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# An integral formula for the coefficients of the inverse cyclotomic polynomial

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**Abstract**

Some recent advances related to an integral formula for the coefficients of inverse cyclotomic polynomials, including applications and numerical simulations are given.

## 1 Introduction

Cyclotomic polynomials play a key role in many areas of mathematics and they are linked to fundamental questions in algebra and number theory. More recently, in [21], inverse cyclotomic polynomials have been defined and investigated. In Section 1.1 we briefly review some properties of cyclotomic polynomials, focusing on the integral formula for coefficients (3) proved in [5], which we use to establish an upper bound. Some known properties of the inverse cyclotomic polynomials are mentioned in Section 1.2. The main result of the paper is the integral formula (13) for the coefficients, proved in Theorem 4. Various applications of this formula are discussed in Section 3 (including an upper bound for coefficients), while numerical simulations related to coefficients and some related integrals are presented in Section 4.

Key Words: cyclotomic polynomials, inverse cyclotomic polynomials, coefficients, integral formula, coefficients upper bounds.
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### 1.1 Cyclotomic polynomials

For an integer $n \geq 1$, an $n$-th root $\zeta$ of the unity is called primitive if $\zeta^n = 1$, while $\zeta^d \neq 1$ for all $1 \leq d < n$. Denoting by $\zeta_n = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ the first root of order $n$ of the unity, the $n$-th cyclotomic polynomial $\Phi_n$ is defined by

$$\Phi_n(z) = \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k,n)=1}} (z - \zeta_n^k) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} z^j, \tag{1}$$

where $\varphi$ is Euler's totient function, also the degree of the polynomial. It can be proved that $\Phi_n$ is monic and palindromic (i.e., $c_j^{(n)} = c_{\varphi(n)-j}^{(n)}$, $j = 0, \ldots, \varphi(n)$) and $c_0^{(n)} = c_{\varphi(n)}^{(n)} = 1$).

The origin of the terminology "cyclotomic" is rooted in the geometric property of the $n$-th roots of unity, which evenly divide the unit circle into $n$ equal arcs. These are the vertices of a regular polygon inscribed within the unit circle, illustrating the concept's underlying connection to both geometry and algebra. The explicit computation of the coefficients of $\Phi_n$ is very difficult (see, for instance, [9] or [16]), but many properties of these polynomials are known [10, 12, 13, 14, 17, 18, 19, 20, 23]. An explicit integral formula for the coefficients was established in the paper [5], using a special version of the Cauchy integral formula detailed in [4], while a recursive formula involving Ramanujan sums (see details in [15] and and [8]) was provided in [7]. By the same techniques, similar formulae have been obtained for the coefficients of Gaussian [1], polygonal [2], and other families of polynomials [3].

Consider the function

$$\Lambda_n(t) = \prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{2}$$

The following result was proved in [5].

**Theorem 1.** *For $n \geq 3$ the coefficients $c_j^{(n)}$ are given by the integral formula:*

$$c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \cos(\varphi(n) - 2j)t \, \mathrm{d}t, \quad j = 0, 1, \ldots, \varphi(n). \tag{3}$$

In the special case $2j = \varphi(n)$, the mid-term coefficient is obtained as

$$c_{\frac{\varphi(n)}{2}}^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \mathrm{d}t. \tag{4}$$

The mid-term coefficient has been intensively studied in many papers (see, e.g., [11]).

**Remark 2.** *Formula* (3) *can help us establish the following upper bound for the coefficients*

$$\left|c_j^{(n)}\right| \leq \frac{2^{\varphi(n)}}{\sqrt{2\pi}} \sqrt{\int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t}, \quad j = 0, 1, \ldots, \varphi(n). \tag{5}$$

*Indeed, by the Cauchy-Schwarz integral inequality we have*

$$\begin{aligned}
\left|c_j^{(n)}\right|^2 &= \frac{2^{2\varphi(n)}}{\pi^2} \left(\int_0^\pi \Lambda_n(t) \cdot \cos\left(\varphi(n) - 2j\right) t\,\mathrm{d}t\right)^2 \\
&\leq \frac{2^{2\varphi(n)}}{\pi^2} \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t \cdot \int_0^\pi \cos^2\left(\varphi(n) - 2j\right) t\,\mathrm{d}t \\
&= \frac{2^{2\varphi(n)}}{\pi^2} \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t \cdot \int_0^\pi \frac{1 + \cos 2\left(\varphi(n) - 2j\right) t}{2}\,\mathrm{d}t \\
&= \frac{2^{2\varphi(n)}}{\pi^2} \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t \left(\frac{\pi}{2} + \left.\frac{\sin 2\left(\varphi(n) - 2j\right) t}{4\left(\varphi(n) - 2j\right)}\right|_{t=0}^{t=\pi}\right) \\
&= \frac{2^{2\varphi(n)}}{2\pi} \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t.
\end{aligned}$$

*Taking the square root we obtain the desired result. In the calculations we used* $\cos^2 x = \frac{1+\cos 2x}{2}$ *to show that* $\int_0^\pi \cos^2\left(\varphi(n) - 2j\right) t\,\mathrm{d}t = \frac{\pi}{2}$.

Notice that the integral $\int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t$ played an important role in establishing the upper bound (5) for the coefficients of $\Phi_n$ and will be examined numerically in Section 4. Here we provide a lower bound.

**Remark 3.** *From* $c_0^{(n)} = 1$ *one obtains*

$$c_0^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cos\varphi(n)t\,\mathrm{d}t = 1, \tag{6}$$

*from where we obtain*

$$\begin{aligned}
\frac{\pi^2}{2^{2\varphi(n)}} &= \left(\int_0^\pi \Lambda_n(t) \cos\varphi(n)t\,\mathrm{d}t\right)^2 \\
&\leq \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t \cdot \int_0^\pi \cos^2\varphi(n)t\,\mathrm{d}t = \frac{\pi}{2} \cdot \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t,
\end{aligned}$$

*which gives the lower bound*

$$\frac{2\pi}{2^{2\varphi(n)}} \leq \int_0^\pi \Lambda_n^2(t)\,\mathrm{d}t. \tag{7}$$

The following formula is known for the alternate sums of the coefficients

$$\Phi_n(-1) = \begin{cases} -2 & \text{if } n = 1; \\ 0 & \text{if } n = 2; \\ p & \text{if } n = 2p^m; \\ 1 & \text{otherwise,} \end{cases} \tag{8}$$

where $p$ is prime and $m \geq 1$ is an integer.

Using (3) for $n \geq 3$, the following formula was established in [5]:

$$\Phi_n(-1) = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \frac{\cos(\varphi(n)+1)t}{\cos t} \, \mathrm{d}t. \tag{9}$$

### 1.2 The inverse cyclotomic polynomial $\Psi_n$

For an integer $n \geq 2$, the $n$-th inverse cyclotomic polynomial $\Psi_n(z)$ is defined by the formulae

$$\Psi_n(z) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n) > 1}} \left( z - e^{\frac{2k\pi i}{n}} \right) = \frac{x^n - 1}{\Phi_n(z)} = \sum_{j=0}^{n-\varphi(n)} d_j^{(n)} z^j. \tag{10}$$

The polynomial $\Psi_n(z)$ is of degree $n - \varphi(n)$ and integer coefficients, which are denoted here by $d_j^{(n)}$, $j = 0, 1, \dots, n - \varphi(n)$. Clearly, if $n$ is a prime, then $\Psi_n(z) = z - 1$. By formula (10), the first inverse cyclotomic polynomials are given by

$$\begin{aligned}
&\Psi_1(z) = 1,\ \Psi_4(z) = z^2 - 1,\ \Psi_6(z) = z^4 + z^3 - z - 1,\ \Psi_8(z) = z^4 - 1, \\
&\Psi_9(z) = z^3 - 1,\ \Psi_{10}(z) = z^6 + z^5 - z - 1,\ \Psi_{12}(z) = z^8 + z^6 - z^2 - 1, \\
&\Psi_{14}(z) = z^8 + z^7 - z - 1,\ \Psi_{15}(z) = z^7 + z^6 + z^5 - z^2 - z - 1, \\
&\Psi_{16}(z) = z^8 - 1,\ \Psi_{18}(z) = z^{12} + z^9 - z^3 - 1,\ \Psi_{20}(z) = z^{12} + z^{10} - z^2 - 1, \\
&\Psi_{21}(z) = z^9 + z^8 + z^7 - z^2 - z - 1,\ \Psi_{22}(z) = z^{12} + z^{11} - z^2 - 1.
\end{aligned}$$

We recall some known properties of $\Psi_n$. For proofs, see [6], [8], or [21].

1° If $n = p^\alpha$ with $p$ prime and $\alpha \geq 1$, then $\Psi_n(z) = z^{p^{\alpha-1}} - 1$.

2° For $n = p_1 \cdots p_k$ square-free, $\deg(\Psi_n) = p_1 \cdots p_k - (p_1 - 1) \cdots (p_k - 1)$.

3° If $p < q$ are primes, then for $n = pq$ one has

$$\Psi_n(z) = \frac{(z^p - 1)(z^q - 1)}{z - 1} = z^{p+q-1} + \cdots + z^{q+1} - z^{p-1} - \cdots - z^2 - z - 1.$$

4° If $p, q, r$ are different primes, then for $n = pqr$ one has

$$\Psi_n(z) = \frac{(z^{pq}-1)(z^{qr}-1)(z^{rp}-1)(z-1)}{(z^p-1)(z^q-1)(z^r-1)}.$$

5° $\Psi_{2n}(z) = (1-z^n)\,\Psi_n(-z)$, if $n$ is odd.
6° $\Psi_{pn}(z) = \Psi_n(z^p)$, if $p \mid n$.
7° $\Psi_{pn}(z) = \Psi_n(z^p)\Phi_n(z)$, if $p \nmid n$.
8° $\Psi_n$ is monic and antipalindromic, that is $d_j^{(n)} = -d_{n-\varphi(n)-j}^{(n)}$, $j = 0, \ldots, n-\varphi(n)$, and $d_0^{(n)} = -1$, while $d_{n-\varphi(n)}^{(n)} = 1$.
9° The number of positive coefficients of $\Psi_n$ is equal to the number of negative coefficients.

## 2 The integral formula for the coefficients of $\Psi_n$

In this section we derive an integral formula for the coefficients of the inverse cyclotomic polynomials. In the proof we use the identity

$$\sum_{\substack{1\le k\le n \\ \gcd(k,n)>1}} k = \frac{n}{2}\left(n+1-\varphi(n)\right), \quad n \ge 3, \tag{11}$$

which directly follows from the well-known formula

$$\sum_{\substack{1\le k\le n-1 \\ \gcd(k,n)=1}} k = \frac{n}{2}\varphi(n).$$

For a proof one can check, for example, [5] or our book [8].

In order to get a unitary formula for $d_j^{(n)}$, we introduce the function

$$\Gamma_n(t) = \prod_{\substack{1\le k\le n \\ \gcd(k,n)>1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{12}$$

Clearly, for $n = 1$, or when $n \ge 2$ is prime, one obtains

$$\Gamma_n(t) = \sin\left(t - \frac{n\pi}{n}\right) = -\sin t.$$

The first composite values are $n = 4$ and $n = 6$ where we have

$$\begin{aligned}\Gamma_4(t) &= \sin\left(t - \frac{2\pi}{4}\right)\sin\left(t - \frac{4\pi}{4}\right) = \sin\left(t - \frac{\pi}{2}\right)\sin\left(t - \pi\right)\\ &= \left(-\cos t\right)\left(-\sin t\right) = \sin t \cdot \cos t = \frac{1}{2}\sin 2t.\\ \Gamma_6(t) &= \sin\left(t - \frac{2\pi}{6}\right)\sin\left(t - \frac{3\pi}{6}\right)\sin\left(t - \frac{4\pi}{6}\right)\sin\left(t - \frac{6\pi}{6}\right)\\ &= \sin\left(t - \frac{\pi}{3}\right)\sin\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{2\pi}{3}\right)\sin\left(t - \pi\right)\\ &= \frac{1}{8}\left(\sin 2t + \sin 4t\right).\end{aligned}$$

In what follows we assume $n \geq 3$.

**Theorem 4.** *The coefficients $d_j^{(n)}$, $j = 0, 1, \ldots, n - \varphi(n)$, of the inverse cyclotomic polynomial $\Psi_n(z)$ are given by the following integral formula:*

$$d_j^{(n)} = (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^{\pi}\Gamma_n(t) \cdot \sin\left(n - \varphi(n) - 2j\right)t\,\mathrm{d}t. \tag{13}$$

*Proof.* Denoting $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and $z = \cos 2t + i\sin 2t$ for $t \in [0, 2\pi]$, $k = 1, \ldots, n$, the previous calculations showed that

$$z - \zeta_n^k = 2i\sin\left(t - \frac{k\pi}{n}\right)\left[\cos\left(t + \frac{k\pi}{n}\right) + i\sin\left(t + \frac{k\pi}{n}\right)\right].$$

To simplify the notation we shall use Euler's notation $\exp(it)$ for $\cos t + i\sin t$.

For $n \geq 3$, one can write the polynomial $\Psi_n(z)$ in the form

$$\begin{aligned}\Psi_n(z) &= \prod_{\substack{1\leq k\leq n\\ \gcd(k,n)>1}}\left(z - \zeta_n^k\right)\\ &= (2i)^{n-\varphi(n)}\prod_{\substack{1\leq k\leq n\\ \gcd(k,n)>1}}\sin\left(t - \frac{k\pi}{n}\right)\exp i\left(t + \frac{k\pi}{n}\right)\\ &= (2i)^{n-\varphi(n)}\Gamma_n(t)\exp i\left(\left[n - \varphi(n)\right]t + \left[n - \varphi(n) + 1\right]\frac{\pi}{2}\right)\\ &= (2i)^{n-\varphi(n)}\Gamma_n(t)\left[\cos\left(n - \varphi(n)t\right) + i\sin\left(n - \varphi(n)t\right)\right]\cdot i^{n-\varphi(n)+1}\\ &= 2^{n-\varphi(n)}(-1)^{n-\varphi(n)}\Gamma_n(t)\left[\cos\left(n - \varphi(n)t\right) + i\sin\left(n - \varphi(n)t\right)\right]\, i\\ &= (-2)^{n-\varphi(n)}\Gamma_n(t)\left[\cos\left(n - \varphi(n)t\right) + i\sin\left(n - \varphi(n)t\right)\right]\, i,\end{aligned}$$

where we have used that $\varphi(n)$ is even for $n \geq 3$, and that $\exp\left(i\frac{\pi}{2}\right) = i$.

For every $j = 0, 1, \ldots, \varphi(n)$, one may write

$$\begin{aligned}
d_j^{(n)} + \sum_{k \neq j} d_k^{(n)} z^{k-j} &= z^{-j} \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)>1}} (z - \zeta_n^k) \\
&= (-2)^{n-\varphi(n)} \Gamma_n(t) \left(\cos 2jt - i \sin 2jt\right) \left[\cos\left(n - \varphi(n)t\right) + i \sin\left(n - \varphi(n)t\right)\right] i \\
&= (-2)^{n-\varphi(n)} \Gamma_n(t) \left[\cos\left(n - \varphi(n) - 2j\right) t + i \sin\left(n - \varphi(n) - 2j\right) t\right] i \\
&= (-1)^{n+1} 2^{n-\varphi(n)} \Gamma_n(t) \left[\sin\left(n - \varphi(n) - 2j\right) t - i \cos\left(n - \varphi(n) - 2j\right) t\right].
\end{aligned}$$

Integrating on the interval $[0, 2\pi]$ we obtain formula (13). This is true since the integral of $z^{k-j}$ over $[0, 2\pi]$ vanishes whenever $k \neq j$. $\square$

In addition, from the proof of the integral formula (13) it follows that

$$\int_0^\pi \Gamma_n(t) \cos(n - \varphi(n) - 2j) t \, \mathrm{d}t = 0, \quad j = 0, 1, \ldots, n - \varphi(n). \tag{14}$$

# 3 Some applications of the integral formula

Using formula (13), we first establish an upper bound for the coefficients. Then we show that the inverse cyclotomic polynomial is antipalindromic, and we derive compact integral formulae for the mid-term coefficient, the direct, and alternate sums of coefficients of the inverse cyclotomic polynomials.

## 3.1 An upper bound for coefficients

An upper bound for the coefficients (13) can be obtained as follows.

**Theorem 5.** *For $n \geq 3$, the following inequality holds:*

$$\left|d_j^{(n)}\right| \leq \frac{2^{n-\varphi(n)}}{\sqrt{2\pi}} \sqrt{\int_0^\pi \Gamma_n^2(t) \, \mathrm{d}t}, \quad j = 0, 1, \ldots, n - \varphi(n). \tag{15}$$

*Proof.* By the Cauchy-Schwarz integral inequality we have

$$\begin{aligned}
\left|d_j^{(n)}\right|^2 &= \frac{2^{2(n-\varphi(n))}}{\pi^2}\left(\int_0^\pi \Gamma_n(t)\cdot\sin\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t\right)^2 \\
&\le \frac{2^{2(n-\varphi(n))}}{\pi^2}\int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t\cdot\int_0^\pi \sin^2\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t \\
&= \frac{2^{2(n-\varphi(n))}}{\pi^2}\int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t\cdot\int_0^\pi \frac{1-\cos 2\left(n-\varphi(n)-2j\right)t}{2}\,\mathrm{d}t \\
&= \frac{2^{2(n-\varphi(n))}}{\pi^2}\int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t\left(\frac{\pi}{2}-\left.\frac{\sin 2\left(n-\varphi(n)-2j\right)t}{4\left(n-\varphi(n)-2j\right)}\right|_{t=0}^{t=\pi}\right) \\
&= \frac{2^{2(n-\varphi(n))}}{2\pi}\int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t.
\end{aligned}$$

Taking the square root we obtain the desired result. In the calculations we used $\sin^2 x = \frac{1-\cos 2x}{2}$ to show that $\int_0^\pi \sin^2\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t = \frac{\pi}{2}$. □

**Remark 6.** *Since* $d_0^{(n)} = -1$, *one obtains the lower bound*

$$\frac{2\pi}{2^{2(n-\varphi(n))}} \le \int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t. \tag{16}$$

The integral $\int_0^\pi \Gamma_n^2(t)\,\mathrm{d}t$ played an important role in establishing the upper bound for the coefficients (15).

### 3.2 $\Psi_n(z)$ is antipalindromic.

Here we give a proof of this property based on the integral formula (13).

The coefficients of $\Psi_n(z)$ satisfy the following relation

$$d_j^{(n)} = -d_{n-\varphi(n)-j}^{(n)}, \quad j = 0,1,\ldots,n-\varphi(n). \tag{17}$$

Using formula (13), for every $j = 0,1,\ldots,\varphi(n)$, we have

$$\begin{aligned}
d_{n-\varphi(n)-j}^{(n)} &= (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(n-\varphi(n)-2(n-\varphi(n)-j)\right)t\,\mathrm{d}t \\
&= (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(2j+\varphi(n)-n\right)t\,\mathrm{d}t \\
&= -(-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t \\
&= -d_j^{(n)}.
\end{aligned}$$

This ends the proof.

### 3.3 The mid-term coefficients of $\Psi_n(z)$.

Recall that $\varphi(n)$ is even for $n \geq 3$. Hence, if $n$ is odd then $n-\varphi(n)$ is odd, and there is no middle coefficient. If $n$ is even, then the middle term coefficient, whose index is given by the formula $j = \dfrac{n-\varphi(n)}{2}$.

Since $n - \varphi(n) - 2j = 0$, from formula (13) we obtain

$$d^{(n)}_{\frac{n-\varphi(n)}{2}} = (-1)^{n+1} \frac{2^{n-\varphi(n)}}{\pi} \int_0^{\pi} \Gamma_n(t) \cdot 0 \, \mathrm{d}t = 0. \tag{18}$$

### 3.4 Direct and alternate sums of the coefficients.

From the definition of $\Psi_n(z)$, one obtains

$$\Psi_n(1) = \sum_{j=0}^{n-\varphi(n)} d_j^{(n)} = 0.$$

This can also be checked by formula (17), which gives

$$\Psi_n(1) = \sum_{j=0}^{n-\varphi(n)} d_j^{(n)} = \sum_{j=0}^{\lfloor \frac{n-\varphi(n)}{2} \rfloor} \left( d_j^{(n)} + d_{n-\varphi(n)-j}^{(n)} \right).$$

Indeed, when $n$ is odd this sum covers all the terms, while when $n$ is even the middle term obtained for $j = \frac{n-\varphi(n)}{2}$ is zero.

This is in contrast with the formula for $\Phi_n(1)$, given by

$$\Phi_n(1) = \begin{cases} 0 & \text{if } n = 1; \\ p & \text{if } n = p^m; \\ 1 & \text{otherwise,} \end{cases} \tag{19}$$

where $m \geq 1$ is an integer and $p$ is prime, labeled A014963 in [22].

By the definition of $\Psi_n(z)$ one has

$$\Psi_n(-1) = \frac{(-1)^n - 1}{\phi_n(-1)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \tag{20}$$

From the expression (13) for the coefficients $d_j^{(n)}$, we obtain an integral formula for $\Psi_n(-1)$, valid for $n \geq 3$, as follows

$$\begin{aligned}
\Psi_n(-1) &= \sum_{j=0}^{n-\varphi(n)} d_j^{(n)}(-1)^j \\
&= (-1)^{n+1} \sum_{j=0}^{n-\varphi(n)} \frac{2^{n-\varphi(n)}}{\pi} \int_0^\pi \Gamma_n(t)(-1)^j \sin(n-\varphi(n)-2j)t \,\mathrm{d}t, \\
&= (-1)^{n+1} \frac{2^{n-\varphi(n)}}{\pi} \int_0^\pi \Gamma_n(t) \left[ \sum_{j=0}^{n-\varphi(n)} (-1)^j \sin(n-\varphi(n)-2j)t \right] \mathrm{d}t \\
&= \left(1+(-1)^{n+1}\right) \frac{2^{n-\varphi(n)-1}}{\pi} \int_0^\pi \Gamma_n(t) \frac{\sin(n-\varphi(n)+1)t}{\cos t} \,\mathrm{d}t. \qquad (21)
\end{aligned}$$

To prove the last identity note that for a fixed $j \in \{0, \ldots, n-\varphi(n)\}$, one has

$$\begin{aligned}
(-1)^j \sin(n-\varphi(n)-2j)t &= \sin\left[(n-\varphi(n)-2j)t - j\pi\right] \\
&= \sin\left[(n-\varphi(n))\,t - 2j\left(t+\frac{\pi}{2}\right)\right].
\end{aligned}$$

Multiplying this expression by $2\sin\left(t+\frac{\pi}{2}\right)$ one obtains

$$\begin{aligned}
&2(-1)^j \sin(n-\varphi(n)-2j)t \cdot \sin\left(t+\frac{\pi}{2}\right) \\
&= \cos\left[nt - \varphi(n)t - (2j+1)\left(t+\frac{\pi}{2}\right)\right] \\
&- \cos\left[nt - \varphi(n)t - (2j-1)\left(t+\frac{\pi}{2}\right)\right].
\end{aligned}$$

Summing for $j \in \{0, \ldots, n-\varphi(n)\}$ we obtain the telescopic sum

$$\begin{aligned}
&\sum_{j=0}^{n-\varphi(n)} (-1)^j \cdot \sin(n-\varphi(n)-2j)t \cdot \sin\left(t+\frac{\pi}{2}\right) \\
&= -\frac{1}{2}\left[\cos\left((n-\varphi(n)+1)t+\frac{\pi}{2}\right)\right] \\
&+ \frac{1}{2}\left[\cos\left(-(n-\varphi(n)+1)t-\frac{\pi}{2}-(n-\varphi(n))\,\pi\right)\right] \\
&= \frac{1}{2}\left[\sin\left((n-\varphi(n)+1)t\right) - (-1)^{n-\varphi(n)} \sin\left((n-\varphi(n)+1)t\right)\right], \\
&= \frac{1+(-1)^{n-\varphi(n)+1}}{2} \cdot \frac{\sin(n-\varphi(n)+1)t}{\cos t},
\end{aligned}$$

where we used $\cos(x + k\pi) = (-1)^k \cos x,\ k \in \mathbb{Z},\ \cos(x + \pi/2) = -\sin x$, $\cos(x - \pi/2) = \sin x$. The result follows, as $\varphi(n)$ is even for $n \geq 3$.

If $n \geq 3$ is odd, the function $f : [0, \pi] \setminus \{\pi/2\} \mapsto \mathbb{R}$ defined by

$$f(t) = \frac{\sin(n - \varphi(n) + 1)t}{\cos t},$$

can be extended by continuity at the point $\pi/2$ as

$$\lim_{t \to \pi/2} f(t) = (-1)^{\frac{n-\varphi(n)-1}{2}} (n - \varphi(n) + 1)\,.$$

Hence, if $n \geq 3$ is odd, then

$$\int_0^\pi \Gamma_n(t) \frac{\sin(n - \varphi(n) + 1)t}{\cos t}\, \mathrm{d}t = \frac{\pi}{2^{n-\varphi(n)-1}}. \tag{22}$$

# 4 Numerical simulations

In this section, we explore the upper bounds of the cyclotomic and inverse cyclotomic polynomial coefficients and investigate some related integrals. As seen earlier, the functions $\Lambda_n(t)$ and $\Gamma_n(t)$ played an important role in the exact integral formulae for the coefficients $c_j^{(n)}$ of $\Phi_n$ and $d_j^{(n)}$ of $\Psi_n$, while and $\Lambda_n^2(t)$ and $\Gamma_n^2(t)$ featured in formulae for their upper bounds. The first few sequence terms ($n = 1, \ldots, 20$) are displayed in Table 1.

The diagrams are obtained in Matlab®, and the integrals are computed by the trapezium rule with 10000 equally spaced points in the interval $[0, \pi]$.

## 4.1 Upper bounds for polynomial coefficients

The upper bound formula (5) for the coefficients $c_j^{(n)}$ of $\Phi_n$, gives the sequence

$$ub(c_j^{(n)}) = \frac{2^{\varphi(n)}}{\sqrt{2\pi}} \sqrt{\int_0^\pi \Lambda_n^2(t)\, \mathrm{d}t}.$$

Similarly, the upper bound (15) of the coefficients $d_j^{(n)}$ of $\Psi_n$ defines

$$ub(d_j^{(n)}) = \frac{2^{n-\varphi(n)}}{\sqrt{2\pi}} \sqrt{\int_0^\pi \Gamma_n^2(t)\, \mathrm{d}t}.$$

These two sequences are plotted in Figure 1 (c).

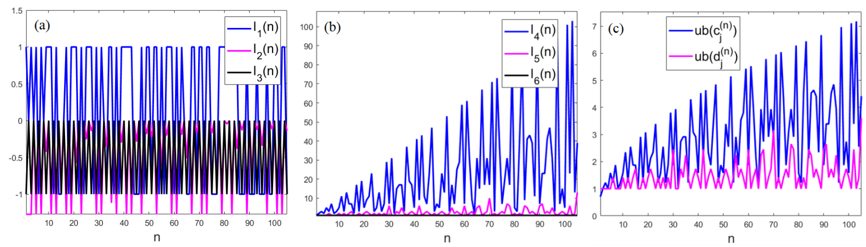

Figure 1: Terms $n = 1, \ldots, 105$ terms of the sequences (a) $I_1(n)$, $I_2(n)$, $I_3(n)$; (b) $I_4(n)$, $I_5(n)$, $I_6(n)$; (c) $ub(c_j^{(n)})$ and $ub(d_j^{(n)})$ .

### 4.2 The normalised integral of $\Lambda_n$

The function $\Lambda_n(t)$ defined in (2) appears in the formula (3) for the coefficients $c_j^{(n)}$, $j = 0, 1, \ldots, \varphi(n)$, of the polynomial $\Phi_n$. The normalised integral produces the sequence of middle term coefficients of $\Phi_n$, seen in (4)

$$I_1(n) = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t)\mathrm{d}t = c_{\frac{\varphi(n)}{2}}^{(n)}, \tag{23}$$

with the numerical values for $n \geq 2$ given by

$$0, 1, 0, 1, -1, 1, 0, 1, 1, \ldots,$$

indexed as A094754 in OEIS.

### 4.3 The normalised integral of $\Gamma_n$

The function $\Gamma_n(t)$ defined in (12) appears in formula (13) for the coefficients $d_j^{(n)}$, $j = 0, 1, \ldots, n - \varphi(n)$, of the polynomial $\Psi_n$. The normalised integral

$$I_2(n) = \frac{2^{n-\varphi(n)}}{\pi} \int_0^{\pi} \Gamma_n(t)\mathrm{d}t, \tag{24}$$

links to formula (18) (although the middle coefficient of $\Psi_n$ vanishes).

### 4.4 The normalised integral of $P_n = \Lambda_n \cdot \Gamma_n$

Consider the function

$$P_n(t) = \Lambda_n(t) \cdot \Gamma_n(t) = \prod_{1 \leq k \leq n} \sin\left(t - \frac{k\pi}{n}\right). \tag{25}$$

| $n$ | $I_1(n)$ | $I_2(n)$ | $I_3(n)$ | $I_4(n)$ | $I_5(n)$ | $I_6(n)$ | $ub(c_j^{(n)})$ | $ub(d_j^{(n)})$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1.273 | -1 | 1 | 1 | 1 | 0.707 | 1 |
| 2 | 0 | -1.273 | 0 | 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | -1.273 | -1 | 3 | 1 | 1 | 1.224 | 1 |
| 4 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| 5 | 1 | -1.273 | -1 | 5 | 1 | 1 | 1.581 | 1 |
| 6 | -1 | 0 | 0 | 3 | 2 | 1 | 1.224 | 1.414 |
| 7 | 1 | -1.273 | -1 | 7 | 1 | 1 | 1.870 | 1 |
| 8 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| 9 | 1 | -0.424 | -1 | 3 | 1 | 1 | 1.224 | 1 |
| 10 | 1 | 0 | 0 | 5 | 2 | 1 | 1.581 | 1.4142 |
| 11 | 1 | -1.273 | -1 | 11 | 1 | 1 | 2.345 | 1 |
| 12 | -1 | 0 | 0 | 3 | 2 | 1 | 1.224 | 1.414 |
| 13 | 1 | -1.273 | -1 | 13 | 1 | 1 | 2.549 | 1 |
| 14 | -1 | 0 | 0 | 7 | 2 | 1 | 1.870 | 1.4142 |
| 15 | -1 | -0.861 | -1 | 7 | 3 | 1 | 1.870 | 1.732 |
| 16 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| 17 | 1 | -1.273 | -1 | 17 | 1 | 1 | 2.915 | 1 |
| 18 | -1 | 0 | 0 | 3 | 2 | 1 | 1.224 | 1.414 |
| 19 | 1 | -1.273 | -1 | 19 | 1 | 1 | 3.082 | 1 |
| 20 | 1 | 0 | 0 | 5 | 2 | 1 | 1.581 | 1.414 |

Table 1: Values of the sequences $I_1(n)$, $I_2(n)$, $I_3(n)$, $I_4(n)$, $I_5(n)$, $I_6(n)$, $ub(c_j^{(n)})$ and $ub(d_j^{(n)})$, computed for $n = 1, \ldots, 20$.

It can be shown that

$$\int_0^\pi P_n(t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n\cdot 2^{n-2}} & \text{if } n \text{ is odd.} \end{cases} \tag{26}$$

In Figure 1 (a) we plot the normalized integral

$$I_3(n) = n2^{n-2} \cdot \int_0^\pi P_n(t)\,\mathrm{d}t. \tag{27}$$

### 4.5 The normalised integral of $\Lambda_n^2$

The integral of $\Lambda_n^2$ featured in the upper bound of the coefficients $c_j^{(n)}$ in (5). In Figure 1 we plot the normalized integral

$$I_4(n) = \frac{2^{2\varphi(n)}}{\pi} \int_0^\pi \Lambda_n^2(t)\mathrm{d}t. \tag{28}$$

The first few terms of this sequence are

$$2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, 7, 7, 2, 17, 3, \ldots,$$

which seem to correspond to A051664 in the OEIS, counting the number of nonzero coefficients in the $n$-th cyclotomic polynomial $\Phi_n$.

### 4.6 The normalised integral of $\Gamma_n^2$

The integral of $\Gamma_n^2$ featured in the upper bound of the coefficients $d_j^{(n)}$ in (15). From (16) we have that $I_5(n) \geq 1$. In Figure 1 we plot the normalized integral

$$I_5(n) = \frac{2^{2(n-\varphi(n))}}{2\pi} \int_0^\pi \Gamma_n^2(t)\mathrm{d}t. \tag{29}$$

The first few terms are

$$1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, \ldots,$$

seeming to indicate the sequence A001221 in OEIS, number of distinct primes dividing $n$, whose terms are half of those in sequence A034444 (representing the number of unitary divisors of $n$ (divisors $d$ of $n$, for which $\gcd(d, n/d) = 1$).

### 4.7 The normalised integral of $P_n^2$

It can be shown that

$$\int_0^\pi P_n^2(t)\,\mathrm{d}t = \int_0^\pi \prod_{1\leq k\leq n} \sin^2\left(t - \frac{k\pi}{n}\right)\mathrm{d}t = \frac{\pi}{2^{2n-1}}. \tag{30}$$

In Figure 1 we plot the normalized integral

$$I_6(n) = \frac{4^n}{2\pi} \int_0^\pi P_n^2(t)\,\mathrm{d}t. \tag{31}$$

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## References

[1] D. Andrica, O. Bagdasar, *A new formula for the coefficients of Gaussian polynomials*, An. Şt. Univ. Ovidius Constanţa **27**(3), 25–35 (2019)

[2] D. Andrica, O. Bagdasar, *On some results concerning the polygonal polynomials*, Carpathian J. Math., **35**(1), 1–12 (2019)

[3] D. Andrica, O. Bagdasar, *Some remarks on a general family of complex polynomials*, Appl. Anal. Discr. Math., **13**(2), 605–618 (2019)

[4] D. Andrica, O. Bagdasar, *Recurrent Sequences: Key Results, Applications and Problems*, Springer (2020)

[5] D. Andrica, O. Bagdasar, *On cyclotomic polynomial coefficients*, Malays. J. Math. Sci., in: Proceedings of “Groups, Group Rings, and Related Topics - 2017” (GGRRT 2017), 19–22 Nov 2017, Khorfakan, UAE **14**(3), 389–402 (2020)

[6] D. Andrica, O. Bagdasar, *Remarks on the coefficients of the inverse cyclotomic polynomials*, Mathematics **11**(17) (2023)

[7] D. Andrica, O. Bagdasar, *Some remarks on the coefficients of cyclotomic polynomials*. In: J. Guàrdia, N. Minculete, D. Savin, M. Vela, A. Zekhnini (eds.), New Frontiers in Number Theory and Applications, Trends in Mathematics, Springer (2023) (to appear)

[8] D. Andrica, O. Bagdasar, G.-C. Ţurcaş, *Topics on Discrete Mathematics and Combinatorics*, Cluj University Press, 318pp (2023)

[9] G. Bachman, *On the Coefficients of Cyclotomic Polynomials*, Mem. Amer. Math. Soc., Vol. **106**, no. 510 (1993)

[10] D. M. Bloom, *On the coefficients of the cyclotomic polynomials*, Amer. Math. Monthly, **75**, 372–377 (1968)

[11] G. P. Dresden, *On the middle coefficient of a cyclotomic polynomial* Amer. Math. Monthly, **111**(6), 531–533 (2004)

[12] M. Endo, *On the coefficients of the cyclotomic polynomials* Comment. Math. Univ. St. Pauli., **23**, 121–126 (1974/75)

[13] P. Erdös, *On the coefficients of the cyclotomic polynomial*, Bull. Amer. Math. Soc., **52**, 179–184 (1946)

[14] P. Erdös, R. C. Vaughan, *Bounds for r-th coefficients of cyclotomic polynomials*, J. London Math. Soc., **8**(2), 393–400 (1974)

[15] G. H. Hardy, P. V. Seshu Aiyar, B. M. Wilson (Eds), *Collected Papers of Srinivasa Ramanujan*, Cambridge University Press, 355pp (2016)

[16] A. Grytezuk, B. Tropak, *A numerical method for the determination of the cyclotomic polynomial coefficients*. In: A. Pethö et al (eds.), Computational Number Theory, Berlin: de Gruyter, 15–20 (1991)

[17] C. G. Ji, W. P. Li, *Values of coefficients of cyclotomic polynomials*, Discrete Math. **308**(23), 5860–5863 (2008)

[18] A. Kosyak, P. Moree, E. Sofos, B. Zhang, *Cyclotomic polynomials with prescribed height and prime number theory*, Mathematika, **67**, 214–234 (2021)

[19] E. Lehmer, *On the magnitude of the coefficients of the cyclotomic polynomial*, Bull. Amer. Math. Soc., **42**, 389–392 (1936)

[20] D. H. Lehmer, *Some properties of the cyclotomic polynomial*, J. Math. Anal. Appl., **42**(1), 105–117 (1966)

[21] P. Moree, *Inverse cyclotomic polynomials*, J. Number Theory, **129**, 667–680 (2009)

[22] The On-Line Encyclopedia of Integer Sequences, http://oeis.org, OEIS Foundation Inc. (2011)

[23] C. Sanna, *A survey on coefficients of cyclotomic polynomials*, Expositiones Mathematicae, **40**(3), 469–494 (2022)

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