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Steiner distances in generalized corona products

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Abstract

The Steiner distance of a subset of vertices in a graph is the minimum size among all the connected subgraphs containing this subset. This paper focuses on the study of Steiner distances in both generalized vertex corona (GVC) and generalized edge corona (GEC) products, and the relationship with their corresponding center and outer graphs. Particularly, we show how Steiner distances in GEC products can be computed from those ones in GVC products, and we also establish sharp bounds for their Steiner numbers, eccentricities, radii, diameters and k-Wiener indices. In this way, we extend some known results on corona products.

1 Introduction and preliminaries

This paper deals with finite, simple and undirected graphs. We denote respectively by $V(G)$ and $E(G)$ the set of vertices and the set of edges of any graph G, so that each pair of adjacent vertices $u, v \in V(G)$ gives rise to the edge $uv \in E(G)$. In 1989, Chartrand et al. [3] introduced the *Steiner distance* $d_G(S)$ of a non-empty subset of vertices $S \subseteq V(G)$ as the minimum size among all connected subgraphs of G containing S. Any such a subgraph of minimum size is necessarily a tree, which is termed *Steiner S-tree* [12, 17].

If $|S| = 2$, then the Steiner distance coincides with the classical geodetic distance. More generally, $d_G(S) \geq |S| - 1$. If the equality holds, then it is

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said that the vertices of S induce a Steiner tree in G. Furthermore, if H is a connected spanning subgraph of G, then $d_G(S) \leq d_H(S)$. Finally, a *Steiner set* of G is any subset $S \subseteq V(G)$ such that every vertex in G is contained in a Steiner S-tree. The minimum cardinality of any such a set is the *Steiner number* $s(G)$.

Let $k \in \{2, \ldots, |V(G)|\}$ be a positive integer. The *Steiner k-eccentricity* of a vertex $v \in V(G)$ is defined as

$$\mathrm{ecc}_k(v, G) := \max_{S \in \mathcal{S}_k(v,G)} d_G(S),$$

where $\mathcal{S}_k(G)$ denotes the set of k-subsets $S \subseteq V(G)$ such that $v \in S$. If $k < |V(G)|$, then

$$k - 1 \leq \mathrm{ecc}_k(v, G) \leq \mathrm{ecc}_{k+1}(v, G) \leq \mathrm{ecc}_{|V(G)|}(v, G) = |V(G)| - 1. \quad (1)$$

The *Steiner k-radius* $\mathrm{srad}_k(G)$ and the *Steiner k-diameter* $\mathrm{sdiam}_k(G)$ are respectively defined as the minimum and maximum Steiner k-eccentricity for every vertex $v \in V(G)$. If $k = 2$, then all the previous parameters coincide respectively with the classical eccentricities, radius and diameter of G.

The problem of computing the Steiner distance of a graph is NP-hard [14]. This problem has relevant applications in designing communication and electrical networks [4, 8, 22], and also in chemical graph theory [10, 20]. In this last regard, the *Steiner k-Wiener index* [18] of the graph G,

$$SW_k(G) := \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d_G(S),$$

is a natural generalization of the classical Wiener index, which arises for $k = 2$. These applications have brought about the study of Steiner distances for different families of graphs. (See [20] for a comprehensive survey on this issue.) Of particular interest for the aims of this paper, we remark those studies dealing with Steiner distances in graph products, and the relationship with their components. In this regard, one may found some results on tensor [2], lexicographic [1, 21], join, Cartesian, vertex corona [24] and edge corona products [21, 23]. This paper delves into this topic by dealing with Steiner distances of generalized vertex corona products and generalized edge corona products.

Recall here that the *corona product* [7] $G \odot H$ of a center graph G and an outer graph H is the graph obtained from G and a set $\{H(v) \colon v \in V(G)\}$ formed by $|V(G)|$ vertex-disjoint copies of H so that every vertex in $H(v)$ is adjacent to v.

Steiner distances in $G \odot H$ do not depend on the topology of H. In this regard, the following theorem was proven in [23, Theorem 4.1] for k-subsets $S \subseteq V(G \odot H)$, with $k > 2$, but it also holds readily for geodetic distances.

Theorem 1. *[23] It is verified that*

$$d_{G\odot H}(S) = d_G(\widetilde{S}) + |S| - |S \cap V(G)|$$

where $\widetilde{S} := \{v \in V(G)\colon\ S \cap (V(H(v)) \cup \{v\}) \neq \emptyset\}$.

The Steiner number of a corona product is established in the next result.

Proposition 2. *[24, Proposition 2.4] It is verified that*

$$s(G \odot H) = |V(G)| \cdot |V(H)|.$$

The next result describes the Steiner k-diameter of a corona product $G\odot H$. It was proved in [23, Proposition 4.1] for the case $k > 2$, but it also holds readily for geodetic distances.

Proposition 3. *[23] It is verified that*

$$\mathrm{sdiam}_k(G \odot H) = \mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G) + \min\{k,\,|V(G)| \cdot |V(H)|\}.$$

Finally, the following theorem establishes the Steiner k-Wiener index of a corona product $G \odot H$.

Theorem 4. *[21, Theorem 2.9] It is verified that*

$$\begin{aligned} SW_k(G \odot H) = &\sum_{\ell=2}^{k} \binom{m+1}{r_1} \cdot \ldots \cdot \binom{m+1}{r_\ell} \cdot SW_\ell(G) + \\ &+ n \cdot \left[\binom{m}{k-1} \cdot (k-1) + \binom{m}{k} \cdot k - x \right] + \\ &+ \sum_{\ell=2}^{k} \binom{n}{\ell} \cdot \left[\sum_{j=1}^{\ell} \prod_{\substack{i=1\\ i\neq j}}^{\ell} \binom{m+1}{r_i} \cdot \left[\binom{m}{r_j - 1} \cdot (r_j - 1) + \binom{m}{r_j} \cdot r_j - x_j \right] \right] \end{aligned}$$

where $\sum_{i=1}^{\ell} r_i = k$, $r_i \geq 1$, *and* x *and each* x_j *are, respectively, the number of* k*- and* r_j*-subsets* $S \subseteq V(H)$ *whose vertices induce a Steiner tree in* H.

A first main aim of this paper is to generalize all the previous results for generalized vertex corona products, and also to describe their relationship with the corresponding Steiner parameters for generalized edge corona products. Recall here that Daykin et al. [5] generalized the corona product $G \odot H$ by replacing each copy $H(v)$ by the graph H_v of an ordered set of vertex-disjoint graphs

$$\mathcal{H} := \left\{H_{v_1}, \ldots, H_{v_{|V(G)|}}\right\},$$

where $V(G) = \{v_1, \ldots, v_{|V(G)|}\}$. The resulting graph is the *generalized vertex corona product* $G \odot \mathcal{H}$. We say that this product is *degenerate* if the ordered set $\mathcal{H}$ contains the order-zero graph K_0 among its components. Otherwise, it is *non-degenerate*. By abuse of notation, we assume that $\mathcal{H}$ is itself a graph. From here on, we term *GVC product* to any degenerate or non-degenerate generalized vertex corona product. Thus, for instance, Figure 1 (left) illustrates the GVC product having the complete graph K_3 as center, and the complete graphs K_1 and K_2, and the path P_3 as outer graphs. Here, and in the subsequent figures, we highlight with dashed lines those edges joining the corresponding center and outer graphs.

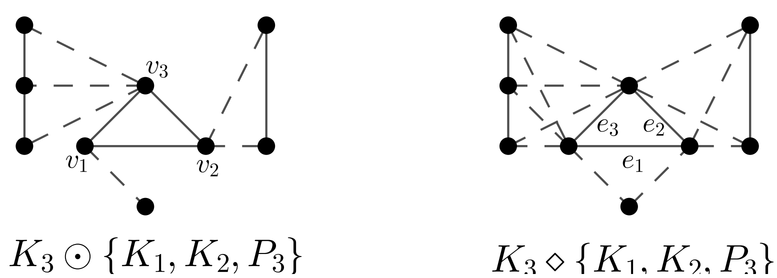


Figure 1: Examples of GVC and GEC products.

In 2010, Hou and Shiu [13] defined the *edge corona product* $G \diamond H$ of a center graph G and an outer graph H as the graph resulting from G and a set $\{H(e)\colon\ e \in E(G)\}$ formed by $|E(G)|$ vertex-disjoint copies of H so that every vertex in $H(e)$ is adjacent to both vertices in e. Luo and Yan [19] generalized this product by replacing each copy $H(e)$ by the graph H_e of an ordered set of vertex-disjoint graphs

$$\mathcal{H} := \{H_{e_1}, \ldots, H_{e_{|E(G)|}}\},$$

where $E(G) = \{e_1, \ldots, e_{|E(G)|}\}$. The resulting graph is the *generalized edge corona product* denoted by $G \diamond \mathcal{H}$.

Thus, for instance, Figure 1 (right) illustrates the GEC product $K_3 \diamond \{K_1, K_2, P_3\}$. Similarly to GVC products, we say that the graph $G \diamond \mathcal{H}$ is *degenerate* if the ordered set $\mathcal{H}$ contains the order-zero graph K_0 among its components. Otherwise, it is *non-degenerate*. By abuse of notation, we assume that $\mathcal{H}$ is itself a graph. In addition, we term *GEC product* to any degenerate or non-degenerate generalized edge corona product.

In recent years, (generalized) corona products have attracted considerable attention because they constitute the topological structure of some communication and distribution networks [6, 9, 11, 15]. Nevertheless, to the best of authors' knowledge, even if one may find in the literature some results concerning Steiner distances in corona products [16, 23, 24], no result exists on Steiner distances in GVC or GEC products. This paper delves into this last topic as follows. Steiner distances in GVC products are studied in Section 2. Particularly, we extend to GVC products all the known results on Steiner distances of corona products that we have enumerated in this introductory section. In addition, we establish some sharp bounds for Steiner eccentricities and Steiner radii of GVC products, which have not previously been studied even for corona products. Exact values of all these parameters are described for non-degenerate GVC products. Finally, Section 3 deals with the study of Steiner distances in GEC products. To this end, we first show how the computation of Steiner distances in GEC products derives from that of GVC products. Then, we establish some sharp bounds on the same Steiner parameters previously considered.

2 Steiner distances in GVC products

In this section, we extend to GVC products those known results on Steiner distances of corona products that we have mentioned in the introductory section. That is, Theorems 1 and 4, and Propositions 2 and 3. In addition, we study Steiner eccentricities and Steiner radii of GVC products, which, to the best knowledge of the authors, have not previously been studied in the literature.

Let $G \odot \mathcal{H}$ be a GVC product. For each vertex $v \in V(G \odot \mathcal{H})$, we define the vertex $\widetilde{v} \in V(G)$ as follows.

$$\widetilde{v} := \begin{cases} v, & \text{if } v \in V(G), \\ u, & \text{if } v \in V(H_u) \text{ for some } u \in V(G). \end{cases}$$

Then, for each non-empty subset $S \subseteq V(G \odot \mathcal{H})$, we also define the subset

$$\widetilde{S} := \{\widetilde{v}\colon\ v \in S\} \subseteq V(G).$$

The next lemma generalizes Theorem 1.

Lemma 5. *If $S \subseteq V(H_v)$ for some $v \in V(G)$, then*

$$d_{G\odot\mathcal{H}}(S) = \min\{d_{H_v}(S),\, |S|\}.$$

Otherwise,

$$d_{G\odot\mathcal{H}}(S) = d_G(\widetilde{S}) + |S \cap V(\mathcal{H})|.$$

Proof. First, we assume that $S \subseteq V(H_v)$ for some $v \in V(G)$. If the vertices of S induce a Steiner tree in H_v, then $d_{G\odot\mathcal{H}}(S) = d_{H_v}(S) = |S| - 1$. Otherwise, the star graph having the vertex v as center and the vertices of S as pendant vertices constitutes a Steiner S-tree in $G \odot \mathcal{H}$ of size $|S|$.

Now, under the assumption of the second statement, let T be a Steiner $\widetilde{S}$-tree in G. Then, the graph resulting after adding to T all the pendant edges vu, with $v \in \widetilde{S}$ and $u \in S \cap V(H_v)$, is a Steiner S-tree in $G \odot \mathcal{H}$ of size $d_G(\widetilde{S}) + |S \cap V(\mathcal{H})|$. □

Based on the previous lemma, the following result establishes lower and upper bounds on the Steiner number of any GVC product. It constitutes a natural generalization of Proposition 2.

Proposition 6. *We have*

$$|V(\mathcal{H})| \leq s(G \odot \mathcal{H}) \leq |V(\mathcal{H})| + |V(G)| - |\widetilde{V(\mathcal{H})}|.$$

Particularly, if the GVC product $G \odot \mathcal{H}$ is non-degenerate, then $s(G \odot \mathcal{H}) = |V(\mathcal{H})|$.

Proof. The lower bound holds because, according to the constructive proof of Lemma 5, every Steiner set of the graph $G \odot \mathcal{H}$ must contain all the vertices in $V(\mathcal{H})$. Then, the upper bound holds because any vertex in $\widetilde{V(\mathcal{H})}$ is superfluous in the description of a Steiner S-set of $G \odot \mathcal{H}$. Finally, the consequence holds because, if the GVC product $G \odot \mathcal{H}$ is non-degenerate, then $\widetilde{V(\mathcal{H})} = V(G)$, and hence, both lower and upper bounds coincide. □

Both bounds in Proposition 6 are sharp. Thus, for instance, we highlight with black triangles ▲ in Figure 2 all the vertices of a Steiner set S in both GVC products $P_3 \odot \{K_1, K_0, K_2\}$ and $P_3 \odot \{K_0, K_1 \cup K_2, K_0\}$. In addition, we highlight with blue color all the edges of a pair of related Steiner S-trees. Particularly,

$$s\left(P_3 \odot \{K_1, K_0, K_2\}\right) = |V\left(\{K_1, K_0, K_2\}\right)| = 3$$

and

$$\begin{gathered} s\left(P_3 \odot \{K_0,\, K_1 \cup K_2,\, K_0\}\right) = \\ = |V(\{K_0,\, K_1 \cup K_2,\, K_0\})| + |V(P_3)| - |V(\{K_0,\, \widetilde{K_1 \cup K_2},\, K_0\})| = \\ = 3 + 3 - 1 = 5. \end{gathered}$$

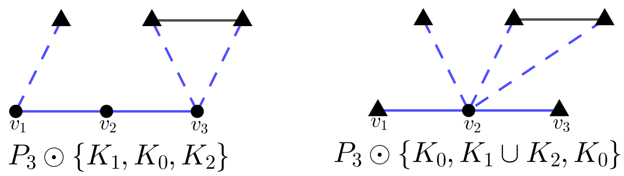


Figure 2: Steiner sets related to minimum and maximum Steiner numbers.

From now on, let $k \in \{2, \ldots, |V(G \odot \mathcal{H})|\}$ be a positive integer. The next result establishes a relationship among Steiner k-eccentricities in the center graph G and the outer graph $\mathcal{H}$.

Proposition 7. *If* $v \in V(G \odot \mathcal{H})$*, then*

$$\operatorname{ecc}_k(\widetilde{v},\, G \odot \mathcal{H}) \leq \operatorname{ecc}_k(v,\, G \odot \mathcal{H}).$$

Proof. The inequality holds readily for $v = \widetilde{v}$. Thus, we focus on the case $v \in V(\mathcal{H})$. Let $S \in \mathcal{S}_k(\widetilde{v},\, G \odot \mathcal{H})$ be such that $d_{G \odot \mathcal{H}}(S) = \operatorname{ecc}_k(\widetilde{v},\, G \odot \mathcal{H})$. If $v \in S$, then $S \in \mathcal{S}_k(v,\, G \odot \mathcal{H})$, and hence, $\operatorname{ecc}_k(\widetilde{v},\, G \odot \mathcal{H}) \leq \operatorname{ecc}_k(v,\, G \odot \mathcal{H})$. Otherwise, if $v \notin S$ then we define $S' := (S \setminus \{\widetilde{v}\}) \cup \{v\} \in \mathcal{S}_k(v,\, G \odot \mathcal{H})$. If $S' \subseteq V(H_v)$, then

$$\operatorname{ecc}_k(\widetilde{v},\, G \odot \mathcal{H}) = d_{G \odot \mathcal{H}}(S) = k - 1 \leq \operatorname{ecc}_k(v,\, G \odot \mathcal{H}).$$

Otherwise, $S' \notin H_w$ for any $w \in V(G)$. Since $S \cap V(G) \neq \emptyset$ and $\widetilde{S'} = \widetilde{S}$, Lemma 5 implies that

$$\operatorname{ecc}_k(\widetilde{v},\, G \odot \mathcal{H}) = d_{G \odot \mathcal{H}}(S) = d_{G \odot \mathcal{H}}(S') - 1 < \operatorname{ecc}_k(v,\, G \odot \mathcal{H}).$$

$\square$

The following result establishes lower and upper bounds on the Steiner k-eccentricities of $G \odot \mathcal{H}$.

Theorem 8. *If $v \in V(G \odot \mathcal{H})$, then*

$$\mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G) \leq \mathrm{ecc}_k(v, G \odot \mathcal{H}) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G) + \min\{k_v,\, |V(\mathcal{H})|\}$$

where $k_v = k - 1$ if $v = \widetilde{v}$, and $k_v = k$ otherwise. The upper bound is reached whenever there is a set $S \in \mathcal{S}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G)$ such that $d_G(S) = \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G)$ and $H_u \neq K_0$ for all $u \in S$. As a consequence, if $G \odot \mathcal{H}$ is non-degenerate, then this upper bound is sharp.

Proof. We claim that

$$\mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G\odot\mathcal{H}) \leq \mathrm{ecc}_k(\widetilde{v}, G\odot\mathcal{H}) \leq \mathrm{ecc}_k(v, G\odot\mathcal{H}).$$

The first inequality holds from the definition of Steiner k-eccentricity, together with the topology of the GVC product $G \odot \mathcal{H}$. The second one follows from (1). Finally, Proposition 7 implies the third one. Thus, the lower bound holds.

In order to prove the upper bound, let $S \in \mathcal{S}_k(v, G \odot \mathcal{H})$ be such that $d_{G\odot\mathcal{H}}(S) = \mathrm{ecc}_k(v, G \odot \mathcal{H})$. Since $|\widetilde{S}| \leq \min\{k, |V(G)|\}$, we have from (1) that $d_G(\widetilde{S}) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(\widetilde{v}, G)$. Then, the upper bound holds readily from Lemma 5.

Now, let $S \subseteq V(G)$ be a subset satisfying the conditions described in the hypothesis. For each vertex $u \in S \setminus \{\widetilde{v}\}$, let $w_u \in V(H_u)$. In addition, we define $w_{\widetilde{v}} = v$. Then, let $S' \subseteq V(\mathcal{H})$ be a k-subset formed by the $|S|$ vertices w_u, with $u \in S$, together with any other $\min\{k, |V(\mathcal{H})|\} - |S|$ vertices in $V(\mathcal{H})$, and $k - \min\{k, |V(\mathcal{H})|\}$ vertices in S. Let T be the Steiner S'-tree in $G \odot \mathcal{H}$ resulting from adding to a given Steiner S-tree in G all the pendant edges $\widetilde{w}w$, with $w \in S' \setminus S$. Then, the size of T coincides with the upper bound. □

Both bounds in Theorem 8 are sharp. Thus, for instance, Figure 3 shows examples for the whole spectrum of Steiner 3-eccentricities of an outer vertex v_2 in the star $K_{1,3}$ for three distinct GVC products $K_{1,3} \odot \mathcal{H}$, with $|V(\mathcal{H})| = 2$. According to Theorem 8, we have

$$3 = \mathrm{ecc}_{\min\{3,\,4\}}(\widetilde{v_2}, K_{1,3}) \leq \mathrm{ecc}_3(v_2, K_{1,3}\odot\mathcal{H}) \leq \mathrm{ecc}_{\min\{3,\,4\}}(\widetilde{v_2}, K_{1,3}) + \min\{3,\, 2\} = 5.$$

In Figure 3, we highlight with a black triangle ▲ all the vertices of the corresponding set S such that $d_{K_{1,3}\odot\mathcal{H}}(S) = \mathrm{ecc}_3(v_2,\, K_{1,3} \odot \mathcal{H})$. In addition, the edges of the corresponding Steiner S-trees are highlighted with blue colour.

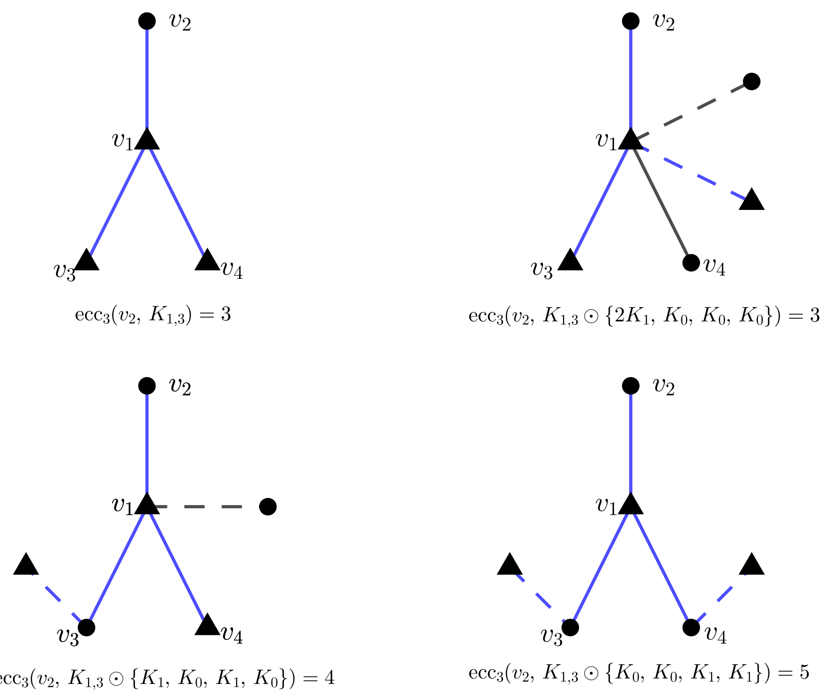


Figure 3: Steiner 3-eccentricities in distinct GVC products.

Based on Theorem 8, we describe a pair of lower and upper bounds on both the Steiner k-radius and the Steiner k-diameter of the graph $G \odot \mathcal{H}$. The lower bounds only depends on the center graph G, whereas the upper bounds also depends on the order, but not the topology, of the outer graph $\mathcal{H}$. The bounds concerning the Steiner k-diameter extends Proposition 3 to GVC products.

Theorem 9. *It is verified that*

$$\mathrm{srad}_{\min\{k,\,|V(G)|\}}(G) \leq \mathrm{srad}_k(G \odot \mathcal{H}) \leq \mathrm{srad}_{\min\{k,\,|V(G)|\}}(G) + \min\{k-1,\, |V(\mathcal{H})|\}$$

and

$$\mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G) \leq \mathrm{sdiam}_k(G \odot \mathcal{H}) \leq \mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G) + \min\{k,\, |V(\mathcal{H})|\}.$$

The first upper bound (respectively, the second one) is reached if there exists a $\min\{k,\, |V(G)|\}$*-subset* $S \subseteq V(G)$ *such that* $d_G(S) = \mathrm{srad}_{\min\{k,\,|V(G)|\}}(G)$

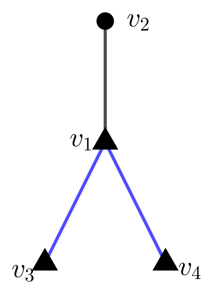
(respectively, $d_G(S) = \mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G)$), and $H_v \neq K_0$ for all $v \in S$. Thus, both upper bounds are reached whenever $G \odot \mathcal{H}$ is non-degenerate.

Proof. The result follows readily from Theorem 8 once we take minimum and maximum values in the lower and upper bounds therein described. Note here from Proposition 7 that the Steiner k-radius requires to deal with a vertex $v \in V(G)$, so that $k_v = k - 1$ in Theorem 8. $\square$

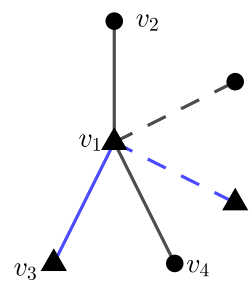
All the bounds in Theorem 9 are sharp. Thus, for instance, all the Steiner 3-eccentricities of the GVC products $K_{1,3} \odot \mathcal{H}$ described in Figure 3 refer indeed to Steiner 3-diameters. For the same graphs, Figure 4 shows the whole spectrum of Steiner 3-radii. According to Theorem 9, we have

$$2 = \mathrm{srad}_{\min\{3,\,4\}}(K_{1,3}) \leq \mathrm{srad}_3(K_{1,3} \odot \mathcal{H}) \leq \mathrm{srad}_{\min\{3,\,4\}}(K_{1,3}) + \min\{2,\,4\} = 4.$$

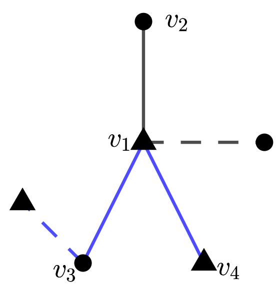
In Figure 4, we highlight with a black triangle $\blacktriangle$ all the vertices of the corresponding set $S \in \mathcal{S}_3(v_1,\, K_{1,3} \odot \mathcal{H})$ such that $d_{K_{1,3} \odot \mathcal{H}}(S) = \mathrm{srad}_3(K_{1,3} \odot \mathcal{H})$, and we highlight with blue color the edges of a related Steiner S-tree.



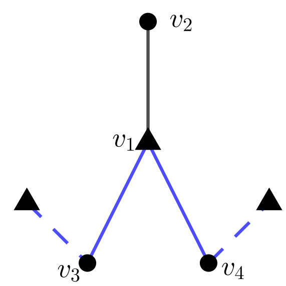
$\mathrm{srad}_3(K_{1,3}) = 2$



$\mathrm{srad}_3(K_{1,3} \odot \{2K_1,\, K_0,\, K_0,\, K_0\}) = 2$



$\mathrm{srad}_3(K_{1,3} \odot \{K_1,\, K_0,\, K_1,\, K_0\}) = 3$



$\mathrm{srad}_3(K_{1,3} \odot \{K_0,\, K_0,\, K_1,\, K_1\}) = 4$

Figure 4: Steiner 3-radius spectrum in distinct GVC products.

We finish this subsection by extending Theorem 4 to GVC products.

Proposition 10. *It is verified that*

$$SW_k(G \odot \mathcal{H}) = \alpha_k + \beta_k$$

where

$$\alpha_k := \sum_{\substack{S \in \mathcal{S}_k(H_v) \\ v \in V(G)}} \min\{d_{H_v}(S),\, |S|\}$$

and

$$\beta_k := \sum_{\substack{S \in \mathcal{S}_k(G \odot \mathcal{H}) \\ |\widetilde{S}|>1}} \Big(d_G(\widetilde{S}) + |S \cap V(\mathcal{H})|\Big) \cdot \prod_{v \in \widetilde{S}} \binom{|V(H_v)|}{|S \cap V(H_v)|}.$$

Proof. The result holds readily from Lemma 5 once we consider all the possibilities of choosing a k-subset of $V(G \odot \mathcal{H})$. Thus, α_k refers to those k-subsets in each set H_v, whereas β_k refers to the remaining k-subsets of $V(G \odot \mathcal{H})$ formed by j vertices in G and $k-j$ vertices in $\mathcal{H}$. $\square$

3 Steiner distances in GEC products

Steiner distances in GEC products can be computed from those ones in GVC products. To prove it, for each GEC product $G \diamond \mathcal{H}$, we denote by $\mathrm{GVC}(G \diamond \mathcal{H})$ the set of GVC products $G \odot \mathcal{H}'$ that are constructed as follows. For each edge $uv \in E(G)$, we choose a vertex $w_{uv} \in \{u, v\}$. Then, we remove from $G \odot \mathcal{H}'$ all those edges $w_{uv}w \in E(G \diamond \mathcal{H})$ such that $w \in V(H_{uv})$. In this way, the graph $H_{uv} \in \mathcal{H}$ becomes a subgraph of the graph $H'_{w_{uv}} \in \mathcal{H}'$. Moreover, $V(G \odot \mathcal{H}') = V(G \diamond \mathcal{H})$ and $E(G \odot \mathcal{H}') \subset E(G \diamond \mathcal{H})$. Thus, for instance, the GVC product $K_3 \odot \{K_1, K_2, P_3\}$ in Figure 1 (left) arises in this way from the GEC product $K_3 \diamond \{K_1, K_2, P_3\}$ in the same figure (right). Furthermore, Figure 2 shows two distinct GVC products, $P_3 \odot \{K_1, \emptyset, K_2\}$ and $P_3 \odot \{\emptyset, K_1 \cup K_2, \emptyset\}$, both of them arisen from the GEC product $P_3 \diamond \{K_1, K_2\}$, which is shown in Figure 5.

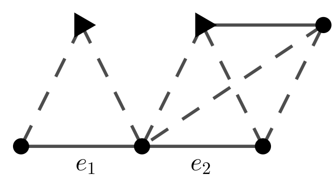


Figure 5: The GVC product $P_3 \diamond \{K_1, K_2\}$.

The following result shows a relevant relationship among Steiner trees in a GEC product $G \diamond \mathcal{H}$ and Steiner trees in its associated set of GVC products.

Lemma 11. *Let $S \subseteq V(G \diamond \mathcal{H})$. If T is a Steiner S-tree in $G \diamond \mathcal{H}$, then there exist a GVC product $G \odot \mathcal{H}' \in \mathrm{GVC}(G \diamond \mathcal{H})$ and a Steiner S-tree T' in $G \odot \mathcal{H}'$ such that $V(T') = V(T)$ and $|E(T')| = |E(T)|$.*

Proof. Let T be a Steiner S-tree in $G \diamond \mathcal{H}$. In what follows, we describe a procedure to construct both the GVC product $G \odot \mathcal{H}' \in \mathrm{GVC}(G \diamond \mathcal{H})$ and the Steiner S-tree T' in the statement. We initialize this procedure by defining the graphs $G_1 := G \diamond \mathcal{H}$ and $G_2 := T$, from which the graphs $G \odot \mathcal{H}'$ and T' will respectively arise. For each edge $uv \in E(G)$, we do the following.

- If there exists one vertex $w \in \{u, v\}$ such that $ww' \notin E(G_2)$ whatever the vertex $w' \in V(H_{uv})$ is, then we remove from G_1 those edges $ww' \in$

$E(G_1)$ such that $w' \in V(H_{uv})$. If both vertices u and v satisfy the mentioned condition, then we only remove all those edges corresponding to one of them.

- Otherwise, if there are two vertices $w_1, w_2 \in H_{uv}$ such that $\{uw_1, vw_2\} \subseteq E(G_2)$, then we remove from G_1 those edges $vw \in E(G_1)$ such that $w \in H_{uv}$. In order to construct the Steiner S-tree T' from G_2, we also modify the set of edges in the graph G_2 as follows. First, we remove from G_2 all those edges appearing also in $E(H_{uv})$, together with all those edges $vw \in E(G_2)$ such that $w \in H_{uv}$. Then, we add to G_2 all the edges in the set

$$\bigcup_{w \in V(H_{uv})} \{uw \colon \exists w' \in V(H_{uv}) \cup \{v\} \text{ such that } ww' \in E(T)\}.$$

If $T \cap ((G \diamond \mathcal{H}) \setminus H_{uv})$ is a connected tree, then the new graph G_2 is a Steiner S-tree in the graph G_1 having the same size than T, and containing its same set of vertices. Otherwise, if $T \cap ((G \diamond \mathcal{H}) \setminus H_{uv})$ is disconnected, then the new graph G_2 has size $|E(T)| - 1$, and it is formed by two disconnected trees. If this is the case, then we also add the edge uv to G_2. In this way, we get a Steiner S-tree in the graph G_1 having the same size than T, and containing its same set of vertices.

Once this procedure is done for every edge in $E(G)$, the resulting graphs G_1 and G_2 constitute, respectively, the GVC product $G \odot \mathcal{H}'$ and the Steiner S-tree T' of the statement. □

In general, if $S \subseteq V(G \diamond \mathcal{H})$ is non-empty, and $G \odot \mathcal{H}'_1$ and $G \odot \mathcal{H}'_2$ are two GVC products in $\mathrm{GVC}(G \diamond \mathcal{H})$, then $d_{G \odot \mathcal{H}'_1}(S)$ and $d_{G \odot \mathcal{H}'_2}(S)$ may be different. Thus, for instance, if S is the set formed by the two vertices highlighted with a black triangle ▶ in Figure 5, then $d_{P_3 \odot \{K_1, \emptyset, K_2\}}(S) = 4$ and $d_{P_3 \odot \{\emptyset, K_1 \cup K_2, \emptyset\}}(S) = 2$ (see Figure 2). The following result shows how the study of Steiner distances in GVC products arisen from the same GEC product is useful to establish Steiner distances in the latter.

Proposition 12. *If $S \subseteq V(G \diamond \mathcal{H})$ is non-empty, then*

$$d_{G \diamond \mathcal{H}}(S) = \min\{d_{G \odot \mathcal{H}'}(S) \colon G \odot \mathcal{H}' \in \mathrm{GVC}(G \diamond \mathcal{H})\}.$$

Proof. Since every graph in $\mathrm{GVC}(G \diamond \mathcal{H})$ is a connected spanning subgraph of $G \diamond \mathcal{H}$, the described minimum value is an upper bound of $d_{G \diamond \mathcal{H}}(S)$. Lemma 11 implies readily that this upper bound is indeed sharp. □

Proposition 12 enables one to compute Steiner parameters on GEC products minimizing those ones on their related GVC products. In general, this procedure requires a comprehensive study of cases, as it happens for computing Steiner k-Wiener indices of GEC products from the computation described in Proposition 10. However, some lower and upper sharp bounds can readily be determined for some Steiner parameters. Thus, for instance, the following result establishes a pair of lower and upper bounds for the Steiner number of the graph $G \diamond \mathcal{H}$.

Theorem 13. *It is verified that*

$$|V(\mathcal{H})| \leq s(G \diamond \mathcal{H}) \leq |V(\mathcal{H})| + |V(G)| - \min\left\{|\widetilde{V(\mathcal{H}')}|:\ G \odot \mathcal{H}' \in \mathrm{GVC}(G \diamond \mathcal{H})\right\}.$$

Proof. The upper bound follows readily from Propositions 6 and 12. To prove the lower bound, let S be a Steiner set of the graph $G \diamond \mathcal{H}$ such that $|S| = s(G \diamond \mathcal{H})$, and let T be a Steiner S-tree in $G \diamond \mathcal{H}$. From Lemma 11, there exists a Steiner S-tree T' in a graph $G \odot \mathcal{H}' \in \mathrm{GVC}(G \diamond \mathcal{H})$ such that $V(T') = V(T) = V(G \odot \mathcal{H}')$. Thus, S is a Steiner set of $G \odot \mathcal{H}'$, and hence, Proposition 6 implies that $|V(\mathcal{H})| \leq s(G \odot \mathcal{H}') \leq s(G \diamond \mathcal{H})$. □

Both bounds in Theorem 13 are sharp. Thus, for instance, we highlight with black triangles ▲ in Figure 6 all the vertices of a Steiner set S in both GEC products $K_3 \diamond \{K_1, K_1, K_1\}$ and $P_3 \diamond \{K_1, K_2\}$. In addition, we highlight with blue color all the edges of some related Steiner S-trees. Particularly, for $K_3 \diamond \{K_1, K_1, K_1\}$ we describe two Steiner S-trees so that every vertex in the GEC product is contained in at least one of them. Note that

$$s\left(K_3 \odot \{K_1,\ K_1,\ K_1\}\right) = \left|V\left(\{K_1,\ K_1,\ K_1\}\right)\right| = 3$$

and

$$\begin{gathered} s\left(P_3 \odot \{K_1,\ K_2\}\right) = |V(\{K_1,\ K_2\})| + |V(P_3)| - \\ - \min\left\{|V(\widetilde{\{K_1,\ K_2\}'})|:\ P_3 \odot \{K_1, K_2\}' \in \mathrm{GVC}(P_3 \diamond \{K_1, K_2\})\right\} = \\ = 3 + 3 - 1 = 5. \end{gathered}$$

From now on, let $k \in \{2, \ldots, |V(G \diamond \mathcal{H})|\}$. We finish our study by describing some lower and upper bounds on the Steiner k-eccentricities, Steiner k-radius and Steiner k-diameter of $G \diamond \mathcal{H}$.

Theorem 14. *Let $v \in V(G \diamond \mathcal{H})$. If $v \in V(G)$, then*

$$\mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v, G) \leq \mathrm{ecc}_k(v, G \diamond \mathcal{H}) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v, G) + \min\{k-1,\ |V(\mathcal{H})|\}.$$

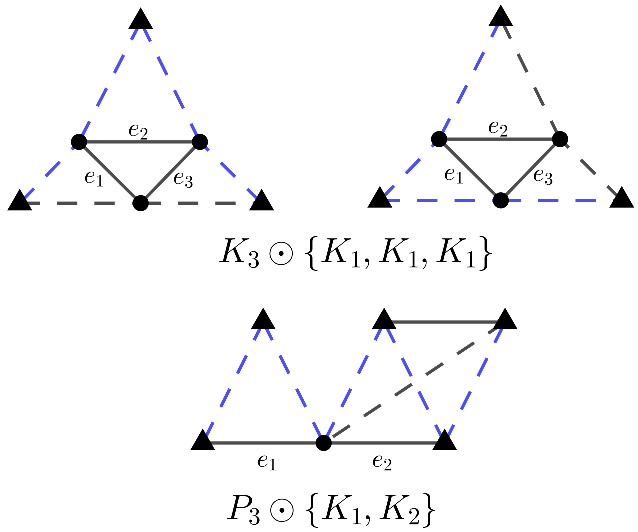


Figure 6: Steiner sets in GEC products.

Otherwise, if $v \in H_{v_1v_2}$ *for some* $v_1v_2 \in E(G)$ *such that*

$$\mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v_1, G) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v_2, G)$$

then

$$\mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v_1, G) \leq \mathrm{ecc}_k(v, G \diamond \mathcal{H}) \leq \mathrm{ecc}_{\min\{k,\,|V(G)|\}}(v_2, G) + \min\{k,\, |V(\mathcal{H})|\}.$$

Proof. The lower bounds hold similarly to the reasoning described in the proof of Theorem 8. Further, the upper bounds hold readily from the mentioned proposition and the fact that every graph in $\mathrm{GVC}(G \diamond \mathcal{H})$ is a connected spanning subgraph of $G \diamond \mathcal{H}$. □

Corollary 15. *It is verified that*

$$\mathrm{srad}_{\min\{k,\,|V(G)|\}}(G) \leq \mathrm{srad}_k(G \diamond \mathcal{H}) \leq \mathrm{srad}_{\min\{k,\,|V(G)|\}}(G) + \min\{k-1,\, |V(\mathcal{H})|\}$$

and

$$\mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G) \leq \mathrm{sdiam}_k(G \diamond \mathcal{H}) \leq \mathrm{sdiam}_{\min\{k,\,|V(G)|\}}(G) + \min\{k,\, |V(\mathcal{H})|\}.$$

Proof. The result follows readily from Theorem 14 once we take minimum and maximum values in the lower and upper bounds therein described. □

Unlike GVC products, the upper bound of the Steiner diameter described in Corollary 15 is not always reached for non-degenerate GEC products. The next result illustrates this fact.

Proposition 16. *If* $\mathrm{srad}_2(G) = 1$, *then*

$$\mathrm{sdiam}_k(G \diamond \mathcal{H}) = \min\{k,\, |V(G \diamond \mathcal{H})| - 1\}.$$

Proof. Let $v \in V(G)$ be such that $\mathrm{ecc}_2(v, G) = \mathrm{srad}_2(G) = 1$. Then, the vertex v is adjacent to any other vertex in G, and hence, to any other vertex in $G \diamond \mathcal{H}$. Thus, v is the center of the star graph $K_{1,|V(G\diamond\mathcal{H})|-1}$, which is a connected spanning subgraph of $G \diamond \mathcal{H}$. Hence,

$$\mathrm{sdiam}_k(G \diamond \mathcal{H}) \leq \mathrm{sdiam}_k(K_{1,|V(G\diamond\mathcal{H})|-1}) = \min\{k,\, |V(G \diamond \mathcal{H})| - 1\}.$$

If $k < |V(G\diamond\mathcal{H})|$, then this upper bound is reached for any k-subset of vertices in $G \diamond \mathcal{H}$ not containing the vertex v, but containing at least one vertex in $\min\{|E(G)|, k\}$ distinct components of the outer graph $\mathcal{H}$. If $k = |V(G \diamond \mathcal{H})|$, then the upper bound is readily reached. □

4 Conclusion and further work

This paper has delved into the study of Steiner distances in generalized vertex corona (GVC) and generalized edge corona (GEC) products. We have extended to GVC products some known results on corona products, and also some related parameters as the Steiner number, the Steiner diameter and the Steiner index of corona products. We have also established some sharp bounds for the Steiner eccentricities and Steiner radius of a GVC product.

Concerning GEC products, we have described how the computation of their Steiner distances derives from those ones in GVC products, and we have established some sharp bounds for the mentioned Steiner parameters. A more comprehensive study is required to deal with some other Steiner concepts and parameters in both GVC and GEC products, as the Steiner center, the Steiner median, the Steiner interval, the Steiner convexity, the Steiner distance hereditary graph, or the Steiner distance stable graph, amongst others. It is proposed as further work.

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