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Square-difference factor absorbing submodules of modules over commutative rings

Hani A. Khashan and Ece Yetkin Celikel

Abstract

Let R be a commutative ring with identity and M an unitary R-module. Recently, in [5], Anderson, Badawi and Coykendalla defined a proper ideal I of R to be a square-difference factor absorbing ideal (sdf-absorbing ideal) of R if whenever $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$. Generally, this article is devoted to introduce and study square-difference factor absorbing submodules. A proper submodule N of M is called square-difference factor absorbing (sdf-absorbing) in M if whenever $m \in M$ and $a, b \in R \backslash Ann_R(m)$ such that $(a^2 - b^2)m \in N$, then $(a + b)m \in N$ or $(a - b)m \in N$. Many properties, examples and characterizations of sdf-absorbing submodules are introduced, especially in multiplication modules. Comparing this new class of submodules with classical prime submodules, we present new characterizations for von-Neumann regular modules in terms of sdf-absorbing submodules. Further characterizations of some special modules in which every nonzero proper submodule is sdf-absorbing are investigated. Finally, the sdf-absorbing submodules in amalgamated modules are studied.

1 Introduction

Throughout this paper, we assume that all rings are commutative with nonzero identity and all modules are unitary. Let R be a ring and M be an R-module.

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By $Char(R)$, $U(R)$, $Z(R)$, we denote the smallest integer which satisfies $n.1_R = 0$, the set of unit elements of R and the set of zero-divisors of R, respectively. Let N and K be submodules of M and I be an ideal of R. The residual of N by K is defined as the set $(N :_R K) = \{r \in R : rK \subseteq N\}$ which is clearly an ideal of R. The residual of N by I is the set $(N :_M I) = \{m \in M : Im \subseteq N\}$ which is clearly a submodule of M containing N. In particular, for $N = \{0\}$, $a \in R$ and $m \in M$, we have $Ann_R(m) = (0 :_R m) = \{r \in R : rm = 0_M\}$ and $Ann_M(a) = (0 :_M a) = \{m \in M : am = 0_M\}$. Recall that an element $m \in M$ is said to be torsion if there exists $r \in R$ such that $rm = 0_M$. The set of torsion elements of M is denoted by $T(M)$. Specially, if $T(M) = M$ or $T(M) = \{0_M\}$, we call M a torsion module or a torsion-free module, respectively.

It is well-known that a proper submodule N of an R-module M is called prime if whenever $r \in R$, $m \in M$ and $rm \in N$, then $r \in (N :_R M)$ or $m \in M$. More generally, the concept of classical prime submodules (or namely weakly prime submodules as in [8]) is introduced and studied by Behboodi in [8]-[10]. A proper submodule P of M is called a classical prime submodule if for $a, b \in R$ and $m \in M$ such that $abm \in N$, we have $am \in N$ or $bm \in N$. The significance of prime submodules transcends mere definition; they serve as crucial tools for understanding the structure of modules, particularly in the context of Artinian and Noetherian modules, where their properties can yield insights into the classification and decomposition of modules. Over the last decades, many generalizations of prime ideals and submodules have been built (see for example, [2], [6], [15]-[18], [21]). Among them, the one originally proposed by Anderson, Badawi and Coykendall offers a new perspective. According to their paper [5], a proper ideal I of R is called a square-difference factor absorbing ideal (sdf-absorbing ideal) of R if for nonzero $a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in I$ or $a - b \in I$. They gave many properties of sdf-absorbing ideals such as showing that a nonzero sdf-absorbing ideal is a radical ideal which leads to a characterization of rings in which every nonzero proper ideal is sdf-absorbing. Moreover, they determine the sdf-absorbing ideals in a PID, direct product of two rings, polynomial rings, idealizations, amalgamation rings and $D + M$ constructions.

This paper aims to extend the notion of sdf-absorbing ideals by establishing the concept of square-difference factor absorbing submodules of an R-module M. We call a proper submodule N of an R-module M an square-difference factor absorbing submodule (briefly, sdf-absorbing submodule) of M if for $m \in M$ and $a, b \in R \backslash Ann_R(m)$, whenever $(a^2 - b^2)m \in N$, then $(a + b)m \in N$ or $(a - b)m \in N$. It is clear that this notion also generalizes the class of classical prime submodules. However, Example 3 shows that this generalization is proper.

Section 2 is started by introducing several properties and characterizations of sdf-absorbing submodules (see Propositions 1-6, Corollary 1). We determine conditions under which sdf-absorbing submodules are classical prime (see Theorem 1, Corollary 3). Moreover, we show that the property "sdf-absorbing" can be carried by a module homomorphism, a localization and a Cartesian product (see Propositions 8, 9, 11, Theorems 6-8, Corollaries 4, 7). We exactly determine the positive integers $n \in \mathbb{N}$ for which the zero submodule of the $\mathbb{Z}$-module $\mathbb{Z}_n$ is sdf-absorbing and then specify all sdf-absorbing submodules of $\mathbb{Z}_n$ (see Proposition 4 and Theorem 3). Furthermore, we present new characterizations for von-Neumann regular modules in terms of semiprime and sdf-absorbing submodules (see Theorems 4, 5). We investigate characterizations for the modules in which every nonzero proper submodule is sdf-absorbing (see Proposition 10, Corollaries 5, 6).

In section 3, we start by considering the idealization ring $R \ltimes M$ of an R-module M. We justify conditions on an ideal I of R and a submodule N of M such that $I \ltimes N$ is an sdf-absorbing ideal of $R \ltimes M$ if and only if N is an sdf-absorbing submodule of M (Proposition 12). Let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. We conclude section 3 by investigating some kinds of sdf-absorbing submodules in the amalgamation $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ of M_1 and M_2 along J with respect to φ (see Theorems 9 and 10).

2 Properties of Sdf-absorbing submodules

In this section, among other results concerning the general properties of sdf-absorbing submodules, some characterizations of this notion will be investigated. Moreover, the relations among sdf-absorbing submodule and some other types of submodules will be clarified.

Definition 1. *Let R be a ring and M be an R-module. A proper submodule N of M is called a square-difference factor absorbing submodule (sdf-absorbing submodule) of M if for $m \in M$ and $a, b \in R \backslash Ann_R(m)$, whenever $(a^2 - b^2)m \in N$, then $(a+b)m \in N$ or $(a-b)m \in N$.*

We note that N is an sdf-absorbing submodule of the R-module R if and only if it is an sdf-absorbing ideal of R. Indeed, suppose N is an sdf-absorbing ideal of R. Let $a, b, m \in R$ where $a, b \notin Ann_R(m)$ and $(a^2 - b^2)m \in N$. Then $am, bm \neq 0$ with $(am)^2 - (bm)^2 \in N$ and by assumption, either $(a+b)m \in N$ or $(a-b)m \in N$ as needed. The converse is clear.

An R-module M is said to be a reduced module if for any $m \in M$ and $a \in R$, $a^2 m = 0$ implies $am = 0$, [19]. We note that in our definition of sdf-absorbing submodules, the hypothesis "$a, b \notin Ann_R(m)$" is needed when $N = \{0\}$ since otherwise, N is not sdf-absorbing in any non-reduced module. Indeed, if M is a non-reduced R-module, then there exist $m \in M$ and $a \in R$ such that $a^2m = 0$ but $am \neq 0$. Then $(a^2 - 0^2)m = 0$ and $am \neq 0$ imply that $N = \{0\}$ is not sdf-absorbing.

Note that if R is a boolean ring, then $a^2 = a$ for all $a \in R$ and so every proper submodule of a module over R is sdf-absorbing. Also, clearly, the zero submodule is always an sdf-absorbing submodule in a torsion-free module.

We start by the following useful characterization of sdf-absorbing submodules.

Proposition 1. *The following statements are equivalent for a proper submodule N of an R-module M.*

1. N is an sdf-absorbing submodule of M.
2. For $a, b \in R$ and a cyclic submodule L of M with $L \nsubseteq Ann_M(a)$, $Ann_M(b)$, whenever $(a^2 - b^2)L \subseteq N$, then $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Suppose N is an sdf-absorbing submodule of M. Let $a, b \in R$ and $L = Rm$ be a cyclic submodule of M with $L \nsubseteq Ann_M(a)$, $Ann_M(b)$. Assume on contrary that $(a^2 - b^2)L \subseteq N$ but $(a+b)l_1 \notin N$ and $(a-b)l_2 \notin N$ for some nonzero $l_1, l_2 \in L$. If $l_1 = rm$ and $a \in Ann_R(l_1)$, then $l_1 \in Ann_M(a)$ and so $L = Rl_1 \subseteq Ann_M(a)$, a contradiction. Thus, $a \notin Ann_R(l_1)$. Similarly, $b \notin Ann_R(l_1)$ and $a, b \notin Ann_R(l_2)$. Since $(a^2 - b^2)l_1 \in N$ and $(a+b)l_1 \notin N$, we have $(a-b)l_1 \in N$. Similarly, since $(a^2 - b^2)l_2 \in N$ and $(a-b)l_2 \notin N$, we have $(a+b)l_2 \in N$. Now, $(a^2 - b^2)(l_1 + l_2) \in N$ and either $l_1 + l_2 = 0$ or $a, b \notin Ann_R(l_1 + l_2)$ which imply $(a+b)(l_1 + l_2) \in N$ or $(a-b)(l_1 + l_2) \in N$. Therefore, $(a+b)l_1 \in N$ or $(a-b)l_2 \in N$, a contradiction. Thus, $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$.

(2) $\Rightarrow$ (1) Suppose that $(a^2 - b^2)m \in N$ for $m \in M$ and $a, b \in R \backslash Ann_R(m)$. Take $L = Rm$ in (2) so that $(a^2 - b^2)L \subseteq N$. If $m = 0$, then we are done. If $m \neq 0$, then $a, b \notin Ann_R(m)$ implies $L \nsubseteq Ann_M(a)$, $Ann_M(b)$. By assumption, $(a+b)L \subseteq N$ or $(a-b)L \subseteq N$. Hence, $(a+b)m \in N$ or $(a-b)m \in N$ as needed. □

In the following proposition, we give two characterizations of nonzero sdf-absorbing submodules.

Proposition 2. *Let N be a proper nonzero submodule of an R-module M. The following statements are equivalent.*

1. N is an sdf-absorbing submodule of M.

2. For every $a, b \in R$, we have $(N :_M a^2 - b^2) = (N :_M a + b)$ or $(N :_M a^2 - b^2) = (N :_M a - b)$.

3. For $a, b \in R$ and a submodule L of M whenever $(a^2 - b^2)L \subseteq N$, then $(a + b)L \subseteq N$ or $(a - b)L \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Suppose N is an sdf-absorbing submodule of M and let $a, b \in R$. Firstly, we prove that $(N :_M a^2 - b^2) = (N :_M a + b) \cup (N :_M a - b)$. Clearly, $(N :_M a + b) \cup (N :_M a - b) \subseteq (N :_M a^2 - b^2)$. Conversely, let $m \in (N :_M a^2 - b^2)$ and suppose $m \notin (N :_R a + b)$. Then $(a^2 - b^2)m \in N$ but $(a+b)m \notin N$ and so by assumption, $(a-b)m \in N$. Thus, $m \in (N :_M a-b) \subseteq (N :_M a + b) \cup (N :_M a - b)$ and the required equality holds. It follows that $(N :_M a^2 - b^2) = (N :_M a + b)$ or $(N :_M a^2 - b^2) = (N :_M a - b)$.

(2) $\Rightarrow$ (3) Let $a, b \in R$ and L be a submodule of M such that $(a^2-b^2)L \subseteq N$ but $(a+b)L \nsubseteq N$. Then $L \subseteq (N :_M a^2 - b^2)\backslash(N :_M a+b)$ and by assumption, we have $L \subseteq (N :_M a - b)$. Therefore, $(a - b)L \subseteq N$ as needed.

(3) $\Rightarrow$ (1) Suppose that $(a^2 - b^2)m \in N$ for $a, b \in R$ and $m \in M$. The claim follows by taking $L = Rm$ in (3). $\square$

The following example shows that the condition " N be nonzero" in Proposition 2 is crucial.

Example 1. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_4$. Then $\{\bar{0}\}$ is an sdf-absorbing submodule of M by Theorem 3. However, choose $L = M$, $a = 4$ and $b = 2$. Then $(a^2 - b^2)L = \{\bar{0}\}$ but $(a + b)L \neq \{\bar{0}\}$ and $(a - b)L \neq \{\bar{0}\}$. Note that $L \subseteq Ann_M(a)$ so that Proposition 1 can not be used.*

Next, we give another characterization of sdf-absorbing submodules satisfying a certain condition.

Proposition 3. *Let N be a proper submodule of an R-module M.*

1. If N is sdf-absorbing in M, then $(N :_R K)$ is an sdf-absorbing ideal of R for every faithful cyclic submodule K of M not contained in N. In particular, if M is faithful and cyclic, then $(N :_R M)$ is an sdf-absorbing ideal of R.

2. Suppose N is nonzero. If N is sdf-absorbing in M, then $(N :_R K)$ is an sdf-absorbing ideal of R for every submodule K of M not contained in N. The converse is true if $(N :_R K) \neq 0$ for every submodule K of M not contained in N.

Proof. (1) Suppose that N is an sdf-absorbing submodule of M and let K be a faithful cyclic submodule of M not contained in N. Let $0 \neq a, b \in R$

such that $a^2 - b^2 \in (N :_R K)$. Then $(a^2 - b^2)K \subseteq N$. If $K \subseteq Ann_M(a)$ or $K \subseteq Ann_M(b)$, then K being faithful implies $a = 0$ or $b = 0$, a contradiction. Thus, $K \nsubseteq Ann_M(a)$, $Ann_M(b)$ and by Proposition 1, $(a + b)K \subseteq N$ or $(a - b)K \subseteq N$, Hence, $a + b \in (N :_R K)$ or $a - b \in (N :_R K)$ as needed. The "in particular" statement is clear.

(2) If N is nonzero sdf-absorbing in M, then $(N :_R K)$ is an sdf-absorbing ideal of R for every submodule K of M not contained in N by Proposition 2. Now, suppose that for every submodule K of M not contained in N, $(N :_R K) \neq 0$ is an sdf-absorbing ideal of R. Let $a, b \in R$ and L be a submodule of M such that $(a^2 - b^2)L \subseteq N$. If $L \subseteq N$, then we are done. If $L \nsubseteq N$, then by assumption, $(N :_R L) \neq 0$ is an sdf-absorbing ideal of R and so $(a + b) \in (N :_R L)$ or $(a - b) \in (N :_R L)$. Therefore, $(a + b)L \subseteq N$ or $(a - b)L \subseteq N$, as needed. □

Corollary 1. *Let N be a nonzero proper submodule of an R-module M.*

1. *If N is an sdf-absorbing submodule of M and A is an ideal of R, then either $(N :_M A) = M$ or $(N :_M A)$ is an sdf-absorbing submodule of M.*

2. *If N is an sdf-absorbing submodule of M, then $(N :_R M)$ is an sdf-absorbing ideal of R. The converse is true if R is a PIR and M is a multiplication module.*

Proof. (1) Let A be an ideal of R and suppose that $(N :_M A)$ is proper in M. Let $m \in M$ and $a, b \in R$ such that $(a^2 - b^2)m \in (N :_M A)$. Then $(a^2 - b^2)Am \subseteq N$ and so by Proposition 2, $(a + b)Am \subseteq N$ or $(a - b)Am \subseteq N$. Thus, $(a + b)m \in (N :_M A)$ or $(a - b)m \in (N :_M A)$ and we are done.

(2) If N is an sdf-absorbing submodule of M, then by Proposition 3(2), $(N :_R M)$ is an sdf-absorbing ideal of R. Now, suppose R is a PIR, M is a multiplication module and $(N :_R M)$ is an sdf-absorbing ideal of R. Note that $(N :_R M) \neq 0$ since $N \neq 0$ and M is multiplication. Let $a, b \in R$ and let $L = xRM$ be a submodule of M ($x \in R$) such that $(a^2 - b^2)L \subseteq N$. Then $(ax)^2 - (bx)^2 \in (a^2 - b^2)xR \subseteq (N :_R M)$ and by assumption, $(a + b)x \in (N :_R M)$ or $(a - b)x \in (N :_R M)$. Therefore, $(a + b)L \subseteq N$ or $(a - b)L \subseteq N$ and the result follows by Proposition 2. □

Next, for all $n \in \mathbb{N}$, we use Corollary 1 and [5, Example 2.8] to justify all nonzero sdf-absorbing submodules of the $\mathbb{Z}$-module $\mathbb{Z}_n$.

Proposition 4. *Let n be a positive integer. Then all nonzero sdf-absorbing submodules of the $\mathbb{Z}$-module $M = \mathbb{Z}_n$ are either prime submodules or of the form $N = \overline{2q}\mathbb{Z}_n$ for some odd prime integer q dividing n.*

Proof. Clearly a prime submodule is sdf-absorbing. Also, for a prime integer q dividing n, $N = \overline{2q}\mathbb{Z}_n$ is sdf-absorbing in M by Corollary 1(2) as $(N :_{\mathbb{Z}} M) = 2q\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$, [5, Example 2.8]. Let N be a nonzero sdf-absorbing submodule of M. Then $(N :_{\mathbb{Z}} M)$ is a nonzero sdf-absorbing ideal of $\mathbb{Z}$ by Corollary 1(2). Thus, $(N :_{\mathbb{Z}} M)$ is a prime ideal or $(N :_{\mathbb{Z}} M) = 2q\mathbb{Z}$ for some odd prime integer q. It follows that N is a prime submodule of M or $N = (N :_{\mathbb{Z}} M)M = \overline{2q}\mathbb{Z}_n$. □

The converse of Corollary 1(1) is not true in general. Moreover, if M is not a multiplication R-module, then the converse of Corollary 1(2) need not be true either.

Example 2.

1. Consider the submodule $N = \langle 6\rangle \times \langle \bar{4}\rangle$ of the non multiplication $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_8$. Then $(N :_{\mathbb{Z}} M) = 6\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ by [5, Example 2.8] but N is not an sdf-absorbing submodule of M. Indeed, $(5^2 - 1^2)(1, \bar{1}) \in N$ but $6.(1, \bar{1}) \notin N$ and $4.(1, \bar{1}) \notin N$.

2. Consider the submodule $N = \langle \bar{4}\rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ and the ideal $A = 2\mathbb{Z}$ of $\mathbb{Z}$. Then clearly $(N :_M A) = \langle \bar{2}\rangle$ is an sdf-absorbing submodule of M. On the other hand N is not sdf-absorbing by Proposition 4.

Let M be an R-module. If for every submodule N of M, there exists an element $r \in R$ such that $N = rM$, then we say that M is a principal ideal multiplication module, [7]. We note clearly that if $(N :_R M) \neq 0$, the converse of Corollary 1(2) is true in the particular case that M is a principal ideal multiplication module. For multiplication modules over principal ideal rings, we have the following characterization for nonzero sdf-absorbing submodules:

Corollary 2. *Let R be a PIR and N be a nonzero proper submodule of a multiplication R-module M. Then N is an sdf-absorbing submodule of M if and only if $(N :_R M)$ is an sdf-absorbing ideal of R.*

Proof. Suppose $(N :_R M)$ is an sdf-absorbing ideal of R. If $(N :_R M) \neq 0$, then the result follows by (2) of Corollary 1. If $(N :_R M) = 0$, then as M is multiplication, $N = (N :_R M)M = 0$ which is a contradiction. The converse is clear. □

It is clear that any classical prime submodule is sdf-absorbing. However, the converse need not be true in general.

Example 3. *Consider the multiplication $\mathbb{Z}$-modules $M_1 = \mathbb{Z}_{24}$ and $M_2 = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $N_1 = \langle \bar{6} \rangle$ and $N_2 = 0_{M_2}$ are sdf-absorbing submodules of M_1 and M_2, respectively since $(N_1 :_{\mathbb{Z}} M_1) = (N_2 :_{\mathbb{Z}} M_2) = 6\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ by [5, Example 2.8]. On the other hand, N_1 and N_2 are not classical prime in M_1 and M_2, respectively. For example, $2.3.\bar{1} \in N_1$ but $2.\bar{1} \notin N_1$, $3.\bar{1} \notin N$. Also, $2.3.(\bar{1}, \bar{1}) = 0_{M_2}$ but $2.(\bar{1}, \bar{1}) \neq 0_{M_2}$ and $3.(1, 1) \neq 0_{M_2}$.*

In [5, Theorem 2.6], it is shown that if I is a nonzero sdf-absorbing ideal of a ring R and $2 \in U(R)$, then I is a prime ideal of R. As an extention of this relationship to modules, the next result gives a case where sdf-absorbing submodules and classical prime submodules coincide.

Theorem 1. *Let N be a nonzero sdf-absorbing submodule of an R-module M. If $2 \in U(R)$, then N is classical prime.*

Proof. Let $a, b \in R$ and $m \in M$ such that $abm \in N$. Choose $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$. Then $x, y \in R$ and $(x^2 - y^2)m = abm \in N$. By assumption, $am = (x + y)m \in N$ or $bm = (x - y)m \in N$ and so N is classical prime in M. □

In the following corollary, we conclude another condition on sdf-absorbing submodules to be classical prime.

Corollary 3. *Let N be a maximal sdf-absorbing submodule of an R-module M with respect to inclusion. Then N is a classical prime submodule of M.*

Proof. If $N = 0$, then N is a maximal submodule of M and so it is classical prime. Suppose N is nonzero. Let $a, b \in R$ and $m \in M$ such that $abm \in N$ and $am \notin N$. Then $m \notin (N :_M a)$ and so $(N :_M a)$ is proper in M. Moreover, $(N :_M a)$ is an sdf-absorbing submodule of M by Corollary 1(1). Since $N \subseteq (N :_M a)$ and by the maximality of N, we conclude that $N = (N :_M a)$. Thus, $bm \in (N :_M a) = N$ and N is a classical prime submodule of M. □

Recall from [22] that a proper submodule N of an R-module M is called semiprime if for $a \in R$ and $m \in M$, we have $a^2m \in N$ implies $am \in N$. Equivalently, N is semiprime in M if and only if for each $a \in R$ and each submodule K of M, $a^2K \subseteq N$ implies that $aK \subseteq N$. More general, for $k, n \in \mathbb{N}$, N is called (k, n)-semiprime if $a^k m \in N$ implies $a^n m \in N$ for $a \in R$ and $m \in M$, [21].

Proposition 5. *Let N be a proper submodule of an R-module M.*

1. If N is a nonzero sdf-absorbing submodule of M, then N is semiprime in M.

2. If $char(R) = 2$ and N is semiprime in M, then N is an sdf-absorbing submodule of M.

Proof. (1) Suppose N is an sdf-absorbing submodule of M and let $a \in R$ and $m \in M$ such that $a^2m \in N$. Then $(a^2 - 0^2)m \in N$ and by assumption, $am \in N$ as needed.

(2) Suppose $char(R) = 2$ and N is semiprime in M. Let $m \in M$ and $a, b \in R\backslash Ann_R(m)$ such that $(a^2 - b^2)m \in N$. Then $char(R) = 2$ implies that $(a+b)^2m \in N$ and by assumption, $(a+b)m \in N$. Thus, N is an sdf-absorbing submodule of M. □

Example 4. *If $char(R) \neq 2$, then the conclusion of Proposition 5(2) need not be true. For example, $N = \{\bar{0}\}$ is clearly a semiprime submodule of the $\mathbb{Z}$-module $M = \mathbb{Z}_{15}$ but N is not sdf-absorbing in $\mathbb{Z}_{15}$ by Proposition 3.*

Analogous to [5, Theorem 2.5] on sdf-absorbing ideal, we have the following characterization of sdf-absorbing submodules N of M such that $2 \in (N :_R M)$.

Theorem 2. *Let N be a proper sdf-absorbing submodule of an R-module M. The following statements are equivalent.*

1. For $m \in M$ and $a, b \in R\backslash Ann_R(m)$, whenever $(a^2 - b^2)m \in N$, then $(a + b)m \in N$ and $(a - b)m \in N$.
2. $2 \in (N :_R M)$.
3. $char(R/(N :_R M)) = 2$.

Proof. (1) ⇔ (2) Suppose (1) holds and let $m \in M$. Choose $a = b = 1$ so that $(a^2 - b^2)m \in N$. If $m = 0$, then $2m \in N$. If $m \neq 0$, then $a, b \in R\backslash Ann_R(m)$ and by assumption, $2m = (a + b)m \in N$. It follows that $2 \in (N :_R M)$ as needed. Conversely, suppose $2 \in (N :_R M)$. Let $m \in M$ and $a, b \in R\backslash Ann_R(m)$ such that $(a^2 - b^2)m \in N$. Then $(a + b)m \in N$ or $(a - b)m \in N$ as N is an sdf-absorbing submodule of M. Suppose $(a + b)m \in N$. Since $2 \in (N :_R M)$, then $2bm \in N$ and so $(a - b)m = (a + b)m - 2bm \in N$. Similarly, if $(a - b)m \in N$, then $(a + b)m \in N$ and the result follows.

(2) ⇔ (3) Clear. □

For the next result, we need to recall the following lemma.

Lemma 1. *[1] For an ideal I of a ring R and a submodule N of a finitely generated faithful multiplication R-module M, the following hold.*

1. *$(IN :_R M) = I(N :_R M)$.*

2. *If I is finitely generated faithful multiplication, then*
   (a) $(IN :_M I) = N$.
   (b) *Whenever* $N \subseteq IM$, *then* $(JN :_M I) = J(N :_M I)$ *for any ideal* J *of* R.

Proposition 6. *Let I be a finitely generated faithful multiplication ideal of a ring R and N a submodule of a finitely generated faithful multiplication R-module M.*

1. Suppose M is cyclic and $Ann_M(I) = 0$. If IN is an sdf-absorbing submodule of M, then either I is an sdf-absorbing ideal of R or N is an sdf-absorbing submodule of M.
2. Suppose N is nonzero. Then N is an sdf-absorbing submodule of IM if and only if $(N :_M I)$ is an sdf-absorbing submodule of M.

Proof. (1) Suppose IN is an sdf-absorbing submodule of M. If $N = M$, then $I = I(N :_R M) = (IN :_R M)$ is an sdf-absorbing ideal of R by Proposition 3(1). Suppose that N is proper in M. Let $m \in M$ and $a, b \in R\backslash Ann_R(m)$ such that $(a^2 - b^2)m \in N$. Then $(a^2 - b^2)Im \subseteq IN$ and clearly $Im \nsubseteq Ann_M(a), Ann_M(b)$ since $Ann_M(I) = 0$. By Proposition 1, $(a + b)Im \subseteq IN$ or $(a - b)Im \subseteq IN$ and so by Lemma 1, $(a + b)m \in (IN :_M I) = N$ or $(a - b)m \in (IN :_M I) = N$ and we are done.

(2) Suppose N is a nonzero sdf-absorbing submodule of IM. Then $(N :_M I)$ is proper in M since otherwise $IM = N$, a contradiction. Moreover, $(N :_M I) \neq 0$ since otherwise by Lemma 1, $N = (IN :_M I) \subseteq (N :_M I) = 0$, a contradiction. Let $m \in M$ and $a, b \in R$ such that $(a^2 - b^2)m \in (N :_M I)$. Then Im is a submodule of IM and $(a^2 - b^2)Im \subseteq N$. By Proposition 2, $(a + b)Im \subseteq N$ or $(a - b)Im \subseteq N$ and so $(a + b)m \in (N :_M I)$ or $(a - b)m \in (N :_M I)$ as required. Conversely, suppose $(N :_M I)$ is an sdf-absorbing submodule of M. Let $a, b \in R$ and $m' \in IM$ such that $(a^2 - b^2)m' \in N$. Then by Lemma 1,

$$(a^2 - b^2)(\langle m' \rangle :_M I) = (\left\langle (a^2 - b^2)m' \right\rangle :_M I) \subseteq (N :_M I).$$

By Proposition 2, $(a + b)(\langle m' \rangle :_M I) \subseteq (N :_M I)$ or $(a - b)(\langle m' \rangle :_M I) \subseteq (N :_M I)$. Therefore, again Lemma 1 implies

$$(a+b)m' \in (I \left\langle (a + b)m' \right\rangle :_M I) = I(\left\langle (a + b)m' \right\rangle :_M I) \subseteq I(N :_M I) = (IN :_M I) = N$$

or

$$(a-b)m' \in (I \left\langle (a - b)m' \right\rangle :_M I) = I(\left\langle (a - b)m' \right\rangle :_M I) \subseteq I(N :_M I) = (IN :_M I) = N$$

and N is an sdf-absorbing submodule of IM. □

In general, if I is an sdf-absorbing ideal of R and N is an sdf-absorbing submodule of an R-module M, then IN need not be sdf-absorbing in M. For example, $I = 6\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ and by Proposition 4, $N = \langle \bar{6} \rangle$ is an sdf-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. But, $IN = \langle \bar{0} \rangle$ is not sdf-absorbing in $\mathbb{Z}_{12}$ by Theorem 3.

Proposition 7. *Let M be a finitely generated faithful principal ideal multiplication R-module and I be an ideal of R. If I is an sdf-absorbing ideal of R, then IM is an sdf-absorbing submodule of M. The converse is true if M is cyclic.*

Proof. Suppose I is an sdf-absorbing ideal of R. If $IM = M$, then Lemma 1 implies $I = (IM :_R M) = R$, a contradiction. Let $a, b \in R$ and $L = rM$ be a submodule of M such that $L \nsubseteq Ann_M(a)$, $Ann_M(b)$ and $(a^2 - b^2)L \subseteq IM$. Then $(a^2 - b^2)r \in (IM :_R M) = I$ and so $(ar)^2 - (br)^2 \in I$. Since clearly, $ar, br \neq 0$, then by assumption, $(a + b)r \in I$ or $(a - b)r \in I$. Thus, $(a + b)L \subseteq IM$ or $(a - b)L \subseteq IM$ as needed. Now, suppose M is cyclic and IM is an sdf-absorbing submodule of M. As $I = (IM :_R M)$, then clearly I is proper. Now, let $0 \neq a, b \in R$ such that $(a^2 - b^2) \in I$. Then $(a^2 - b^2)M \subseteq IM$ with $M \nsubseteq Ann_M(a)$, $Ann_M(b)$ as M is faithful. By Proposition 1, $(a + b)M \subseteq IM$ or $(a - b)M \subseteq IM$. It follows that $(a + b) \in (IM :_R M) = I$ or $(a - b) \in (IM :_R M) = I$ and I is an sdf-absorbing ideal of R. □

Next, we verify that "sdf-absorbing" property in modules can carried with a module homomorphism.

Proposition 8. *Let M_1 and M_2 be R-modules and $\varphi : M_1 \to M_2$ be a module homomorphism.*

1. If φ is one to one and N_2 is an sdf-absorbing submodule of M_2 with $\varphi^{-1}(N_2) \neq M_1$, then $\varphi^{-1}(N_2)$ is an sdf-absorbing submodule of M_1.

2. If N_2 is a nonzero sdf-absorbing submodule of M_2 with $\varphi^{-1}(N_2) \neq M_1$, then $\varphi^{-1}(N_2)$ is an sdf-absorbing submodule of M_1.

3. If φ is onto and N_1 is an sdf-absorbing submodule of M_1 containing $Ker(\varphi)$, then $\varphi(N_1)$ is an sdf-absorbing submodule of M_2.

Proof. (1) Let $m_1 \in M_1$ and $a, b \in R \backslash Ann_R(m_1)$ such that $(a^2 - b^2)m_1 \in \varphi^{-1}(N_2)$. Then $a, b \in R \backslash Ann_R(\varphi(m_1))$ as φ is one to one and $(a^2 - b^2)\varphi(m_1) = \varphi((a^2 - b^2)m_1) \in N_2$. By assumption, $(a + b)\varphi(m_1) = \varphi((a + b)m_1) \in N_2$ or $(a - b)\varphi(m_1) = \varphi((a - b)m_1) \in N_2$. Thus, $(a + b)m_1 \in \varphi^{-1}(N_2)$ or $(a - b)m_1 \in \varphi^{-1}(N_2)$ as needed.

(2) Similar to (1).

(3) Suppose φ is onto. Let $m_2 \in M_2$ and $a, b \in R \backslash Ann_R(m_2)$ such that $(a^2 - b^2)m_2 \in \varphi(N_1)$. Choose $m_1 \in M_1$ such that $m_2 = \varphi(m_1)$. Since $Ker(\varphi) \subseteq N_1$, we have clearly, $(a^2 - b^2)m_1 \in N_1$ and $a, b \in R \backslash Ann_R(m_1)$. Therefore, either $(a + b)m_1 \in N_1$ or $(a - b)m_1 \in N_1$. Therefore, $(a + b)m_2 \in \varphi(N_1)$ or $(a - b)m_2 \in \varphi(N_1)$ and $\varphi(N_1)$ is an sdf-absorbing submodule of M_2. $\square$

In the following example, we show that the hypothesis "$Ker(\varphi) \subseteq N_1$" is needed in Proposition 8(3).

Example 5. *Consider the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}[x]$ and the module epimorphism $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\varphi(f(x)) = f(0)$. Then the submodule $N = \langle x + 18\rangle$ of $\mathbb{Z}[x]$ is clearly an sdf-absorbing submodule of $\mathbb{Z}[x]$. But, $\varphi(N) = 18\mathbb{Z}$ is not sdf-absorbing in $\mathbb{Z}$ by [5, Example 2.8]. Note that $Ker(\varphi) = \langle x\rangle \nsubseteq N$.*

As a direct consequence of Proposition 8, we have the following.

Corollary 4. *Let R be a ring and M_1, M_2 be nonzero R-modules.*

1. If $M_1 \subseteq M_2$ and N is a nonzero sdf-absorbing submodule of M_2, then $N \cap M_1$ is an sdf-absorbing submodule of M_1.
2. Let $K \subseteq N$ be submodules of M_1. Then N is an sdf-absorbing submodule of M_1 if and only if N/K is an sdf-absorbing submodule of M_1/K.

In the following theorem which is an analogy of [5, Theorem 4.11] on sdf-ideal, we completely determine when $\{0\}$ is an sdf-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}_n$ for each $n \in \mathbb{N}$.

Theorem 3. *Let $n \in \mathbb{N}$. Then $\{0\}$ is an sdf-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}_n$ if and only if $n = 4$, $n = 9$, $n = p$ is prime, or $n = 2p$ for some odd prime p.*

Proof. If $n = p$ is prime, then $\{0\}$ is an sdf-absorbing submodule of $\mathbb{Z}_n$ since it is maximal in $\mathbb{Z}_n$. Suppose $n = 2p$ for some odd prime p. Since $2p\mathbb{Z}$ is an sdf-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}$, then by Corollary 4, $\{0\} = 2p\mathbb{Z}/2p\mathbb{Z}$ is an sdf-absorbing submodule of $\mathbb{Z}/2p\mathbb{Z} \cong \mathbb{Z}_{2p}$. Now, let $n = 9$. Let $m \in \mathbb{Z}_9$ and $a, b \in \mathbb{Z} \backslash Ann_{\mathbb{Z}}(m)$ such that $(a^2 - b^2)m = 0$. Then $a^2 - b^2 \in 9\mathbb{Z}$ with $a, b \notin 9\mathbb{Z}$. Suppose $a + b \notin 9\mathbb{Z}$ and $a - b \notin 9\mathbb{Z}$. Then $a + b \in 3\mathbb{Z}$ and $a - b \in 3\mathbb{Z}$ as $9\mathbb{Z}$ is primary in $\mathbb{Z}$. Write $a + b = 3k$ where $k \equiv 1 (\bmod 3)$ or $k \equiv 2 (\bmod 3)$ and $a - b = 3l$ where $l \equiv 1 (\bmod 3)$ or $l \equiv 2 (\bmod 3)$. If $k \equiv 1 (\bmod 3)$ and $l \equiv 1 (\bmod 3)$, then $3 \mid (k - l)$ and so $b = 3(\frac{k-l}{2}) \in 9\mathbb{Z}$, a contradiction. If $k \equiv 1 (\bmod 3)$ and $l \equiv 2 (\bmod 3)$, then $3 \mid (k + l)$ and so $a = 3(\frac{k+l}{2}) \in 9\mathbb{Z}$, a contradiction. The remaining two cases give similar contradictions. Therefore,

$a+b \in 9\mathbb{Z}$ or $a-b \in 9\mathbb{Z}$. Hence, $(a+b)m = 0$ or $(a-b)m = 0$ and we are done. By using a similar argument, one can prove that $\{0\}$ is an sdf-absorbing submodule of $\mathbb{Z}_4$.

Conversely, let $n \in \mathbb{N}$ such that $n \neq 4$, $n \neq 9$, $n \neq p$ (prime) and $n \neq 2p$ for odd prime p. Suppose on contrary that $\{0\}$ is an sdf-absorbing submodule of $\mathbb{Z}_n$. Then we have four cases.

Case 1: Let $n = kl$ for some distinct odd integers $k, l \neq 1$.

Let $a = \frac{k+l}{2}$ and $b = \frac{k-l}{2}$. Then $a, b \notin Ann_{\mathbb{Z}}(\bar{1})$ and $(a^2 - b^2).\bar{1} = kl.\bar{1} = \bar{0}$ but $k.\bar{1} \neq \bar{0}$ and $l.\bar{1} \neq \bar{0}$, a contradiction.

Case 2: $n = p^k$ for a prime p and $k \geq 3$ and $p \neq 2, 3$ if $k = 2$.

Let $a = \frac{p^{k-1}+3p}{2}$ and $b = \frac{p^{k-1}-3p}{2}$. Then $a, b \notin Ann_{\mathbb{Z}}(\bar{1})$ and $(a^2 - b^2).\bar{1} = p^{k-1}.3p.\bar{1} = \bar{0}$ but $p^{k-1}.\bar{1} \neq \bar{0}$ and $3p.\bar{1} \neq \bar{0}$, a contradiction.

Case 3: $n = 2kl$ where k, l are distinct integers such that $kl \neq 1, 2$.

Let $a = k + l$ and $b = k - l$. Then $a, b \notin Ann_{\mathbb{Z}}(\bar{1})$ and $(a^2 - b^2).\bar{1} = 2k.2l.\bar{1} = \bar{0}$ but $2k.\bar{1} \neq \bar{0}$ and $2l.\bar{1} \neq \bar{0}$, a contradiction.

Case 4: $n = 2k^n$ where $k \neq 1$ for each n and $k \neq 2$ for $n = 1$.

Let $a = k^{n-1} + k$ and $b = k^{n-1} - k$. Then $a, b \notin Ann_{\mathbb{Z}}(\bar{1})$ and $(a^2 - b^2).\bar{1} = 2k^{n-1}.2k = \bar{0}$ but $2k^{n-1}.\bar{1} \neq \bar{0}$ and $2k.\bar{1} \neq \bar{0}$, a contradiction.

Therefore, $\{0\}$ is not an sdf-absorbing submodule of $\mathbb{Z}_n$. □

Let N be a submodule of an R-module M. By $Z_N(M)$, we denote the set $\{r \in R : rm \in N \text{ for some } m \in M \backslash N\}$.

Proposition 9. *Let N be a proper submodule of an R-module M and S be a multiplicatively closed subset of R such that $(N :_R M) \cap S = \emptyset$. Then*

1. If N is an sdf-absorbing submodule of M, then $S^{-1}N$ is an sdf-absorbing submodule of $S^{-1}M$.
2. If $S^{-1}N$ is an sdf-absorbing submodule of $S^{-1}M$ and $Z_N(M) \cap S = Z(M) \cap S = \emptyset$, then N is an sdf-absorbing submodule of M.

Proof. (1) Let $\frac{m}{u} \in S^{-1}M$ and $\frac{a}{s}, \frac{b}{t} \in S^{-1}R \backslash Ann_{S^{-1}R}(\frac{m}{u})$ such that $((\frac{a}{s})^2 - (\frac{b}{t})^2)\frac{m}{u} \in S^{-1}N$. Then there exists $v \in S$ such that $v(t^2a^2 - s^2b^2)m \in N$. If $vta \in Ann_R(m)$, then $\frac{a}{s}\frac{m}{u} = \frac{vtam}{vtsu} = \frac{0}{1}$, a contradiction. Thus, $vta \notin Ann_R(m)$ and similarly $vsb \notin Ann_R(m)$. Since N is sdf-absorbing, then either $(vta+vsb)m \in N$ or $(vta-vsb)m \in N$. Thus, $(\frac{a}{s}+\frac{b}{t})\frac{m}{u} \in S^{-1}N$ or $(\frac{a}{s}-\frac{b}{t})\frac{m}{u} \in S^{-1}N$ and $S^{-1}N$ is sdf-absorbing in $S^{-1}M$.

(2) Let $m \in M$ and $a, b \in R \backslash Ann_R(m)$ such that $(a^2 - b^2)m \in N$. Then $((\frac{a}{1})^2 - (\frac{b}{1})^2)\frac{m}{1} \in S^{-1}N$. If $\frac{a}{1} \in Ann_{S^{-1}R}(\frac{m}{1})$, then there exists $t \in S$ such that $tam = 0$ and $Z(M) \cap S = \emptyset$ implies $am = 0$, a contradiction. Thus, $\frac{a}{1} \notin Ann_{S^{-1}R}(\frac{m}{1})$ and similarly $\frac{b}{1} \notin Ann_{S^{-1}R}(\frac{m}{1})$. By assumption, either $(\frac{a}{1} + \frac{b}{1})\frac{m}{1} \in S^{-1}N$ or $(\frac{a}{1} - \frac{b}{1})\frac{m}{1} \in S^{-1}N$. Thus, there exist $u, v \in S$ such that

$u(a+b)m \in N$ or $v(a-b)m \in N$. Since $Z_N(M) \cap S = \emptyset$, then we have either $(a+b)m \in N$ or $(a-b)m \in N$ and N is an sdf-absorbing submodule of M. $\square$

Recently, Jayaram and Tekir [14] extended the concept of von Neumann regular ring to modules by defining M-von Neumann regular elements of a module. Let M be an R-module. Then an element $a \in R$ is called an M-von Neumann regular if $aM = a^2M$. An R-module M is said to be a von Neumann regular module if for each $m \in M$, $Rm = aM = a^2M$ for some $a \in R$. In particular, if M is finitely generated, then M is a multiplication von Neumann regular module if and only if every element in R is M-von Neumann regular, [14, Theorem 2]. In [21], the finitely generated von Neumann regular modules are characterized as multiplication reduced modules in which every proper submodule is (k,n)-semiprime where $k, n \in \mathbb{N}$.

However, in semiprime submodules, we can drop the "multiplication reduced" condition in the above characterization analogous to [5, Theorem 3.1]:

Theorem 4. *Let M be a finitely generated R-module. The following statements are equivalent.*

1. M is a von Neumann regular module.
2. Every proper submodule of M is semiprime.

Proof. (1) $\Rightarrow$ (2) Since M is a finitely generated von Neumann regular module, then clearly M is a multiplication module. Let N be a proper submodule of M. Let $a \in R$ and K be a submodule of M such that $a^2K \subseteq N$. By assumption, $aM = a^2M$ and so $aK = a(K :_R M)M = a^2(K :_R M)M = a^2K \subseteq N$, as needed.

(2) $\Rightarrow$ (1) Let $a \in R$. If $a^2M = M$, then $aM = a^2M$. Suppose $a^2M \subsetneq M$. Then by assumption, a^2M is semiprime in M. Since $a^2M \subseteq a^2M$, then $aM \subseteq a^2M$ so that $aM = a^2M$ and M is a von Neumann regular module by [14, Theorem 1]. $\square$

Recall that an element m in an R-module M is nilpotent if $r^k m = 0$ for some $r \in R$ and $k \in \mathbb{N}$. The set of all nilpotent elements of M is denoted by $N(M)$. Note that M is reduced if $N(M) = 0$ and moreover, $M/N(M)$ is reduced for any R-module M.

Analogous to [5, Theorem 3.1] on sdf-absorbing ideal, we have the following result.

Theorem 5. *Let M be a finitely generated R-module.*

1. If every nonzero proper submodule of M is sdf-absorbing, then $M/N(M)$ is a von Neumann regular module.

2. If $char(R) = 2$ and M is a von Neumann regular module, then every proper submodule of M is sdf-absorbing.

Proof. (1) If every nonzero proper submodule of M is sdf-absorbing, then by Proposition 5, every nonzero proper submodule of M is semiprime. Thus, $M/N(M)$ is a von Neumann regular module by Theorem 4.

(2) If $char(R) = 2$, then every semiprime submodule of M is sdf-absorbing by Proposition 5. In this case, the claim follows again by Theorem 4. □

Note that the conclusion of Theorem 5(2) is not true in general. For example, the $\mathbb{Z}$-module $M = \mathbb{Z}_{15}$ is von Neumann regular by [14, Exanple 5]. But the submodule $N = \langle \bar{0} \rangle$ is not sdf-absorbing in M by Theorem 3.

The proof of the following lemma can be seen in [4, Corrolary 3]

Lemma 2. *[4]Let R be a quasi-local ring and M an R-module. Then M is von Neumann regular if and only if M is trivial or simple.*

Therefore, we have the following result which is an analogy of [5, Theorem 3.3] on sdf-ideal.

Proposition 10. *Let R be a quasi-local ring and M be a finitely generated R-module. If every proper submodule of M is sdf-absorbing, then $N(M) = M$ or $N(M)$ is a maximal submodule of M.*

For the case of M being a faithful multiplication R-module and R satisfying certain conditions, we have the following corollaries.

Corollary 5. *Let R be a reduced ring with $2 \in U(R)$ and M be a faithful principal ideal multiplication R-module. The following are equivalent.*

1. Every nonzero proper submodule of M is sdf-absorbing.
2. Every nonzero proper ideal of R is an sdf-absorbing ideal of R.
3. R is a field or R is isomorphic to $F_1 \times F_2$ for fields F_1, F_2.

Proof. (1) $\Leftrightarrow$ (2): Follows directly by Proposition 7.

(2) $\Leftrightarrow$ (3): [5, Theorem 3.5]. □

Corollary 6. *Let R be a von Neumann regular ring with $0 \neq 2 \in Z(R)$ and M be a faithful principal ideal multiplication R-module. The following are equivalent.*

1. Every nonzero proper submodule of M is sdf-absorbing.
2. Every nonzero proper ideal of R is sdf-absorbing.

3. Exactly one maximal ideal M of R has $char(R/M) \neq 2$.

Proof. (1) $\Leftrightarrow$ (2): Proposition 7.
(2) $\Leftrightarrow$ (3): [5, Theorem 3.7]. □

As we conclude this section, we turn our attention to examining the relationship between sdf-absorbing submodules within modules and those within the Cartesian products of these modules.

Proposition 11. *Let N_1 and N_2 be nonzero proper submodules of R-modules M_1 and M_2, respectively. Consider the R-module $M_1 \times M_2$.*

1. $N_1 \times M_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if N_1 is an sdf-absorbing submodule of M_1.
2. $M_1 \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if N_2 is an sdf-absorbing submodule of M_2.
3. If $N_1 \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$, then N_1 and N_2 are sdf-absorbing in M_1 and M_2, respectively. The converse holds provided that $2 \in (N_1 :_R M_1)$ or $2 \in (N_2 :_R M_2)$.

Proof. (1) and (2) are clear by using Corollary 4(2).
(3) Suppose $N_1 \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$. Let $a, b \in R$, $m \in M$ such that $(a^2 - b^2)m \in N_1$. Then $(a^2 - b^2)(m, 0) \in N_1 \times N_2$ implies either $(a+b)(m, 0) \in N_1 \times N_2$ or $(a-b)(m, 0) \in N_1 \times N_2$. Thus, $(a+b)m \in N_1$ or $(a - b)m \in N_1$ and N_1 is an sdf-absorbing submodule of M_1. Similarly, N_2 is an sdf-absorbing submodule of M_2. Conversely, assume $2 \in (N_1 :_R M_1)$. Let $a, b \in R$, $(m_1, m_2) \in M_1 \times M_2$ such that $(a^2 - b^2)(m_1, m_2) \in N_1 \times N_2$. Since N_1 is sdf-absorbing, $(a^2 - b^2)m_1 \in N_1$ and $2 \in (N_1 :_R M_1)$, we have $(a + b)m_1 \in N_1$ and $(a - b)m_1 \in N_1$ by Theorem 2. On the other hand, since N_2 is sdf-absorbing in M_2, we get either $(a + b)m_2 \in N_2$ or $(a - b)m_2 \in N_2$. Thus, $(a+b)(m_1, m_2) \in N_1 \times N_2$ or $(a-b)(m_1, m_2) \in N_1 \times N_2$ and $N_1 \times N_2$ is sdf-absorbing in $M_1 \times M_2$. The proof of the case $2 \in (N_2 :_R M_2)$ is similar. □

The condition "$2 \in (N_1 :_{R_1} M_1)$ or $2 \in (N_2 :_{R_2} M_2)$" in Proposition 11(3) is crucial. For example, consider the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$ and the submodules $N_1 = 3\mathbb{Z}$ and $N_2 = 7\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$. Then N_1 and N_2 are sdf-absorbing submodules of $\mathbb{Z}$ but $N_1 \times N_2$ is not sdf-absorbing in $\mathbb{Z} \times \mathbb{Z}$ as $(5^2 - 2^2)(1, 1) \in N_1 \times N_2$, $7(1, 1) \notin N_1 \times N_2$ and $3(1, 1) \notin N_1 \times N_2$. The following theorem is an analogy of [5, Theorem 4.12] on sdf-absorbing ideal.

Theorem 6. *Let M_1 be an R_1-module, M_2 an R_2-module and N_1, N_2 be nonzero submodules of M_1, M_2, respectively. Then $N_1 \times N_2$ is an sdf-absorbing submodule of the $R_1 \times R_2$-module $M_1 \times M_2$ if and only if one of the following holds.*

1. $N_1 = M_1$ and N_2 is an sdf-absorbing submodule of M_2.
2. $N_2 = M_2$ and N_1 is an sdf-absorbing submodule of M_1.
3. N_1, N_2 are sdf-absorbing submodules of M_1, M_2, respectively and $2 \in (N_1 :_{R_1} M_1)$ or $2 \in (N_2 :_{R_2} M_2)$.

Proof. Suppose $N_1 \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ and N_1, N_2 are proper. Then N_1 and N_2 are sdf-absorbing submodules of M_1 and M_2, respectively by Corollary 4(2). Now, suppose $2 \notin (N_2 :_{R_2} M_2)$. Choose $m_2 \in M_2$ such that $2m_2 \notin N_2$ and let $0_{M_1} \neq m_1 \in M_1$. Since $((1,1)^2 - (1,-1)^2)(m_1, m_2) \in N_1 \times N_2$ and $(0,2)(m_1, m_2) \notin N_1 \times N_2$, then by assumption, $(2,0)(m_1, m_2) \in N_1 \times N_2$ and so $2m_1 \in N$. Thus, $2 \in (N_1 :_{R_1} M_1)$ as required. The converse part can be proved by using the same argument of the proof of Proposition 11. □

In general, we can use the mathematical induction to characterize sdf-absorbing submodules of Cartesian product of finitely many R_i-modules.

Corollary 7. *Let $N_1, N_2, \cdots, N_k$ be nonzero submodules of R_i-modules $M_1, M_2, \cdots, M_k$ where $k \geq 2$. Let $R = \times_{i=1}^{k} R_i$, $M = \times_{i=1}^{k} M_i$ and $N = \times_{i=1}^{k} N_i$. Suppose that $N_1, ..., N_t$ are proper for some $1 \leq t \leq k$ and $N_j = M_j$ for all $t < j \leq k$. Then N is an sdf-absorbing submodule of M if and only if N_i 's are sdf-absorbing submodules of M_i for all $i \in \{1, ..., t\}$ and at most for one of $i \in \{1, ..., t\}$, $2 \notin (N_i :_{R_i} M_i)$.*

Now, we consider the case when one or both of the submodules in the direct product is zero analogous to [5, Theorem 4.13].

Theorem 7. *Let M_1 be an R_1-module, M_2 an R_2-module and N_1, N_2 be nonzero submodules of M_1, M_2, respectively.*

1. $\{0\} \times M_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if $\{0\}$ is an sdf-absorbing submodule of M_1 and M_1 is reduced. A similar result holds for $M_1 \times \{0\}$.
2. $\{0\} \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if $\{0\}$ is an sdf-absorbing submodule of M_1, N_2 is an sdf-absorbing submodule of M_2, M_1 is reduced and $2 \in (0 :_{R_1} M_1)$ or $2 \in (N_2 :_{R_2} M_2)$. A similar result holds for $N_1 \times \{0\}$.
3. $\{0\} \times \{0\}$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if M_1, M_2 are reduced, $\{0\}$ is an sdf-absorbing submodule of M_1, M_2 and $2 \in (0 :_{R_1} M_1)$ or $2 \in (0 :_{R_2} M_2)$.

Proof. (1) Note that $\{0\} \times M_2$ is a nonzero submodule of $M_1 \times M_2$. It is clear that $\{0\}$ is an sdf-absorbing submodule of M_1. Now, let $m_1 \in M_1$ and $a_1 \in R_1$ such that $a_1^2 m_1 = 0$. Then $((a_1, 0)^2 - (0, 0)^2)(m_1, 0) \in \{0\} \times M_2$. Thus, our assumption implies that $a_1 m_1 = 0$ and so M_1 is reduced. Conversely, suppose $\{0\}$ is sdf-absorbing in M_1 and M_1 is reduced. Let $(a_1, a_2), (b_1, b_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ such that $((a_1, a_2)^2 - (b_1, b_2)^2)(m_1, m_2) \in \{0\} \times M_2$. Then $(a_1^2 - b_1^2)m_1 = 0$. If $a_1 \in Ann_{R_1}(m_1)$, then $b_1^2 m_1 = 0$ and so $b_1 m_1 = 0$ as M_1 is reduced. Similarly, if $b_1 \in Ann_{R_1}(m_1)$, then $a_1 m_1 = 0$. Therefore, $(a_1 + b_1)m_1 = (a_1 - b_1)m_1 = 0$. If $a_1, b_1 \notin Ann_{R_1}(m_1)$, then by assumption, we have also $(a_1 + b_1)m_1 = 0$ or $(a_1 - b_1)m_1 = 0$. Therefore, $((a_1, a_2) + (b_1, b_2))(m_1, m_2) \in \{0\} \times M_2$ or $((a_1, a_2) - (b_1, b_2))(m_1, m_2) \in \{0\} \times M_2$ as needed.

(2) If $\{0\} \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$, then similar to the proof of (1), $\{0\}$ is sdf-absorbing in M_1, N_2 is sdf-absorbing in M_2 and M_1 is reduced. Let $m_1 \in M_1$ and $m_2 \in M_2$. Since $((1, 1)^2 - (1, -1)^2)(m_1, m_2) \in N_1 \times N_2$, then either $(2, 0)(m_1, m_2) \in \{0\} \times N_2$ or $(0, 2)(m_1, m_2) \in \{0\} \times N_2$. Hence, $2m_1 = 0$ or $2m_2 \in N_2$ and so $2 \in (0 :_{R_1} M_1)$ or $2 \in (N_2 :_{R_2} M_2)$. Conversely, let $(a_1, a_2), (b_1, b_2) \in R_1 \times R_2$, $(m_1, m_2) \in M_1 \times M_2$ such that $((a_1, a_2)^2 - (b_1, b_2)^2)(m_1, m_2) \in \{0\} \times N_2$. If $2 \in (N_2 :_{R_2} M_2)$, then by using Theorem 2, $(a_2 + b_2)m_2 \in N_2$ and $(a_2 - b_2)m_2 \in N_2$. Since also, $(a_1+b_1)m_1 = 0$ or $(a_1-b_1)m_1 = 0$, then $((a_1, a_2)+(b_1, b_2))(m_1, m_2) \in \{0\}\times N_2$ or $((a_1, a_2) - (b_1, b_2))(m_1, m_2) \in \{0\} \times N_2$. If $2 \in (0 :_R M_1)$, we also get the same conclusion. Therefore, $\{0\} \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$.

(3) Suppose $\{0\} \times \{0\}$ is an sdf-absorbing submodule of $M_1 \times M_2$. Then clearly, $\{0\}$ is an sdf-absorbing submodule of M_1 and M_2, and by using a similar argument to that in part (2), we have M_1 and M_2 are reduced. Let $0 \neq m_1 \in M_1$ and $0 \neq m_2 \in M_2$. Since $((1, 1)^2 - (1, -1)^2)(m_1, m_2) \in \{0\}\times\{0\}$ and clearly $(1, 1), (1, -1) \notin Ann_{R_1\times R_2}((m_1, m_2))$, we have $2 \in (0 :_{R_1} M_1)$ or $2 \in (0 :_{R_2} M_2)$. Conversely, let $(m_1, m_2) \in M_1 \times M_2$ and $(a_1, a_2), (b_1, b_2) \in R_1 \times R_2 \backslash Ann_{R_1\times R_2}((m_1, m_2))$ such that $((a_1, a_2)^2 - (b_1, b_2)^2)(m_1, m_2) \in \{0\} \times \{0\}$. Since M_1 and M_2 are reduced, it is enough to consider the case $a_1, b_1 \notin Ann_{R_1}(m_1)$ and $a_2, b_2 \notin Ann_{R_2}(m_2)$. By using a similar argument to that in the proof of part (2), we conclude that $\{0\}\times\{0\}$ is an sdf-absorbing submodule of $M_1 \times M_2$. □

The proof of the following theorem is similar to that of Theorem 7 and is left to the reader.

Theorem 8. *Let N_1 and N_2 be nonzero proper submodules of nonzero R-modules M_1 and M_2.*

1. $\{0\} \times M_2$ is an sdf-absorbing submodule of $M_1 \times M_2$ if and only if $\{0\}$ is an sdf-absorbing submodule of M_1 and M_1 is reduced. A similar result holds for $M_1 \times \{0\}$.

2. If $\{0\} \times N_2$ is an sdf-absorbing submodule of $M_1 \times M_2$, then $\{0\}$ is an sdf-absorbing submodule of M_1, N_2 is an sdf-absorbing submodule of M_2 and M_1 is reduced. The converse is true if $2 \in (N_2 :_R M_2)$. A similar result holds for $N_1 \times \{0\}$.

3. If $\{0\} \times \{0\}$ is an sdf-absorbing submodule of $M_1 \times M_2$, then $\{0\}$ is an sdf-absorbing submodule of M_1, M_2 and M_1, M_2 are reduced. The converse is true if $2 \in (0 :_R M_1)$ or $2 \in (0 :_R M_2)$.

3 Sdf-absorbing Submodules of Amalgamation Modules

Let R be a ring and M be an R-module. Recall that the idealization of M in R denoted by $R \ltimes M$ is the commutative ring $R \oplus M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ [20]. For an ideal I of R and a submodule N of M, The set $I \ltimes N = I \oplus N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$ [3, Theorem 3.1].

It is proved in [5, Theorem 4.16] that for a nonzero ideal I of a ring R, $I \ltimes N$ is an sdf-absorbing ideal of $R \ltimes M$ if and only if I is an sdf-absorbing ideal of R and $N = M$. Therefore, for every sdf-absorbing submodule N of M and a nonzero ideal I of R, $I \ltimes N$ is never an sdf-absorbing ideal of $R \ltimes M$.

For $I = \{0\}$, we have the following result analogous to [5, Remark 4.18].

Proposition 12. *If N is a proper submodule of an R-module M and $\{0\} \ltimes N$ is an sdf-absorbing ideal of $R \ltimes M$, then $N = \{0_M\}$ is an sdf-absorbing submodule of M.*

Proof. Suppose $\{0\} \ltimes N$ is an sdf-absorbing ideal of $R \ltimes M$. Then $N = \{0_M\}$ by [5, Remark 4.18]. Now, let $m \in M$ and $a, b \in R \backslash Ann_R(m)$ such that $(a^2 - b^2)m = 0_M$. Then $(0, am)^2 - (0, bm)^2 = (0, 0_M)$ with $(0, am), (0, bm) \neq (0, 0_M)$. Thus, $[(0, am) + (0, bm)] = (0, (a + b)m) = (0, 0_M)$ or $[(0, am) - (0, bm)] = (0, (a - b)m) = (0, 0_M)$. Hence, $(a + b)m = 0_M$ or $(a - b)m = 0_M$ as needed. □

Let R be a ring, J an ideal of R and M an R-module. The amalgamated duplication of R along J is defined as

$$R \bowtie J = \{(r, r + j) : r \in R \ , \ j \in J\}$$

which is a subring of $R \times R$, see [12]. The duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined recently in [11] as

$$M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM\}$$

which is an $(R \bowtie J)$-module with scalar multiplication defined by $(r, r + j)(m, m') = (rm, (r + j)m')$ for $r \in R$, $j \in J$ and $(m, m') \in M \bowtie J$. Many properties and results concerning this kind of modules can be found in [11].

Let N be a submodule of an R-module M and J be an ideal of R. Then clearly

$$N \bowtie J = \{(a, b) \in N \times M : a - b \in JM\}$$

and

$$\bar{N} = \{(b, a) \in M \times N : b - a \in JM\}$$

are submodules of $M \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1,\ j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [13], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scalar product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2)$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, clearly the sets

$$N_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : m_1 \in N_1\}$$

and

$$\overline{N_2}^{\varphi} = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 :\ \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^{\varphi} JM_2$. The above notation will be be fixed throughout the rest of this section

In the following two theorems, we justify conditions under which the submodules $N_1 \bowtie^{\varphi} JM_2$ and $\overline{N_2}^{\varphi}$ are sdf-absorbing in $M_1 \bowtie^{\varphi} JM_2$ analogous to [5, Theorem 4.19].

Theorem 9. *Consider the* $(R_1 \bowtie^f J)$*-module* $M_1 \bowtie^\varphi JM_2$ *defined as above and let* N_1 *be a submodule of* M_1*. If* $N_1 \bowtie^\varphi JM_2$ *is an sdf-absorbing submodule of* $M_1 \bowtie^\varphi JM_2$*, then* N_1 *is an sdf-absorbing submodule of* M_1*. The converse is true if* N_1 *is nonzero.*

Proof. Firstly, we note that N_1 is a proper submodule of M_1 if and only if $N_1 \bowtie^\varphi JM_2$ is a proper submodule of $M_1 \bowtie^\varphi JM_2$. Suppose $N_1 \bowtie^\varphi JM_2$ is sdf-absorbing in $M_1 \bowtie^\varphi JM_2$ and let $m_1 \in M_1$ and $r_1, s_1 \in R_1 \backslash Ann_{R_1}(m_1)$ such that $(r_1^2 - s_1^2)m_1 \in N_1$. Then $(r_1, f(r_1)), (s_1, f(s_1)) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^\varphi JM_2$. Moreover, clearly $(r_1, f(r_1)), (s_1, f(s_1)) \notin Ann_{R_1 \bowtie^f J}((m_1, \varphi(m_1)))$ and

$$\left[(r_1, f(r_1))^2 - (s_1, f(s_1))^2\right](m_1, \varphi(m_1)) = ((r_1^2-s_1^2)m_1, \varphi((r_1^2-s_1^2)m_1)) \in N_1 \bowtie^\varphi JM_2.$$

By assumption, we conclude that

$$\begin{aligned}((r_1 + s_1)m_1, \varphi((r_1 + s_1)m_1)) &= (r_1 + s_1, f(r_1 + s_1))(m_1, \varphi(m_1)) \\ &= \left[(r_1, f(r_1)) + (s_1, f(s_1))\right](m_1, \varphi(m_1)) \in N_1 \bowtie^\varphi JM_2\end{aligned}$$

or

$$\begin{aligned}((r_1 - s_1)m_1, \varphi((r_1 - s_1)m_1)) &= (r_1 - s_1, f(r_1 - s_1)(m_1, \varphi(m_1)) \\ &= \left[(r_1, f(r_1)) - (s_1, f(s_1))\right](m_1, \varphi(m_1)) \in N_1 \bowtie^\varphi JM_2\end{aligned}$$

Thus, $(r_1 + s_1)m_1 \in N_1$ or $(r_1 - s_1)m_1 \in N_1$ and N_1 is an sdf-absorbing submodule of M_1. Now, suppose N_1 is a nonzero sdf-absorbing in M_1. Let $(r_1, f(r_1) + j), (s_1, f(s_1) + j') \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie JM_2$ such that

$$\left[(r_1, f(r_1) + j)^2 - (s_1, f(s_1) + j')^2\right](m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^\varphi JM_2$$

Then $(r_1^2 - s_1^2)m_1 \in N_1$ and so $(r_1 + s_1)m_1 \in N_1$ or $(r_1 - s_1)m_1$. It follows that

$$\left[(r_1, f(r_1) + j) + (s_1, f(s_1) + j')\right](m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^\varphi JM_2$$

or

$$\left[(r_1, f(r_1) + j) - (s_1, f(s_1) + j')\right](m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^\varphi JM_2$$

and $N_1 \bowtie^\varphi JM_2$ is an sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$. □

The next result follows as an analogy of [5, Theorem 4.19] on sdf-ideal.

Theorem 10. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 9 where f and φ are epimorphisms and let N_2 be a submodule of M_2. Then*

1. If $\overline{N_2}^\varphi$ is an sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$, then N_2 is an sdf-absorbing submodule of M_2. The converse is true if N_2 is nonzero.

2. Suppose $Ann_{M_2}(J) = 0$. If $\overline{N_2}^\varphi$ is an sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$ and $J \nsubseteq (N_2 :_{R_2} M_2)$, then $(N_2 :_{M_2} J)$ is an sdf-absorbing submodule of M_2.

Proof. (1) We firstly prove that N_2 is proper in M_2 if and only if $\overline{N_2}^\varphi$ is proper in $M_1 \bowtie^\varphi JM_2$. If $N_2 = M_2$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2$, then $\varphi(m_1)+m_2 \in N_2$. Thus, $(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi$ and so $\overline{N_2}^\varphi = M_1 \bowtie^\varphi JM_2$. Next, suppose $\overline{N_2}^\varphi = M_1 \bowtie^\varphi JM_2$ and let $m_2 = \varphi(m_1) \in M_2$ for some $m_1 \in M_1$. Then $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2 = \overline{N_2}^\varphi$ and so $m_2 \in N_2$. Hence, $N_2 = M_2$. Suppose $\overline{N_2}^\varphi$ is an sdf-absorbing submodule of $M_1 \bowtie^\varphi JM_2$. Let $m_2 = \varphi(m_1) \in M_2$ and $r_2 = f(r_1)$, $s_2 = f(s_1) \in R_2 \backslash Ann_{R_2}(m_2)$ such that $(r_2^2 - s_2^2)m_2 \in N_2$. Then (r_1, r_2), $(s_1, s_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$ with $\left[(r_1, r_2)^2 - (s_1, s_2)^2\right](m_1, m_2) \in \overline{N_2}^\varphi$ and clearly (r_1, r_2), $(s_1, s_2) \notin Ann_{R_1 \bowtie^f J}((m_1, m_2))$. By assumption,

$$[(r_1, r_2) + (s_1, s_2)]\,(m_1, m_2) \in \overline{N_2}^\varphi \mathit{or}\, [(r_1, r_2) - (s_1, s_2)]\,(m_1, m_2) \in \overline{N_2}^\varphi$$

Thus, $(r_2 + s_2)m_2 \in N_2$ or $(r_2 - s_2)m_2 \in N_2$ and N_2 is an sdf-absorbing submodule of M_2. Now, suppose N_2 is a nonzero sdf-absorbing submodule of M_2 and note that $\overline{N_2}^\varphi$ is also nonzero. Let $(r_1, f(r_1) + j), (s_1, f(s_1) + j') \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie JM_2$ such that

$$\left[(r_1, f(r_1) + j)^2 - (s_1, f(s_1) + j')^2\right](m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi$$

Then $\left[(f(r_1) + j)^2 - (f(s_1) + j')^2\right](\varphi(m_1) + m_2) \in N_2$ and so we have either

$$[(f(r_1) + j) + (f(s_1) + j')]\,(\varphi(m_1) + m_2) \in N_2$$

or

$$[(f(r_1) + j) - (f(s_1) + j')]\,(\varphi(m_1) + m_2) \in N_2.$$

It follows that either

$$[(r_1, f(r_1) + j) + (s_1, f(s_1) + j')]\,(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi$$

or

$$[(r_1, f(r_1) + j) + (s_1, f(s_1) + j')]\,(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi$$

and we are done.

(2) Since $J \nsubseteq (N_2 :_{R_2} M_2)$, $(N_2 :_{M_2} J)$ is proper in M_2. Suppose $\overline{N_2}^{\varphi}$ is an sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$. Let $m_2 \in M_2$ and $r_2 = f(r_1), s_2 = f(s_1) \in R_2 \backslash Ann_{R_2}(m_2)$ such that $(r_2^2 - s_2^2)m_2 \in (N_2 :_{M_2} J)$. Then $(r_2^2 - s_2^2)Jm_2 \subseteq N_2$ and so $\left[(r_1, r_2)^2 - (s_1, s_2)^2\right](0, Jm_2) \subseteq \overline{N_2}^{\varphi}$. Moreover, $(r_1, r_2), (s_1, s_2) \notin Ann_{R_1 \bowtie^f J}((0, Jm_2))$ as $Ann_{M_2}(J) = 0$. By assumption, $[(r_1, r_2) + (s_1, s_2)](0, Jm_2) \subseteq \overline{N_2}^{\varphi}$ or $[(r_1, r_2) - (s_1, s_2)](0, Jm_2) \subseteq \overline{N_2}^{\varphi}$. Thus, either $(r_2 + s_2)Jm_2 \subseteq N_2$ or $(r_2 - s_2)Jm_2 \subseteq N_2$. It follows that $(r_2 + s_2)m_2 \in (N_2 :_{M_2} J)$ or $(r_2 - s_2)m_2 \in (N_2 :_{M_2} J)$ and so $(N_2 :_{M_2} J)$ is an sdf-absorbing submodule of M_2. $\square$

In Theorem 10, if φ is not an epimorphism, then N_2 is an sdf-absorbing submodule of M_2 does not imply that $\overline{N_2}^{\varphi}$ is an sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$.

Example 6. *Let $R_1 = M_1 = R_2 = \mathbb{Z}$, $M_2 = \mathbb{Z}_{12}$ and $J = \langle 0_{R_2} \rangle$. Define $\varphi : M_1 \to M_2$ by $\varphi(x) = 6x$ and let $N_2 = \langle \bar{6} \rangle$. Then N_2 is an sdf-absorbing submodule of M_2 by Proposition 4. On the other hand $\overline{N_2}^{\varphi} = \{(m_1, \varphi(m_1)) : \varphi(m_1) \in N_2\} = \{(m_1, 6m_1) : 6m_1 \in \langle \bar{6} \rangle\} = M_1 \bowtie^{\varphi} JM_2$ is not an sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$.*

Also, the converse of (2) of Theorem 10 need not be true in general.

Example 7. *Let $R_1 = M_1 = R_2 = \mathbb{Z}$, $M_2 = \mathbb{Z}_{12}$ and $J = 2\mathbb{Z}$. Let $f : R_1 \to R_2$, $\varphi : M_1 \to M_2$ be the identity epimorphisms and $N_2 = \langle \bar{4} \rangle$. Then $(N_2 :_{M_2} J) = \langle \bar{2} \rangle$ is clearly an sdf-absorbing submodule of M_2. But, N_2 is not sdf-absorbing submodule of M_2 by Proposition 4. Thus, $\overline{N_2}^{\varphi}$ is not an sdf-absorbing submodule of $M_1 \bowtie^{\varphi} JM_2$ by (1) of Theorem 10.*

The following example shows that it is crucial that N be a nonzero submodule in (1) of Theorem 9 and (1) of Theorem 10.

Example 8. *Let $R = J = \mathbb{Z}$ and $M = \mathbb{Z}_4$. Then $N = \langle \bar{0} \rangle$ is an sdf-absorbing submodule of M by Theorem 3. On the other hand, $0 \bowtie J$ is not an sdf-absorbing submodule of the $(R \bowtie J)$-module $M \bowtie J$. Indeed, let $m = (\bar{1}, \bar{1}) \in M \bowtie J = \mathbb{Z}_4 \times \mathbb{Z}_4$, $a = (2, 0)$ and $b = (0, 2)$. Then $a, b \notin Ann_{\mathbb{Z}}(m)$ and $(a^2 - b^2)m = (\bar{0}, \bar{0}) \in 0 \bowtie J$ but $(a + b)m = (a - b)m = (\bar{2}, \bar{2}) \notin 0 \bowtie J$.*

In particular, we have the following corollaries of the previous two theorems.

Corollary 8. *Let N be a submodule of an R-module M and J be an ideal of R. If $N \bowtie J$ is an sdf-absorbing submodule of $M \bowtie J$, then N is an sdf-absorbing submodule of M. The converse is true if N is nonzero.*

Corollary 9. *Let N be a submodule of an R-module M and J be an ideal of R. Then*

1. If $\overline{N}$ is an sdf-absorbing submodule of $M \bowtie J$, then N is an sdf-absorbing submodule of M. The converse is true if N is nonzero.
2. If $\overline{N}$ is an sdf-absorbing submodule of $M \bowtie J$, $J \nsubseteq (N :_R M)$ and $Ann_{M_2}(J) = 0$, then $(N :_M J)$ is an sdf-absorbing submodule of M.

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Hani A. KHASHAN,
Department of Mathematics,
Al al-Bayt University,
Al Mafraq, Jordan.
Email: hakhashan@aabu.edu.jo

Ece YETKIN CELIKEL,
Department of Software Engineering,
Hasan Kalyoncu University,
Gaziantep, Türkiye.
Email:ece.celikel@hku.edu.tr, yetkinece@gmail.com