. Sheffer introduced Sheffer stroke (or Sheffer operation) [24] which can states all mathematically definable connectives in Boolean logic [14]. In other words, this operation can independently construct a logical formal system without requiring any other logical operations, a property that other unary or binary connectives do not possess. Therefore, there exist its some applications in computer science, technology and industry. For instance, The Sheffer stroke operates all diodes on the chip forming the processor in a computer, making it

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easier and cheaper than producing different diodes for other operation. Since this operator is a commutative, applying to logical algebras leads to many useful results, and it reducts axiom systems of many algebraic structures. Thus, the Sheffer stroke replaces unary and binary operations in many algebraic structures such as orthoimplication algebras [1], ortholattices [3], strong Sheffer stroke non-associative MV-algebras [4] and filters [18], fuzzy filters of Sheffer stroke Hilbert algebras [19], neutrosophic $\mathcal{N}$-structures of Sheffer stroke BL-algebras [12], and Sheffer stroke Hilbert algebras [20]. Also, Molkhasi et al studied representations of strongly algebraically closed algebras, Sheffer stroke algebras and Visser algebras ([15], [16]), and Veroff present a shortest 2-basis for Boolean algebra in terms of the Sheffer stroke [25]. Currently, Chajda and Länger extend the notion of Sheffer operation and orthomodular lattices to arbitrary directed in relational systems with involution [5].

Ewald introduced intuitionistic tense logic IKt as a generalization of intuitionistic propositional logic with the unary operators P, F, H, and G [6]. Additionally, Figallo and Pelaitay introduced the variety IKt of IKt-algebras and demonstrated that the IKt system has IKt-algebras as its algebraic counterpart [6]. Later, H. Rasiowa introduced the concept of Nelson algebras, or N-lattices [21], and referred to them as quasi-pseudo Boolean algebras [22], which are the algebraic counterparts of the constructive logic with strong negation introduced by Nelson [17] and Markov [13]. Figallo et al. introduced and investigated the variety of tense Nelson algebras, which are Nelson algebras with two tense operators, and analyzed the relationships between (centered tense) Nelson algebras and IKt-algebras [8]. Rivieccio et al. generalized Nelson algebras by considering a negation that is not necessarily involutive [9], introducing the variety of quasi-Nelson algebras and fragments [10]. Recently, Gomez et al. studied the relationship between strong and weak (or intuitionistic) negation in Nelson algebras [11].

We present basic definitions of Sheffer stroke and Nelson algebras. Then Sheffer stroke Nelson algebras (in short, s-Nelson algebras) and a partial order on this algebra are defined, and some of the properties are investigated. It is proved that an s-Nelson algebra is a bounded distributive modular lattice. In addition, we indicate that every s-Nelson algebra is a Nelson algebra but the special conditions are necessary to prove the inverse. Also, an (ultra) ideal of an s-Nelson algebra is introduced and it is presented the statements which are equivalent to the definition of an (ultra) ideal of this algebraic structure. Indeed, it is demonstrated that every ultra ideal of an s-Nelson algebra is the ideal but the inverse is not true in general. Besides, a congruence relation on an s-Nelson algebra is defined by its ideal, and so, it is illustrated that a quotient set of an s-Nelson algebra defined by this congruence relation is an s-Nelson algebra. Consequently, it is showed that a partial order on the quotient s-Nelson algebra defined by an ultra ideal is a totally order on this algebraic structure.

2 Preliminaries

In this section, we present basic definitions and notions about Sheffer operation and Nelson algebras.

Definition 2.1. *[3] Let* $\mathcal{N} = \langle N, | \rangle$ *be a groupoid. The operation* $|$ *is said to be a Sheffer operation (or Sheffer stroke) if it satisfies the following conditions:*
(S1) $n_1|n_2 = n_2|n_1$,
(S2) $(n_1|n_1)|(n_1|n_2) = n_1$,
(S3) $n_1|((n_2|n_3)|(n_2|n_3)) = ((n_1|n_2)|(n_1|n_2))|n_3$,
(S4) $(n_1|((n_1|n_1)|(n_2|n_2)))|(n_1|((n_1|n_1)|(n_2|n_2))) = n_1$,
for all $n_1, n_2, n_3 \in N$.

Definition 2.2. *[8] A Nelson algebra is an algebra* $\mathcal{N} = \langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ *of type* $(2, 2, 2, 1, 0)$ *that satisfy the conditions:*

(N1) $n \vee 1 = 1$,
(N2) $n_1 \wedge (n_1 \vee n_2) = n_1$,
(N3) $n_1 \wedge (n_2 \vee n_3) = (n_3 \wedge n_1) \vee (n_2 \wedge n_1)$,
(N4) $\sim\sim n = n$,
(N5) $\sim (n_1 \wedge n_2) =\sim n_1 \vee \sim n_2$,
(N6) $n_1 \wedge \sim n_1 = (n_1 \wedge \sim n_1) \wedge (n_2 \vee \sim n_2)$, *(N7)* $n \rightarrow n = 1$,
(N8) $n_1 \wedge (n_1 \rightarrow n_2) = n_1 \wedge (\sim n_1 \vee n_2)$,
(N9) $(n_1 \wedge n_2) \rightarrow n_3 = n_1 \rightarrow (n_2 \rightarrow n_3)$,
(N10) $(n_1 \rightarrow n_2) \wedge (\sim n_1 \vee n_2) =\sim n_1 \vee n_2$,
(N11) $n_1 \rightarrow (n_2 \wedge n_3) = (n_1 \rightarrow n_2) \wedge (n_1 \rightarrow n_3)$,
for all $n, n_1, n_2, n_3 \in N$.

Moreover, it follows from axioms (N2) and (N3) that $\langle N, \wedge, \vee, \sim, 1 \rangle$ *is a Kleene algebra.*

Definition 2.3. *[23] A modular lattice is any lattice which satisfies* $x \leq y \longrightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$.

Theorem 2.1. *[23] Every distributive lattice is a modular lattice.*

3 Sheffer stroke Nelson algebras

In this section, we introduce Sheffer stroke Nelson algebras and give some properties.

Definition 3.1. *A Sheffer stroke Nelson algebra (shortly, s-Nelson algebra) is an algebra* $\mathcal{N} = (N, |, 1)$ *of type* $(2, 0)$ *satisfying the following conditions:*

(sn1) $n|(n|n) = 1$,
(sn2) $n_1|(n_2|n_3) = ((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))$,
for all $n, n_1, n_2, n_3 \in N$.

Notation: $0 = 1|1$.

Remark 3.1. *The axioms (sn1) and (sn2) are independent.*

1. Independence of (sn1): Given a set $N = \{0, \alpha, 1\}$ with the following Cayley table:

Table 1:

$\vert$	0	α	1
0	1	1	1
α	1	α	α
1	1	α	0

Then (sn2) holds while (sn1) does not since $1 \neq \alpha = \alpha|(\alpha|\alpha)$.

2. Independence of (sn2): Consider a set $N = \{0, \alpha, 1\}$ with Cayley table as follows:

Table 2:

$\vert$	0	α	1
0	1	1	1
α	1	0	α
1	1	α	0

Then (sn1) is satisfied (sn2) is not since $\alpha = \alpha|(0|1) \neq 0 = \alpha|\alpha = (\alpha|1)|(\alpha|1) = ((\alpha|(0|0))|(\alpha|(1|1)))|((\alpha|(0|0))|(\alpha|(1|1)))$.

Example 3.1. *Consider a structure* $(N, |, 1)$ *where* $N = \{0, w_1, w_2, 1\}$ *and a binary operation* $|$ *on* N *with Cayley table (Table* 3*). Then it is an s-Nelson algebra.*

Proposition 3.1. *Let* $(N, |, 1)$ *be an s-Nelson algebra. Then*

Table 3: Cayley table of the Sheffer operation on N in Example 3.1

$\mid$	0	w_1	w_2	1
0	1	1	1	1
w_1	1	w_2	1	w_2
w_2	1	1	w_1	w_1
1	1	w_2	w_1	0

1. $n|(1|1) = 1$,
2. $1|(n|n) = n$,
3. $(n|1)|(n|1) = n$,
4. $n|n = n|1$,
5. $(n|1)|1 = n$,
6. $n|(n|1) = 1$,
7. $((n_1|n_2)|(n_1|n_3))|((n_1|n_2)|(n_1|n_3)) = n_1|((n_2|n_2)|(n_3|n_3))$,
8. $((n_1|n_2)|n_2)|n_2 = n_1|n_2$,
9. $n_1|n_1 = n_1|n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$,
10. $n_1|(n_1|n_2) = n_1|(n_2|n_2)$,
11. $n|0 = 1$,
12. $0|0 = 1$,
13. $0|n = 1$,

for all $n, n_1, n_2, n_3 \in N$.

Proof. Let $(N, |, 0, 1)$ be an s-Nelson algebra.

1. $n|(1|1) = n|((n|(n|n))|(n|(n|n))) = ((n|n)|(n|n))|(n|n) = n|(n|n) = 1$ from (sn1), (S2) and (S3).
2. $1|(n|n) = (n|(n|n))|(n|n) = n$ from (sn1), (S1) and (S2), respectively.
3. $(n|1)|(n|1) = (n|(n|(n|n)))|(n|(n|(n|n))) = (((n|n)|(n|n))|((n|n)|n))|(((n|n)|(n|n))|((n|n)|n)) = n$ from (sn1), (S1) and (S2).

4. $n|n = ((n|1)|(n|1))|((n|1)|(n|1)) = n|1$ from (3) and (S2), respectively.

5. $(n|1)|1 = 1|(((n|1)|(n|1))|((n|1)|(n|1))) = 1|(n|n) = n$ from (S1), (S2), (3) and (2), respectively.

6. $n|(n|1) = (n|1)|((n|1)|(n|1)) = 1$ from (S1), (3) and (sn1), respectively.

7.
$$\begin{aligned}
&((n_1|n_2)|(n_1|n_3))|((n_1|n_2)|(n_1|n_3)) \\
&= ((n_1|((n_2|n_2)|(n_2|n_2)))|(n_1|((n_3|n_3)|(n_3|n_3)))) \\
&\quad |((n_1|((n_2|n_2)|(n_2|n_2)))|(n_1|((n_3|n_3)|(n_3|n_3)))) \\
&= n_1|((n_2|n_2)|(n_3|n_3))
\end{aligned}$$

   from (S2) and (sn2).

8.
$$\begin{aligned}
((n_1|n_2)|n_2)|n_2 &= n_2|(n_2|(n_2|n_1)) \\
&= ((n_2|(n_2|n_2))|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
&\quad |((n_2|(n_2|n_2))|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
&= (1|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
&\quad |(1|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
&= n_2|((n_2|n_1)|(n_2|n_1)) \\
&= ((n_2|n_2)|(n_2|n_2))|n_1 \\
&= n_1|n_2
\end{aligned}$$

   from (S1)-(S3) and (sn1)-(sn2).

9. Let $n_1|n_1 = n_1|n_2$. Then it follows from (S2), (S3), (sn1) and (1) that

$$\begin{aligned}
n_1|(n_2|n_2) &= ((n_1|n_1)|(n_1|n_1))|(n_2|n_2) \\
&= ((n_1|n_2)|(n_1|n_2))|(n_2|n_2) \\
&= n_1|((n_2|(n_2|n_2))|(n_2|(n_2|n_2))) \\
&= n_1|(1|1) \\
&= 1,
\end{aligned}$$

   for all $n_1, n_2 \in N$.

   Conversely, let $n_1|(n_2|n_2) = 1$. Thus, it is obtained from (2)-(3), (sn1)-(sn2) and (S1)-(S2) that

$$\begin{aligned}
n_1|n_1 &= 1|((n_1|n_1)|(n_1|n_1)) \\
&= 1|n_1 \\
&= n_1|(n_1|(n_2|n_2)) \\
&= ((n_1|(n_1|n_1))|(n_1|((n_2|n_2)|(n_2|n_2)))) \\
&\quad |((n_1|(n_1|n_1))|(n_1|((n_2|n_2)|(n_2|n_2)))) \\
&= (1|(n_1|n_2))|(1|(n_1|n_2)) \\
&= n_1|n_2,
\end{aligned}$$

for all $n_1, n_2 \in N$.

10. We have from (3), (sn1)-(sn2) and (S1) that

$$\begin{aligned} n_1|(n_1|n_2) &= ((n_1|(n_1|n_1))|(n_1|(n_2|n_2))) \\ &\quad |((n_1|(n_1|n_1))|(n_1|(n_2|n_2))) \\ &= (1|(n_1|(n_2|n_2)))|(1|(n_1|(n_2|n_2))) \\ &= n_1|(n_2|n_2) \end{aligned}$$

for all $n_1, n_2 \in N$.

11. It follows from (1) that $n|0 = n|(1|1) = 1$, for all $n \in N$.
12. It is clear by (12).
13. We have $0|n = (1|1)|n = 1|(n|n) = n$ by (10) and (2), respectively.

□

Lemma 3.1. *Let* $(N, |1)$ *be an s-Nelson algebra. Then a binary operation* $\leq$ *on* N *defined by*

$$n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$$

is a partial order on N.

Proof. Let $(N, |, 0, 1)$ be an s-Nelson algebra and a binary operation $\leq$ on N be defined by $n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$, for any $n_1, n_2 \in N$.

- Reflexive: since $n|(n|n) = 1$ from (sn1), we have $n \leq n$, for all $n \in N$.
- Antisymmetric: let $n_1 \leq n_2$ and $n_2 \leq n_1$. Then $n_1|(n_2|n_2) = n_2|(n_1|n_1) = 1$. Since

$$\begin{aligned} n_1 &= 1|(n_1|n_1) \\ &= (n_2|(n_2|n_2))|(n_1|n_1) \\ &= (n_1|n_1)|(n_2|(n_2|n_2)) \\ &= (((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|((n_2|n_2)|(n_2|n_2))))| \\ &\quad (((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|((n_2|n_2)|(n_2|n_2)))) \\ &= (((n_1|n_1)|(n_2|n_2))|(n_2|(n_1|n_1)))| \\ &\quad (((n_1|n_1)|(n_2|n_2))|(n_2|(n_1|n_1))) \\ &= (n_1|n_1)|(n_2|n_2) \end{aligned}$$

  and similarly, $n_2 = (n_1|n_1)|(n_2|n_2)$ from (S1)-(S2), (sn1)-(sn2) and (2)-(3), it follows that $n_1 = n_2$.

- Transitive: let $n_1 \leq n_2$ and $n_2 \leq n_3$. Then $n_1|(n_2|n_2) = n_2|(n_3|n_3) = 1$. Since

$$\begin{aligned} 1 &= n_1|(n_2|n_2) \\ &= n_1|(n_2|1) \\ &= n_1|(n_2|(n_2|(n_3|n_3))) \\ &= n_1|(((n_2|(n_2|n_2))|(n_2|((n_3|n_3)|(n_3|n_3)))) \\ &\quad |((n_2|(n_2|n_2))|(n_2|((n_3|n_3)|(n_3|n_3))))) \\ &= n_1|((1|(n_2|n_3))|(1|(n_2|n_3))) \\ &= n_1|(n_2|n_3) \\ &= ((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3))) \\ &= (1|(n_1|(n_3|n_3)))|(1|(n_1|(n_3|n_3))) \\ &= n_1|(n_3|n_3) \end{aligned}$$

from (S1)-(S2), (sn1)-(sn2) and (3), it is obtained that $n_1 \leq n_3$.

It is clear that 0 is the least element and 1 is the greatest element of N. □

Lemma 3.2. *Let $(N, |, 1)$ be an s-Nelson algebra and $\leq$ be a partial order on N. Then $(N, \leq)$ (or $(N, \vee, \wedge, 0, 1)$) is a bounded distributive modular lattice where $n_1 \vee n_2 = (n_1|n_1)|(n_2|n_2)$ and $n_1 \wedge n_2 = (n_1|n_2)|(n_1|n_2)$, for all $n_1, n_2 \in N$.*

Proof. Let $(N, |, 0, 1)$ be an s-Nelson algebra and $\leq$ be a partial order on N. Since $((n_1|n_2)|(n_1|n_2))|(n_2|n_2) = n_1|((n_2|(n_2|n_2))|(n_2|(n_2|n_2))) = n_1|(1|1) = 1$ and $((n_1|n_2)|(n_1|n_2))|(n_1|n_1) = n_2|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))) = n_2|(1|1) = 1$ from (S1), (S3) and Proposition 3.1 (1), we get $(n_1|n_2)|(n_1|n_2) \leq n_1$ and $(n_1|n_2)|(n_1|n_2) \leq n_2$. So, $(n_1|n_2)|(n_1|n_2)$ is a lower bound of n_1 and n_2. Let $n_3 \leq n_1, n_2$. Then $n_3|(n_1|n_1) = n_3|(n_2|n_2) = 1$. Since

$$\begin{aligned} n_3|(((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))) &= n_3|(n_1|n_2) \\ &= ((n_3|(n_1|n_1))|(n_3|(n_2|n_2)))| \\ &\quad ((n_3|(n_1|n_1))|(n_3|(n_2|n_2))) \\ &= (1|1)|(1|1) \\ &= 1 \end{aligned}$$

from (S2) and (sn2), it follows that $n_3 \leq (n_1|n_2)|(n_1|n_2) \leq n_1, n_2$. Thus, $n_1 \wedge n_2 = (n_1|n_2)|(n_1|n_2)$ is the greatest lower bound (infimum) of n_1 and n_2. Similarly, $n_1 \vee n_2 = (n_1|n_1)|(n_2|n_2)$ is the least upper bound (supremum) of n_1 and n_2. Hence, $(N, \leq)$ is a lattice.

Since

$$\begin{aligned} n_1 \vee (n_2 \wedge n_3) &= (n_1|n_1)|(n_2|n_3) \\ &= (((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_3|n_3))) \\ &\quad |(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_3|n_3))) \\ &= (n_1 \vee n_2) \wedge (n_1 \vee n_3) \end{aligned}$$

and

$$\begin{aligned} n_1 \wedge (n_2 \vee n_3) &= (n_1|((n_2|n_2)|(n_3|n_3)))|(n_1|((n_2|n_2)|(n_3|n_3))) \\ &= (n_1|n_2)|(n_1|n_3) \\ &= (((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))) \\ &\quad |(((n_1|n_3)|(n_1|n_3))|((n_1|n_3)|(n_1|n_3))) \\ &= (n_1 \wedge n_2) \vee (n_1 \wedge n_3) \end{aligned}$$

from (sn2), Proposition 3.1 (6) and (S2), $(N, \leq)$ is distributive. By Theorem 2.1, it is modular lattice.

Moreover, since $n \vee 1 = (n|n)|(1|1) = 1$, $n \vee 0 = (n|n)|(0|0) = 1|(n|n) = n$ and $n \wedge 0 = (n|0)|(n|0) = (n|(1|1))|(n|(1|1)) = 1|1 = 0$, $n \wedge 1 = (n|1)|(n|1) = n$ from Proposition 3.1 (1)-(3) and (S1), $(N, \leq)$ is bounded.

Therefore, $(N, \leq)$ is a bounded distributive modular lattice. $\square$

Lemma 3.3. *Let $(N, |, 1)$ be an s-Nelson algebra. Then*

1. $n_1 \leq n_2 \Leftrightarrow n_2|n_2 \leq n_1|n_1$,
2. $n_1 \leq n_2$ *implies* $n_2|n_3 \leq n_1|n_3$,
3. $n_1 \leq n_2|(n_1|n_1)$,
4. $n_2|n_2 \leq n_2|(n_1|n_1)$,
5. $n_1 \leq (n_1|n_2)|n_2$,
6. $n_1 \leq n_2|(n_3|n_3) \Leftrightarrow (n_1|n_2)|(n_1|n_2) \leq n_3$,
7. $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$,
8. $n_1|(n_2|n_2) \leq (n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))$,
9. $n_1|(n_2|n_2) \leq (n_3|(n_1|n_1))|((n_3|(n_2|n_2))|(n_3|(n_2|n_2)))$,

for all $n_1, n_2, n_3 \in N$.

Proof. Let $(N, |, 1)$ be an s-Nelson algebra.

1. $n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1 \Leftrightarrow (n_2|n_2)|((n_1|n_1)|(n_1|n_1)) = n_1|(n_2|n_2) = 1 \Leftrightarrow n_2|n_2 \leq n_1|n_1$ from Lemma 3.2 and (S1)-(S2).

2. Let $n_1 \leq n_2$. Then $n_1|(n_2|n_2) = 1$ from Lemma 3.1. Since

$$\begin{aligned}
(n_2|n_3)|((n_1|n_3)|(n_1|n_3)) &= ((n_1|n_3)|(n_1|n_3))|(n_2|n_3) \\
&= ((((n_1|n_3)|(n_1|n_3))|(n_2|n_2))| \\
&\quad (((n_1|n_3)|(n_1|n_3))|(n_3|n_3)))| \\
&\quad ((((n_1|n_3)|(n_1|n_3))|(n_2|n_2))| \\
&\quad (((n_1|n_3)|(n_1|n_3))|(n_3|n_3))) \\
&= ((n_3|((n_1|(n_2|n_2))|(n_1|(n_2|n_2))))| \\
&\quad (n_1|((n_3|(n_3|n_3))|(n_3|(n_3|n_3)))))| \\
&\quad ((n_3|((n_1|(n_2|n_2))|(n_1|(n_2|n_2))))| \\
&\quad (n_1|((n_3|(n_3|n_3))|(n_3|(n_3|n_3))))) \\
&= ((n_3|(1|1))|(n_1|(1|1)))|((n_3|(1|1))|(n_1|(1|1))) \\
&= 1
\end{aligned}$$

from (S1)-(S3), (sn1)-(sn2) and Proposition 3.1 (1), it follows that $n_2|n_3 \leq n_1|n_3$.

3. Since

$$\begin{aligned}
n_1|((n_2|(n_1|n_1))|(n_2|(n_1|n_1))) &= ((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|n_2 \\
&= n_2|(1|1) \\
&= 1
\end{aligned}$$

from (S1), (S3) and Proposition 3.1 (1), it is obtained from Lemma 3.1 that $n_1 \leq n_2|(n_1|n_1)$.

4. $n_2|n_2 \leq (n_1|n_1)|((n_2|n_2)|(n_2|n_2)) = n_2|(n_1|n_1)$ from (3), (S1) and (S2), respectively.

5. Since $n_1|(((n_1|n_2)|n_2)|((n_1|n_2)|n_2)) = (n_1|n_2)|((n_1|n_2)|(n_1|n_2)) = 1$ from (S1), (S3) and (sn1), it follows from Lemma 3.1 that $n_1 \leq (n_1|n_2)|n_2$.

6. $n_1 \leq n_2|(n_3|n_3) \Leftrightarrow (n_1|n_2)|(n_1|n_2) \leq n_3$ is proved from Lemma 3.1 and (S3).

7. Since

$$
\begin{aligned}
&((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))|(((n_1|(n_2|n_2))|\\
&(n_2|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_2|(n_3|n_3))))\\
&= (n_1|(n_2|n_2))|(((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|\\
&\quad n_3))))|((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))))\\
&= (n_1|(n_2|n_2))|((n_1|(((n_2|(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3|\\
&\quad n_3))))|(n_1|(((n_2|(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3)))))\\
&= ((n_1|(n_1|(n_2|n_2)))|(n_1|(n_1|(n_2|n_2))))|(((n_2\\
&\quad |(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3)))\\
&= ((n_1|n_2)|(n_1|n_2))|(((n_2|(n_3|n_3))|\\
&\quad (n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3)))\\
&= ((((n_1|n_2)|(n_1|n_2))|(n_3|n_3))|(((n_1|\\
&\quad n_2)|(n_1|n_2))|(n_3|n_3)))|(n_2|(n_3|n_3))\\
&= ((n_1|((n_2|(n_3|n_3))|(n_2|(n_3|n_3))))|(n_1|((n_2\\
&\quad |(n_3|n_3))|(n_2|(n_3|n_3)))))|(n_2|(n_3|n_3))\\
&= n_1|(((n_2|(n_3|n_3))|((n_2|(n_3|n_3))|(n_2|(n_3|n_3))))\\
&\quad |((n_2|(n_3|n_3))|((n_2|(n_3|n_3))|(n_2|(n_3|n_3)))))\\
&= n_1|(1|1)\\
&= 1
\end{aligned}
$$

from (S1)-(S3), (sn1), Proposition 3.1 (1) and (10), it follows from Lemma 3.1 that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$.

8. Since

$$
\begin{aligned}
&(n_1|(n_2|n_2))|(((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|\\
&n_3))))|((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))))\\
&= ((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))|(((n_1|(n_2|n_2))\\
&\quad |(n_2|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_2|(n_3|n_3))))\\
&= 1
\end{aligned}
$$

from (S1), (S3), (7) and Lemma 3.1, we have from Lemma 3.1 that

$$n_1|(n_2|n_2) \leq (n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3))).$$

9. By substituting $[n_1 := n_2|n_2]$, $[n_2 := n_1|n_1]$ and $[n_3 := n_3|n_3]$ in (8), simultaneously, it follows from (S1) and (S2) that

$$n_1|(n_2|n_2) \leq (n_3|(n_1|n_1))|((n_3|(n_2|n_2))|(n_3|(n_2|n_2))).$$

□

Theorem 3.1. *Let $(N, |, 1)$ be an s-Nelson algebra. If $n_1 \to n_2 := n_1|(n_2|n_2)$, $n_1 \vee n_2 := (n_1|n_1)|(n_2|n_2)$, $n_1 \wedge n_2 := (n_1|n_2)|(n_1|n_2)$ and $\sim n := n|n$, for all $n, n_1, n_2 \in N$, then a structure $\mathcal{N} = \langle N, \wedge, \vee, \to, \sim, 1\rangle$ states a Nelson algebra.*

Proof. Let $(N, |, 0, 1)$ be an s-Nelson algebra, and $n_1 \to n_2 := n_1|(n_2|n_2)$, $n_1 \vee n_2 := (n_1|n_1)|(n_2|n_2)$, $n_1 \wedge n_2 := (n_1|n_2)|(n_1|n_2)$ and $\sim n := n|n$, for all $n, n_1, n_2 \in N$. Then

(N1) $n \vee 1 = (n|n)|(1|1) = 1$ from Proposition 3.1 (1),

(N2) $n_1 \wedge (n_1 \vee n_2) = (n_1|((n_1|n_1)|(n_2|n_2)))|(n_1|((n_1|n_1)|(n_2|n_2))) = n_1$ from (S2),

(N3) $n_1 \wedge (n_2 \vee n_3) = (n_3 \wedge n_1) \vee (n_2 \wedge n_1)$ from Lemma 3.1,

(N4) $\sim\sim n = (n|n)|(n|n)$ from (S2),

(N5) $\sim (n_1 \wedge n_2) = n_1|n_2 = ((n_1|n_1)|(n_1|n_1))|((n_2|n_2)|(n_2|n_2)) =\sim n_1 \vee \sim n_2$ from (S2),

(N6) Since $n_1 \wedge \sim n_1 = (n_1|(n_1|n_1))|(n_1|(n_1|n_1)) = 1|1 = 0$ and $n_2 \vee \sim n_2 = n_2|(n_2|n_2) = 1$ from (sn1), (S1) and (S2), it follows from Proposition 3.1 (3) that $n_1 \wedge \sim n_1 = 0 = (0|1)|(0|1) = 0 \wedge 1 = (n_1 \wedge \sim n_1) \wedge (n_2 \vee \sim n_2)$,

(N7) $n \to n = n|(n|n) = 1$ from (sn1),

(N8)
$$\begin{array}{rcl} n_1 \wedge (n_1 \to n_2) & = & (n_1|(n_1|(n_2|n_2)))|(n_1|(n_1|(n_2|n_2))) \\ & = & (n_1|(((n_1|n_1)|(n_1|n_1))|(n_2|n_2))) \\ & & |(n_1|(((n_1|n_1)|(n_1|n_1))|(n_2|n_2))) \\ & = & n_1 \wedge (\sim n_1 \vee n_2) \end{array}$$
from (S2),

(N9) $(n_1 \wedge n_2) \to n_3 = ((n_1|n_2)|(n_1|n_2))|n_3 = n_1|((n_2|n_3)|(n_2|n_3)) = n_1 \to (n_2 \to n_3)$ from (S3),

(N10) $(n_1 \to n_2) \wedge (\sim n_1 \vee n_2) = n_1|(n_2|n_2) = ((n_1|n_1)|(n_1|n_1))|(n_2|n_2) =\sim n_1 \vee n_2$ from (S2), and

(N11) $n_1 \to (n_2 \wedge n_3) = n_1|(n_2|n_3) = ((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3))) = (n_1 \to n_2) \wedge (n_1 \to n_3)$ from (sn2).

□

Example 3.2. *Consider the s-Nelson algebra $(N, |, 1)$ in Example 3.1. Then $\langle N, \wedge, \vee, \to, \sim, 1\rangle$ defined by the s-Nelson algebra is a Nelson algebra with the following Cayley tables:*

Theorem 3.2. *Let $\mathcal{N} = \langle N, \wedge, \vee, \to, \sim, 1\rangle$ be a Nelson algebra with 0. If $n_1|n_2 :=\sim (n_1 \wedge n_2) =\sim n_1 \vee \sim n_2 = n_1 \to\sim n_2$, for all $n_1, n_2 \in N$, then a structure $(N, |, 1)$ is an s-Nelson algebra*

Table 4:

$\to$	0	w_1	w_2	1
0	1	1	1	1
w_1	w_2	1	w_2	1
w_2	w_1	w_1	1	1
1	0	w_1	w_2	1

$\vee$	0	w_1	w_2	1
0	0	w_1	w_2	1
w_1	w_1	w_1	1	1
w_2	w_2	1	w_2	1
1	1	1	1	1

p	$\sim p$
0	1
w_1	w_2
w_2	w_1
1	0

$\wedge$	0	w_1	w_2	1
0	0	0	0	0
w_1	0	w_1	0	w_1
w_2	0	0	w_2	w_2
1	0	w_1	w_2	1

Proof. Let $\mathcal{N} = \langle N, \wedge, \vee, \to, \sim, 1\rangle$ be an Nelson algebra with 0 and $n_1|n_2 :=\sim (n_1 \wedge n_2) =\sim n_1 \vee \sim n_2 = n_1 \to\sim n_2$, for all $n_1, n_2 \in N$. Then

(S1) $n_1|n_2 =\sim n_1 \vee \sim n_2 =\sim n_2 \vee \sim n_1 = n_2|n_1$,

(S2) $(n_1|n_1)|(n_1|n_2) =\sim (\sim n_1 \wedge (\sim n_1 \vee \sim n_2)) =\sim\sim n_1 = n_1$ from (N2) and (N4),

(S3) $n_1|((n_2|n_3)|(n_2|n_3)) =\sim n_1 \vee (\sim n_2 \vee \sim n_3) = (\sim n_1 \vee \sim n_2) \vee \sim n_3 = ((n_1|n_2)|(n_1|n_2))|n_3$ from (N4),

(S4) $(n_1|((n_1|n_1)|(n_2|n_2)))|(n_1|((n_1|n_1)|(n_2|n_2))) = n_1 \wedge (n_1 \vee n_2) = n_1$, from (N2) and (N4),

(sn1) $n|(n|n) = n \to n = 1$ from (N4) and (N7), and

(sn2)

$$\begin{aligned} n_1|(n_2|n_3) &= n_1 \to (n_2 \wedge n_3) \\ &= (n_1 \to n_2) \wedge (n_1 \to n_3) \\ &= (n_1 \to\sim\sim n_2) \wedge (n_1 \to\sim\sim n_3) \\ &= ((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3))) \end{aligned}$$

from (N4) and (N11).

□

Example 3.3. *Consider a Nelson algebra $\langle N, \wedge, \vee, \to, \sim, 1\rangle$ in which $N = \{0, a, b, c, d, e, f, 1\}$, binary operations $\wedge, \vee, \to$ and an unary operation $\sim$ on N with the following Cayley tables:*

Table 5:

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	f	1	1	f	1
b	e	e	1	e	1	e	1	1
c	d	d	d	1	d	1	1	1
d	c	e	f	c	1	e	f	1
e	b	d	b	f	d	1	f	1
f	a	a	d	e	d	e	1	1
1	0	a	b	c	d	e	f	1

p	$\sim p$
0	1
a	f
b	e
c	d
d	c
e	b
f	a
1	0

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

Then a structure $(N, |, 1)$ *defined by this Nelson algebra is an s-Nelson algebra with Cayley table as follow:*

Table 6:

$\mid$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

4 Ideals of s-Nelson algebras

In this section, we define a quasi-subalgebra and an ideal of an s-Nelson algebra, and examine some features. When not otherwise specified, N is an s-Nelson algebra.

Definition 4.1. *Let N be an s-Nelson algebra. Then a nonempty subset M of N is called a quasi-subalgebra of N if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in M$, for all $n_1, n_2 \in M$.*

Example 4.1. *Consider the s-Nelson algebra N in Example 3.3. Then a subset $M = \{0, b, e, 1\}$ of N is a quasi-subalgebra of N.*

Definition 4.2. *Let N be an s-Nelson algebra. Then a nonempty subset I of N is called an ideal of N if*

(NI1) $0 \in I$
(NI2) $n_2 \in I$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ imply $n_1 \in I$,
for all $n_1, n_2 \in N$.

Example 4.2. *Consider the s-Nelson algebra N in Example 3.3. Then N itself, $\{0\}$, $\{0, a, d, b\}$ and $\{0, c\}$ are ideals of N.*

Lemma 4.1. *Let I be a nonempty subset of an s-Nelson algebra N. Then I is an ideal of N if and only if*

(NI3) $n_1 \leq n_2$ and $n_2 \in I$ imply $n_1 \in I$,
(NI4) $n_1, n_2 \in I$ implies $n_1 \vee n_2 \in I$,
for all $n_1, n_2 \in N$.

Proof. Assume that I is an ideal of N. Let $n_1 \leq n_2$ and $n_2 \in I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) = 1|1 = 0 \in I$ from Lemma 3.1 and (NI1). Thus, $n_1 \in I$ by (NI2). Let $n_1, n_2 \in I$. Since $((((n_1 \vee n_2)|(n_2|n_2))|((n_1 \vee n_2)|(n_2|n_2)))|(n_1|n_1))|((((n_1 \vee n_2)|(n_2|n_2))|((n_1 \vee n_2)|(n_2|n_2)))|(n_1 n_1)) = ((n_1 \vee n_2)|((n_1 \vee n_2)|(n_1 \vee n_2)))|((n_1 \vee n_2)|((n_1 \vee n_2)|(n_1 \vee n_2))) = 1|1 = 0 \in I$ from (S1), (S3), Lemma 3.2, (sn1) and (NI1), we have from (NI2) that $n_1 \vee n_2 \in I$.

Conversely, let I be a nonempty subset of N satisfying (NI3) and (NI4). Since $0 \leq n$ and $n \in I$, we get from (NI3) that $0 \in I$. Let $n_2 \in I$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$. Since

$$\begin{aligned} n_1 \vee n_2 &= (n_1|n_1)|(n_2|n_2) \\ &= (((n_1|n_1)|(n_2|n_2))|1)|(((n_1|n_1)|(n_2|n_2))|1) \\ &= (((n_2|n_2)|(n_1|n_1))|((n_2|n_2)|((n_2|n_2)|(n_2|n_2)))) \\ &\quad |(((n_2|n_2)|(n_1|n_1))|((n_2|n_2)|((n_2|n_2)|(n_2|n_2)))) \\ &= (n_1|(n_2|n_2))|(n_2|n_2) \\ &= ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee n_2 \in I \end{aligned}$$

from Lemma 3.2, Proposition 3.1 (1), (S1)-(S2), (sn2) and (NI4), and $n_1 \leq n_1 \vee n_2$, it follows from (NI3) that $n_1 \in I$. $\square$

Lemma 4.2. *Every ideal of an s-Nelson algebra N is a quasi-subalgebra of N.*

Proof. Let I be an ideal of N and $n_1, n_2 \in I$. Since

$$\begin{aligned}
&((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|n_2)))\\
&= (n_2|n_2)|((n_1|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|\\
&\quad n_2))))|(n_1|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|n_2)))))\\
&= (n_2|n_2)|((((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|(n_2|\\
&\quad n_2))|(((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|(n_2|n_2)))\\
&= (n_2|n_2)|(((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1)))\\
&= 1
\end{aligned}$$

from (S1), (S3), (sn1) and Proposition 3.1 (1), it follows from Lemma 3.1 and Lemma 3.2 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \leq (n_1|n_1)|(n_2|n_2) = n_1 \vee n_2$. Thus, we have from Lemma 4.1 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$, and so, I is a quasi-subalgebra of N. $\square$

However, the inverse of Lemma 4.2 is not true in general.

Example 4.3. *Consider the s-Nelson algebra N in Example 3.3. Then $I = \{0, c, d, 1\}$ is a quasi-subalgebra of N but it is not an ideal of N since $a \notin I$ when $(a|(d|d))|(a|(d|d)) = 0 \in I$ and $d \in I$.*

Theorem 4.1. *The family $\mathfrak{I}_N$ of all ideals of an s-Nelson algebra N forms a complete distributive modular lattice.*

Proof. Let $\{I_i : i \in \mathbb{N}\}$ be a family of ideals of N. Since $0 \in I_i$, for all $i \in \mathbb{N}$, it follows that $0 \in \bigcup_{i\in\mathbb{N}} I_i$ and $0 \in \bigcap_{i\in\mathbb{N}} I_i$. Assume that $n_2 \in \bigcap_{i\in\mathbb{N}} I_i$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in \bigcap_{i\in\mathbb{N}} I_i$. Since $n_2 \in I_i$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I_i$, for all $i \in \mathbb{N}$, it is obtained that $n_1 \in I_i$, for all $i \in \mathbb{N}$, which implies that $n_1 \in \bigcap_{i\in\mathbb{N}} I_i$. Let X be the family of all ideals of N containing $\bigcup_{i\in\mathbb{N}} I_i$. Thus, $\bigcap X$ is also an ideal of N.

If $\bigwedge_{i\in\mathbb{N}} I_i = \bigcap_{i\in\mathbb{N}} I_i$ and $\bigvee_{i\in\mathbb{N}} I_i = \bigcap X$, then $(\mathfrak{I}_N, \bigvee, \bigwedge)$ is a complete lattice. Moreover, $(\mathfrak{I}_N, \bigvee, \bigwedge)$ is distributive from the definitions of $\bigvee$ and $\bigwedge$, and so, it is modular by the Theorem 2.1. $\square$

Corollary 4.1. *Let V be a subset of an s-Nelson algebra N. Then there exists the minimal ideal $\langle V\rangle$ of V containing the subset V.*

Proof. Let $V = \{I : I \textit{ is an ideal of } N \textit{ containing } M \subseteq N\}$. Then $\bigcap V = \{n \in N : n \in \bigcap_{I\in V} I\}$ is the minimal ideal of N containing $M \subseteq N$. $\square$

Definition 4.3. *Let I be an ideal of an s-Nelson algebra N. Then I is called an ultra ideal of N if $n_1 \wedge n_2 \in I$ implies $n_1 \in I$ or $n_2 \in I$, for all $n_1, n_2 \in N$.*

Example 4.4. *Consider the s-Nelson algebra N in Example 3.3. Then $\{0, a, b, d\}$ is an ultra ideal of N.*

Remark 4.1. *Every ultra ideal of N is an ideal of N by Definition 4.3, but the inverse does not usually hold.*

Example 4.5. *Consider the s-Nelson algebra N in Example 3.3. Then $I = \{0, b\}$ is an ideal of N but it is not ultra since $d, f \notin I$ when $d \wedge f = b \in I$.*

Proposition 4.1. *Let I be an ideal of an s-Nelson algebra N. Then I is an ultra ideal of N if and only if $n \in I$ or $n|n \in I$, for all $n \in N$.*

Proof. ($\Rightarrow$) Let I be an ultra ideal of N. Since $n \wedge (n|n) = (n|(n|n))|(n|(n|n)) = 1|1 = 0 \in I$ from Lemma 3.2 and (sn1), it is obtained that $n \in I$ or $n|n \in I$, for all $n \in N$.

($\Leftarrow$) Let I be an ideal of N such that $n \in I$ or $n|n \in I$, for all $n \in N$, and $n_1 \wedge n_2 \in I$. Assume that $n_1 \notin I$, for some $n_1 \in N$. Then $n_1|n_1 \in I$. Since $(n_2|((n_1|n_1)|(n_1|n_1)))|(n_2|((n_1|n_1)|(n_1|n_1))) = (n_1|n_2)|(n_1|n_2) = n_1 \wedge n_2 \in I$ from (S1), (S2) and Lemma 3.2, it follows from (NI2) that $n_2 \in I$. Suppose that $n_2 \notin I$, for some $n_2 \in N$. Then $n_2|n_2 \in I$. Similarly, it is obtained from (NI2) that $n_1 \in I$ by substituting $[n_1 := n_2]$ and $[n_2 := n_1]$ in above statement, simultaneously. Thus, I is an ultra ideal of N. □

Proposition 4.2. *Let I be an ideal of an s-Nelson algebra N. Then I is an ultra ideal of N if and only if $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Proof. Let I be an ultra ideal of N, and $n_1, n_2 \notin I$. By Proposition 3.1, $n_1|n_1 \in I$ and $n_2|n_2 \in I$. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \leq n_2|n_2$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \leq n_1|n_1$ from Lemma 3.3 (1) and (3), it follows from (NI3) that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.

Conversely, let I be an ideal of N satisfying that $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for any $n_1, n_2 \in N$. Suppose that $n \notin I$ and $n|n \notin I$, for some $n \in N$. Then we have from (S2) that $(n|((n|n)|(n|n)))|(n|((n|n)|(n|n))) = (n|n)|(n|n) = n \in I$ and $((n|n)|(n|n))|((n|n)|(n|n)) = n|n \in I$, which are contradictions. Thus, $n \in I$ or $n|n \in I$. Thereby, I is an ultra ideal of N from Proposition 3.1. □

Proposition 4.3. *Let I be an ideal of an s-Nelson algebra N. Then I is an ultra ideal of N if and only if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Proof. Let I be an ultra ideal of N. Since

$$\begin{aligned}
&((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \wedge ((n_2|(n_1|n_1))|(n_2|(n_1|n_1)))\\
&= (((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|((n_2|(n_1|n_1))|(n_2|(n_1|n_1))))\\
&\quad |(((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|((n_2|(n_1|n_1))|(n_2|(n_1|n_1))))\\
&= (((((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|(n_1|n_1))|(((n_1|(n_2|n_2))|\\
&\quad (n_1|(n_2|n_2)))|(n_1|n_1)))|n_2)|(((((n_1|(n_2|n_2))|(n_1|(n_2|\\
&\quad n_2)))|(n_1|n_1))|(((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|(n_1|n_1)))|n_2)\\
&= (((n_2|n_2)|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))))|((n_2|n_2)|((n_1|\\
&\quad (n_1|n_1))|(n_1|(n_1|n_1))))|n_2)|(((n_2|n_2)|((n_1|(n_1|n_1))|(n_1\\
&\quad |(n_1|n_1))))|((n_2|n_2)|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))))|n_2)\\
&= (((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1))|n_2)|(((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1))|n_2)\\
&= 1|1\\
&= 0 \in I
\end{aligned}$$

from Lemma 3, (S1), (S3), (sn1) and Proposition 3.1 (1), it follows from Definition 4.3 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.

Conversely, let I be an ideal of N such that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for any $n_1, n_2 \in N$. Assume that $n_1 \wedge n_2 \in I$. Since

$$\begin{aligned}
&(n_1|((n_1 \wedge n_2)|(n_1 \wedge n_2)))|(n_1|((n_1 \wedge n_2)|(n_1 \wedge n_2)))\\
&= (n_1|(((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))))\\
&\quad |(n_1|(((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))))\\
&= (n_1|(n_1|n_2))|(n_1|(n_1|n_2))\\
&= (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I
\end{aligned}$$

or

$$\begin{aligned}
&(n_2|((n_1 \wedge n_2)|(n_1 \wedge n_2)))|(n_2|((n_1 \wedge n_2)|(n_1 \wedge n_2)))\\
&= (n_2|((n_2 \wedge n_1)|(n_2 \wedge n_1)))|(n_2|((n_2 \wedge n_1)|(n_2 \wedge n_1)))\\
&= (n_2|(((n_2|n_1)|(n_2|n_1))|((n_2|n_1)|(n_2|n_1))))\\
&\quad |(n_2|(((n_2|n_1)|(n_2|n_1))|((n_2|n_1)|(n_2|n_1))))\\
&= (n_2|(n_2|n_1))|(n_2|(n_2|n_1))\\
&= (n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I
\end{aligned}$$

from Lemma 3.2, (S2) and Proposition 3.1 (10), it follows from (NI2) that $n_1 \in I$ or $n_2 \in I$, for all $n_1, n_2 \in N$. Therefore, I is an ultra ideal of N. □

5 Quotient s-Nelson algebra by ideals

In this section, a congruence relation on an s-Nelson algebra is defined by means of ideals and a quotient s-Nelson algebra is determined via this congruence relation. Also some of properties of the quotient algebra are given.

Definition 5.1. *Let R be an equivalence relation on N and $(n_1, n_2) \in R$ imply $(n_1|n_3, n_2|n_3) \in R$, for all $n_1, n_2, n_3 \in N$. Then R is called a congruence relation on N.*

Example 5.1. *Consider the s-Nelson algebra N in Example 3.1. Then $R = \{(0,0),(w_1,w_1),(w_2,w_2),(1,1),(0,w_1),(w_1,0),(w_2,1),(1,w_2)\}$ is a congruence relation on N.*

Lemma 5.1. *An equivalence relation R is a congruence relation on N if and only if $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$ imply $(n_1|n_2, m_1|m_2) \in R$, for all $n_1, n_2, m_1,$
$m_2 \in N$.*

Proof. Let R be a congruence relation on N and $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$. Since $n_1|n_2 \beta m_1|n_2$ and $m_1|n_2 \beta m_1|m_2$ from (S1), it is obtained from the transitivity of R that $(n_1|n_2, m_1|m_2) \in R$, for all $n_1, n_2, m_1,$
$m_2 \in N$.

Conversely, let R be an equivalence relation on N such that $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$ imply $(n_1|n_2, m_1|m_2) \in R$, for any $n_1, n_2, m_1, m_2 \in N$. Suppose that n_1, n_2 and n_3 be arbitrary elements of N such that $(n_1, n_2) \in R$. Since $(n_3, n_3) \in R$, it follows that $(n_1|n_3, n_2|n_3) \in R$, for all $n_1, n_2, n_3 \in N$. Hence, R is a congruence relation on N. □

The following lemma shows that a binary relation on an s-Nelson algebra defined by ideals states a congruence relation on this algebra.

Lemma 5.2. *Let I be an ideal of an s-Nelson algebra N and a binary relation λ_I on N be defined by*

$$(n_1, n_2) \in \lambda_I \Leftrightarrow (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I \\ \text{and } (n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I,$$

for all $n_1, n_2 \in N$. Then λ_I is a congruence relation on N.

Proof. Let I be an ideal of N.

•: The reflexivity property is obtained from (sn1).

•: The symmetry property is obvious from the definition of λ_I.

•: Transitivity: let $(n_1, n_2) \in \lambda_I$ and $(n_2, n_3) \in \lambda_I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)), (n_2|(n_1|n_1))|(n_2|(n_1|n_1)), (n_2|(n_3|n_3))|(n_2|(n_3|n_3)), (n_3|(n_2|n_2))|(n_3|(n_2|n_2)) \in I$. Since $(n_1|(n_2|n_2))|(n_2|(n_3|n_3)) = ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee ((n_2|(n_3|n_3))|(n_2|(n_3|n_3))) \in I$ from Lemma 3.2, (S2) and (NI4), it follows from Lemma 3.3 (7) and (NI3) that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \in I$. Similarly, it is obtained from Lemma 3.3 (7) and (NI3) that $(n_3|(n_1|n_1))|(n_3|(n_1|n_1)) \in I$ since $(n_3|(n_2|n_2))|(n_2|(n_1|n_1)) = ((n_3|(n_2|n_2))|(n_3|(n_2|n_2))) \vee ((n_2|(n_1|n_1))|(n_2|(n_1|n_1))) \in I$ from Lemma 3.2, (S2) and (NI4). So, $(n_1, n_3) \in \lambda_I$.

Hence, λ_I is an equivalence relation on N.

Let $(n_1, n_2) \in \lambda_I$ and $(m_1, m_2) \in \lambda_I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2))$, $(n_2|(n_1|n_1))|(n_2|(n_1|n_1))$, $(m_1|(m_2|m_2))|(m_1|(m_2|m_2))$, $(m_2|(m_1|m_1))|(m_2|(m_1|m_1)) \in I$.

1. Since

$$\begin{aligned} n_1|(n_2|n_2) &= (n_2|n_2)|((n_1|n_1)|(n_1|n_1)) \\ &\leq (m_1|((n_2|n_2)|(n_2|n_2)))|((m_1|((n_1|n_1) \\ &\quad |(n_1|n_1)))|(m_1|((n_1|n_1)|(n_1|n_1)))) \\ &= (m_1|n_2)|((m_1|n_1)|(m_1|n_1)) \end{aligned}$$

and

$$\begin{aligned} n_2|(n_1|n_1) &= (n_1|n_1)|((n_2|n_2)|(n_2|n_2)) \\ &\leq (m_1|((n_1|n_1)|(n_1|n_1)))|((m_1|((n_2|n_2) \\ &\quad |(n_2|n_2)))|(m_1|((n_2|n_2)|(n_2|n_2)))) \\ &= (m_1|n_1)|((m_1|n_2)|(m_1|n_2)) \end{aligned}$$

from (S1), (S2) and Lemma 3.3 (9), it is obtained Lemma 3.3 (1) and (S1) that $((n_2|m_1)|((n_1|m_1)|(n_1|m_1)))|((n_2|m_1)|((n_1|m_1)|(n_1|m_1))) \leq (n_1|(n_2|n_2))|(n_1|(n_2|n_2))$ and $((n_1|m_1)|((n_2|m_1)|(n_2|m_1)))|((n_1|m_1)|((n_2|m_1)|(n_2|m_1))) \leq (n_2|(n_1|n_1))|(n_2|(n_1|n_1))$. Thus, $((n_2|m_1)|((n_1|m_1)|(n_1|m_1)))|((n_2|m_1)|((n_1|m_1)|(n_1|m_1))) \in I$ and $((n_1|m_1)|((n_2|m_1)|(n_2|m_1)))|((n_1|m_1)|((n_2|m_1)|(n_2|m_1))) \in I$ from (NI3), and so, $(n_1|m_1, n_2|m_1) \in \lambda_I$.

2. By substituting $[n_1 := m_1]$, $[n_2 := m_2]$ and $[m_1 := n_2]$ in (1), it follows from (S1) that $(n_2|m_1, n_2|m_2) \in \lambda_I$.

   Hence, $(n_1|m_1, n_2|m_2) \in \lambda_I$ from the transitivity of λ_I.

Therefore, λ_I is a congruence relation on N. □

The following theorem states that a quotient set of an s-Nelson algebra is determined by means of a congruence relation defined as in Lemma 5.2 and the quotient set is an s-Nelson algebra.

Theorem 5.1. *Let I be an ideal of an s-Nelson algebra N and λ_I be a congruence relation on N defined by I. Then $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is an s-Nelson algebra where the quotient set $N/\lambda_I = \{[n]_{\lambda_I} : n \in N\}$, the binary operation $|_{\lambda_I}$ is defined by $[n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I} = [n_1|n_2]_{\lambda_I}$ and $[1]_{\lambda_I}|_{\lambda_I}[1]_{\lambda_I} = [0]_{\lambda_I} = I$.*

Proof. Let I be an ideal of an s-Nelson algebra N and λ_I be a congruence relation on N defined by I.

(S1): $[n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I} = [n_1|n_2]_{\lambda_I} = [n_2|n_1]_{\lambda_I} = [n_2]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I}$

(S2): $([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}) = [(n_1|n_1)|(n_1|n_2)]_{\lambda_I} = [n_1]_{\lambda_I}$,

(S3):
$$
\begin{aligned}
&[n_1]_{\lambda_I}|_{\lambda_I}(([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I}))\\
&= [n_1|((n_2|n_3)|(n_2|n_3))]_{\lambda_I}\\
&= [((n_1|n_2)|(n_1|n_2))|n_3]_{\lambda_I}\\
&= (([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I})|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I}[n_3]_{\lambda_I},
\end{aligned}
$$

(S4): $([n_1]_{\lambda_I}|_{\lambda_I}(([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I})))|_{\lambda_I}$
$([n_1]_{\lambda_I}|_{\lambda_I}(([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I})))$
$= [(n_1|((n_1|n_1)|(n_2|n_2)))|(n_1|((n_1|n_1)|(n_2|n_2)))]_{\lambda_I}$
$= [n_1]_{\lambda_I}$,

(sn1): $[n]_{\lambda_I}|_{\lambda_I}([n]_{\lambda_I}|_{\lambda_I}[n]_{\lambda_I}) = [n|(n|n)]_{\lambda_I} = [1]_{\lambda_I}$,

(sn2): $[n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})$
$= [n_1|(n_2|n_3)]_{\lambda_I}$
$= [((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))]_{\lambda_I}$
$= (([n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}([n_3]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})))|_{\lambda_I}$
$(([n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}([n_3]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})))$,

for all $n_1, n_2, n_3 \in N$.

Thereby, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is an s-Nelson algebra.

Moreover, $[1]_{\lambda_I}|_{\lambda_I}[1]_{\lambda_I} = [1|1]_{\lambda_I} = [0]_{\lambda_I}$. Assume that $n \in [0]_{\lambda_I}$. Since $(n, 0) \in \lambda_I$, we have from Proposition 3.1 (1), (3) and (S1) that $n = (n|1)|(n|1) = (n|(0|0))|(n|(0|0)) \in I$ and $0 = 1|1 = ((n|n)|(1|1))|((n|n)|(1|1)) = (0|(n|n))|(0|(n|n)) \in I$. So, $[0]_{\lambda_I} \subseteq I$. Suppose that $n \in I$. Since $(n|(0|0))|(n|(0|0)) = (n|1)|(n|1) = n \in I$ and $(0|(n|n))|(0|(n|n)) = ((n|n)|(1|1))|((n|n)|(1|1)) = 1|1 = 0 \in I$ from Proposition 3.1 (1), (3) and (S1), it follows that $(n, 0) \in \lambda_I$, and so, $I \subseteq [0]_{\lambda_I}$. Thus, $[0]_{\lambda_I} = I$. □

Example 5.2. *Consider the s-Nelson algebra N in Example 3.3. For an ideal $I = \{0, c\}$ of N, a relation $\lambda_I = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (1,1), (0,c), (c,0), (d,1), (1,d), (a,e), (e,a), (b,f), (f,b)\}$ is a congruence relation on N. Then $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is an s-Nelson algebra where $N/\lambda_I = \{[0]_{\lambda_I}, [b]_{\lambda_I}, [e]_{\lambda_I}, [1]_{\lambda_I}\}$ and the Sheffer operation $|_{\lambda_I}$ on N/λ_I have Cayley table as follow:*

Table 7:

$\vert_{\lambda_I}$	$[0]_{\lambda_I}$	$[b]_{\lambda_I}$	$[e]_{\lambda_I}$	$[1]_{\lambda_I}$
$[0]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$
$[b]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$
$[e]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[b]_{\lambda_I}$	$[b]_{\lambda_I}$
$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$	$[b]_{\lambda_I}$	$[0]_{\lambda_I}$

Lemma 5.3 demonstrates that a relation on a quotient s-Nelson algebra defined by means of a congruence relation and ideals is partial order on the quotient algebra.

Lemma 5.3. *Let I be an ideal of an s-Nelson algebra N and λ_I be a congruence relation on N defined by I. Then a relation $\leq$ on N/λ_I defined by*

$$[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I} \Leftrightarrow (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$$

is a partial order on N/λ_I.

Proof. Let I be an ideal of an s-Nelson algebra N and λ_I be a congruence relation on N defined by I. By Theorem 5.1, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is an s-Nelson algebra.

• Since $(n|(n|n))|(n|(n|n)) = 1|1 = 0$ from (sn1), it follows that $[n]_{\lambda_I} \leq [n]_{\lambda_I}$.

• Let $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, it is obtained that $(n_1, n_2) \in \lambda_I$, which means that $[n_1]_{\lambda_I} = [n_2]_{\lambda_I}$.

• Let $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_3]_{\lambda_I}$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_3|n_3))|(n_2|(n_3|n_3)) \in I$. Since $(n_1|(n_2|n_2))|(n_2|(n_3|n_3)) = ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee ((n_2|(n_3|n_3))|(n_2|(n_3|n_3))) \in I$ and $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$ from (S2), Lemma 3.2 and Lemma 3.3 (7), it is obtained from (NI3) that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \in I$. Thus, $[n_1]_{\lambda_I} \leq [n_3]_{\lambda_I}$.

Hence, a relation $\leq$ on N/λ_I is a partial order on N/λ_I. □

Theorem 5.2. *Let I be an ideal of an s-Nelson algebra N. Then I is an ultra ideal of N if and only if N/λ_I is a totally ordered s-Nelson algebra and $\mid N/\lambda_I \mid \leq 2$.*

Proof. ($\Rightarrow$) Let I be an ultra ideal of an s-Nelson algebra N. By Theorem 5.1, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is an s-Nelson algebra. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$ from Proposition 4.3, it follows that $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$, for all $n_1, n_2 \in N$. So, N/λ_I is a totally ordered s-Nelson algebra. Suppose that $\mid N/\lambda_I \mid > 2$. Let $[n]_{\lambda_I} \in N/\lambda_I$ such that $[0]_{\lambda_I} < [n]_{\lambda_I} < [1]_{\lambda_I}$. Since I is an ultra ideal of N, we get from Proposition 4.1 that $n \in I$ or $n|n \in I$. Assume that $n|n \in I$. Since $(1|(n|n))|(1|(n|n)) = n|n \in I$ and $(n|(1|1))|(n|(1|1)) = 1|1 = 0 \in I$ from Proposition 3.1 (1)-(2) and (NI1), it is obtained that $(n, 1) \in \lambda_I$. Thus, $[n]_{\lambda_I} = [1]_{\lambda_I}$ which is a contradiction. Hence, $\mid N/\lambda_I \mid \leq 2$.

($\Leftarrow$) Let I be an ideal of N and N/λ_I be a totally ordered s-Nelson algebra. Since $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ or $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$, we get that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in$

I or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$. Therefore, I is an ultra ideal of N by Proposition 4.3. □

Corollary 5.1. *Let I be an ideal of an s-Nelson algebra N. Then N/λ_I is a totally ordered s-Nelson algebra if and only if $n \in I$ or $n|n \in I$, for all $n \in N$.*

Corollary 5.2. *Let I be an ideal of an s-Nelson algebra N. Then N/λ_I is a totally ordered s-Nelson algebra if and only if $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Corollary 5.3. *Let I be an ideal of an s-Nelson algebra N. Then N/λ_I is a totally ordered s-Nelson algebra if and only if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

6 Conclusion

In the study, we have given a Sheffer stroke Nelson algebra, some types of ideals, a quasi-subalgebra and a quotient set of this algebraic structure. We introduce a Sheffer stroke Nelson algebra (for short, s-Nelson algebra) and show that the axioms are independent. Then a partial order on an s-Nelson algebra is defined and some properties are examined. We indicate that an s-Nelson algebra states a bounded distributive modular lattice. Also, it is proved that every s-Nelson algebra is a Nelson algebra and a Nelson algebra with zero is an s-Nelson algebra under the special conditions. Indeed, we describe an (ultra) ideal of an s-Nelson algebra and present some properties. It is demonstrated that every ideal of an s-Nelson algebra is its quasi-subalgebra but the inverse is generally not true. In fact, it is indicated that a family of all ideals of an s-Nelson algebra forms a complete distributive modular lattice, and so, a minimal ideal of an s-Nelson algebra containing its subset is determined. Besides, it is illustrated that every ultra ideal of an s-Nelson algebra is the ideal but the inverse does not mostly hold. It is given the statements which are equivalent to the definition of an (ultra) ideal of an s-Nelson algebra. Moreover, a congruence relation on an s-Nelson algebra is defined by means of its ideal. Thus, a quotient s-Nelson algebra is described via this congruence relation and a partial order on the quotient structure is determined. It is proved that a quotient s-Nelson algebra is totally ordered if and only if an ideal of an s-Nelson algebra is ultra, and some results are presented.

In future works, we wish to investigate fuzzy and neutrosophic structures on s-Nelson algebras.

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References

[1] J. C. Abbott, *Implicational algebras*, Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie **11**(59) (1967), 3-23.

[2] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts Math **78** (1981).

[3] I. Chajda, *Sheffer operation in ortholattices*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **44**(1) (2005), 19-23.

[4] I. Chajda, R. Halaš and H. Länger, *Operations and structures derived from non-associative MV-algebras*, Soft Computing **23** (2019), 3935-3944.

[5] I. Chajda and H. Länger, *Sheffer operation in relational systems*, Soft Computing **26** (2022), 89-97.

[6] W. B. Ewald, *Intuitionistic tense and modal logic*, Journal of Symbolic Logic **51**(1) (1986), 166-179.

[7] A. V. Figallo and G. Pelaitay, *An algebraic axiomatization of the Ewalds intuitionistic tense logic*, Soft Computing **18** (2014), 1873-1883..

[8] A. V. Figallo, G. Pelaitay and J. Sarmiento, *An algebraic study of tense operators on Nelson algebras*, Studia Logica **19** (2021), 285-312.

[9] U. Rivieccio and M. Spinks, *Quasi-Nelson; or, non-involutive Nelson algebras*. In: D. Fazio, A. Ledda, F.Paoli (eds.), Algebraic Perspectives on Substructural Logics. Trends in Logic, Springer **55** (2020), 133-168.

[10] U. Rivieccio and R. Jansana, *Quasi-Nelson algebras and fragments*, Mathematical Structures in Computer Science **31**(3) (2021), 257-285.

[11] C. Gomez, M. A. Marcos and H. J. San Martin, *On the relation of negations in Nelson algebras*, Reports on Mathematical Logic **56** (2022), 15-56.

[12] T. Katican, T. Oner, A. Rezaei and F. Smarandache, *Neutrosophic N-structures applied to Sheffer stroke BL-algebras*, Computer Modeling in Engineering and Sciences **129**(1) (2021), 355-372.

[13] A. A. Markov, *A constructive logic*, Journal of Sysmbolic Logic **18**(3) (1953), 257-257.

[14] W. McCune, R. Veroff, B. Fitelson, K. Harris, A. Feist and L. Wos, *Short single axioms for Boolean algebra*, Journal of Automated Reasoning **29** (2002), 1-16.

[15] A. Molkhasi and K. P. Shum, *Representations of strongly algebraically closed algebras*, Algebra and Discrete Mathematics **28**(1) (2019), 130-143.

[16] A. Molkhasi, *Representations of Sheffer stroke algebras and Visser algebras*, Soft Computing **25** (2021), 85338538.

[17] D. Nelson, *Constructible falsity*, Journal of Symbolic Logic **14**(1) (1949), 16-26.

[18] T. Oner, T. Katican, A. Borumand Saeid and M. Terziler, *Filters of strong Sheffer stroke non-associative MV-algebras*, Analele Stiintifice ale Universitatii Ovidius Constanta **29**(1) (2021), 143-164.

[19] T. Oner, T. Katican and A. Borumand Saeid, *Fuzzy Filters of Sheffer stroke Hilbert algebras*, Journal of Intelligent and Fuzzy Systems **40**(1) (2021), 759-772.

[20] T. Oner, T. Katican and A. Borumand Saeid, *Relation between Sheffer stroke and Hilbert algebras*, Categories and General Algebraic Structures with Applications **14**(1) (2021), 245-268.

[21] H. Rasiowa, *N-lattices and constructive logic with strong negation*, Fundamenta Mathematicae **46**(1) (1958), 61-80.

[22] H. Rasiowa, *An algebraic aproach to non-classic logics*, Warzawa & Amsterdam: North-Holland Publishing Company, (1974).

[23] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts Math, **78** (1981).

[24] H. M. Sheffer, *A set of five independent postulates for Boolean algebras, with application to logical constants*, Transactions of the American Mathematical Society **14**(4) (1913), 481-488.

[25] R. Veroff, *A shortest 2-basis for Boolean algebra in terms of the Sheffer stroke*, Journal of Automated Reasoning **31** (2003), 1-9.

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