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# Skew cyclic codes over $\mathbb{F}_4R$ and their applications to DNA codes construction

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**Abstract**

The fundamental aim of this research is to analyze the configuration of $\mathbb{F}_4R$ submodules, skew cyclic codes over $\mathbb{F}_4R$ and establish their connection with DNA codes, where $\mathbb{F}_4$ is a field of order 4 and $R = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4 + w\mathbb{F}_4$ with $u^2 = u,\ v^2 = v,\ w^2 = w,\ uv = vu = 0,\ vw = wv = 0,\ wu = uw = 0$ is a finite ring. This is achieved by examining particular subclasses like reversible codes. Ultimately, this study aims to utilize Gray maps to derive codes that possess the characteristics of DNA structures. At the end of this paper, we have provided the necessary and sufficient condition for skew cyclic codes to be reversible complement.

## 1 Introduction

Cyclic codes, a significant category of block codes have been researched for over fifty years. Various rings, including those referenced as [10, 14, 16, 19], have been used to investigate cyclic codes. Apart from cyclic and negacyclic codes, constacyclic and quasi-cyclic codes are generalizations within this field. Many coding theory articles employ the non-commutative ring, also known as the skew polynomial ring. One particular generalization of cyclic codes is the skew cyclic code, introduced by Boucher et al. in [8] using the skew polynomial ring. In addition, Ulmer et al. [9] focused on studying skew constacyclic codes utilizing the Galois ring. Irfan Siap et al. [18] examined the structure

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of skew cyclic codes of arbitrary length.

Furthermore, San Ling et al. [13] investigated skew constacyclic codes over the finite chain ring. Many authors studied skew cyclic codes over the rings $\mathbb{F}_2 + v\mathbb{F}_2,\ \mathbb{F}_q + v\mathbb{F}_q,\ \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, where $u^2 = u,\ v^2 = v,\ uv = vu = 0$ and $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u,\ v^2 = v,\ uv = vu$ in [2, 4, 12, 20]. In the beginning, specifically in 1997, Rifa et al. [17] established the concept of codes using a mixed alphabet. Subsequently, Borges et al. [6, 7] explored additive codes and additive cyclic codes over $\mathbb{Z}_2\mathbb{Z}_4$.

In [3], Adleman studies on DNA computing by solving an instance of an NP-complete problem over DNA molecules. A single DNA strand is a sequence of four possible nucleotides: adenine ($A$), guanine ($G$), cytosine ($C$) and thymine ($T$). DNA has two strands governed by the rule called Watson Crick complement (WCC), i.e., $A$ pairs with $T$ and $G$ pairs with $C$. We denote the WCC as $\overline{A} = T,\ \overline{T} = A,\ \overline{C} = G,\ \overline{G} = C$. The structure of DNA is used as a model for constructing good error-correcting codes. Conversely, error-correcting codes with similar properties to DNA structure are also used to understand DNA. Several papers have proposed different techniques to construct a set of DNA codeword. Several authors have also extensively used linear and cyclic codes to construct DNA codes.

There are various constraints that a DNA code must satisfy, such as the Hamming constraint for minimum distance, the reverse constraint, the reverse-complement constraint, the GC-content constraint, the melting temperature constraint, the thermodynamic constraint, and the uncorrelated-correlated constraint. The challenge for DNA code design is constructing a DNA code of a given length, size, and distance that satisfies the maximum set of constraints. Classical algebraic block codes have been extensively used to construct DNA codes. In this approach, a block code that satisfies the reverse-complement constraint is usually called a DNA code [15]. Among many methods of constructing DNA codes from classical codes is using skew cyclic codes over various fields and rings [5, 15]. In [5], the authors show how to construct DNA codes from skew cyclic codes over the mixed alphabet $\mathbb{F}_4(\mathbb{F}_4 + v\mathbb{F}_4)$, where $v^2 = v$. They state a condition on the associated generator polynomial of a skew cyclic code that guarantees the code to be a reversible complement. Further, Dertli et al. [11] investigated the utilization of skew cyclic codes for DNA codes over the mixed alphabet $\mathbb{F}_4(\mathbb{F}_4+u\mathbb{F}_4+v\mathbb{F}_4)$, where $u^2 = u,\ v^2 = v$. Motivated by this work, In this research article, we examine the application of skew cyclic codes over $\mathbb{F}_4(\mathbb{F}_4+u\mathbb{F}_4+v\mathbb{F}_4+w\mathbb{F}_4)$,where $u^2 = u,\ v^2 = v,\ w^2 = w$ to construct DNA codes.

## 2 Preliminaries

Let $\mathbb{F}_4$ be defined as the set $\{0, 1, \hbar, \hbar^2 = \hbar + 1\}$ be the field with order 4 and let $R = \{a + ub + vd + we : a, b, d, e \in \mathbb{F}_4\}$, where $u^2 = u$, $v^2 = v$, $w^2 = w$, $uv = vu = 0$, $vw = wv = 0$, $wu = uw = 0$ be the finite commutative ring with ideals $\langle 1 + u\rangle, \langle 1 + v\rangle, \langle 1 + w\rangle$ and $\langle u + v + w\rangle$. Let $\mu_1 = u, \mu_2 = v, \mu_3 = w$ and $\mu_4 = 1 + u + v + w$. Then, we can show that

$$\mu_\iota \mu_\jmath = \begin{cases} \mu_\iota; & \text{if } \iota = \jmath \\ 0; & \text{if } \iota \neq \jmath \end{cases}$$

and $\sum_{\iota=1}^{4} \mu_\iota = 1$. Therefore, we have $R = \mu_1 R \oplus \mu_2 R \oplus \mu_3 R \oplus \mu_4 R$ and $\mu_\iota R \cong \mu_\iota \mathbb{F}_4$ for $\iota \in \{1, \ldots, 4\}$. In other words, any element $x \in R$ can be uniquely expressed as $x = \sum_{\iota=1}^{4} \mu_\iota a_\iota$, where $a_\iota \in \mathbb{F}_4$ for $\iota \in \{1, \ldots, 4\}$. Now, the Gray map is defined as follows:

$$\phi : R \longrightarrow \mathbb{F}_4^4$$

$$a + ub + vd + we \longmapsto (a, b, a + d, d + e) \tag{1}$$

$$\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4 \longmapsto (a_1, a_1 + a_2, a_3, a_3 + a_4) \tag{2}$$

The Lee weight of $x \in R$ is defined as the Hamming weight of $\phi(x)$ denoted as $wt_L(x) = wt_H(\phi(x))$, where $wt_L$ and $wt_H$ denote the Lee weight and the Hamming weight, respectively. We can extend $\phi$ componentwise to $R^n$ as follows:

$$\phi : R^n \longrightarrow \mathbb{F}_4^{4n}$$

Let $x = (x_1, x_2, \ldots, x_n) \in R^n$, then $\phi(x) = (\phi(x_1), \phi(x_2), \ldots, \phi(x_n)) \in \mathbb{F}_4^{4n}$. Furthermore, $wt_L(x) = \sum_{\iota=1}^{n} wt_L(x_\iota) = \sum_{\iota=1}^{n} wt_H(\phi(x_\iota))$. The map $\phi$ serves as an isometry from $(R^n, d_L)$ to $(\mathbb{F}_4^{4n}, d_H)$. In other words, for any $x, y \in R^n$, $d_L(x, y) = d_H(\phi(x), \phi(y))$.

Throughout this article, the ring homomorphism $\theta$ on $R$ is defined as follows:

$$\theta : R \longrightarrow R$$

$$\theta(a + ub + vd + we) = a^2 + ub^2 + vd^2 + we^2 \tag{3}$$

$$\theta(\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4) = \mu_1 \theta(a_1) + \mu_2 \theta(a_2) + \mu_3 \theta(a_3) + \mu_4 \theta(a_4) \tag{4}$$

Note that, the order of the homomorphism $\theta$ is two and the subring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2$ remains fixed under $\theta$.

**Definition 2.1.** Let $\mathcal{A}_\iota(\iota = 1, 2)$ be codes over $R$. Then, its direct sum and the Plotkin sum are defined as follows:

$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \{(u_1 + u_2) \ : \ u_1 \in \mathcal{A}_1, \ u_2 \in \mathcal{A}_2\} \text{ and}$$

$$\mathcal{A}_1 \oplus_p \mathcal{A}_2 = \{(u_1, u_1 + u_2) \ : \ u_\iota \in \mathcal{A}_\iota, \ \iota = 1, 2\}.$$

**Definition 2.2.** Let $C$ be a linear code of length $n$ over $R$. Then we define

$$C_\iota = \{a_\iota \in \mathbb{F}_4^n \mid \exists \ a_\jmath \in \mathbb{F}_4^n, \ \jmath \neq \iota \mid \mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4 \in C\},$$

for $\iota, \jmath = 1, 2, 3, 4$. Clearly, $C_\iota$ for $\iota \in \{1, \ldots, 4\}$ is a linear code over $\mathbb{F}_4$, $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ and $|C| = |C_1||C_2||C_3||C_4|$.

**Lemma 2.1.** *[20] Let $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ be a linear code of length $n$ over $R$ and $G_\iota$ be the generator matrices of $C_\iota$ for $\iota \in \{1, \ldots, 4\}$, respectively. Then, the generator matrix of $C$ is*

$$G = \begin{pmatrix} \mu_1 G_1 \\ \mu_2 G_2 \\ \mu_3 G_3 \\ \mu_4 G_4 \end{pmatrix}.$$

**Lemma 2.2.** *Let $C$ be a linear code of length $n$ over $R$ with generator matrix $G$ as given in Lemma 2.1. Then, the generator matrix of $\phi(C)$ is*

$$\phi(G) = \begin{pmatrix} \phi(\mu_1 G_1) \\ \phi(\mu_2 G_2) \\ \phi(\mu_3 G_3) \\ \phi(\mu_4 G_4) \end{pmatrix} = \begin{pmatrix} G_1 & G_1 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_3 & G_3 \\ 0 & 0 & 0 & G_4 \end{pmatrix}.$$

*Moreover, $\phi(C) = (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$, where $\otimes$ and $\oplus_p$ stand for direct product and the Plotkin sum, respectively.*

## 3 Skew Cyclic Codes Over $R$

**Definition 3.1.** The set $R[x, \theta] = \{a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} : a_\iota \in R, \ 0 \leq \iota \leq n - 1, \ n \in \mathbb{N}\}$ of polynomials constitutes a ring referred to as a skew polynomial ring with the usual addition of polynomials and the multiplication is defined as follows: $(ax^r)(bx^s) = a\theta^r(b)x^{r+s}$, where $\theta^r$ is the composition of $\theta$(repeated r-times).

For an element $x = (x_1, x_2, \ldots, x_n) \in R^n$, the cyclic shift $T(x)$ and the skew cyclic shift $T_\theta(x)$ of $x$ are defined by $T(x) = (x_n, x_1, x_2, \ldots, x_{n-1})$ and $T_\theta(x) = (\theta(x_n), \theta(x_1), \ldots, \theta(x_{n-1}))$, respectively.

**Definition 3.2.** A linear code $C \subseteq R^n$ is said to be cyclic over $R$ if for any $x = (x_1, x_2, \ldots, x_n) \in C$, the cyclic shift $T(x) = (x_n, x_1, x_2, \ldots, x_{n-1}) \in C$ and $C$ is called a skew cyclic code over $R$ if for any $x = (x_1, x_2, \ldots, x_n) \in C$, the skew cyclic shift $T_\theta(x) = (\theta(x_n), \theta(x_1), \ldots, \theta(x_{n-1})) \in C$.

**Theorem 3.3.** *Let $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ be a linear code over $R$, where $C_\iota$ is a linear code over $\mathbb{F}_4$ for each $\iota \in \{1, \ldots, 4\}$. Then, $C$ is a skew cyclic code over $R$ if and only if $C_\iota$ is a skew cyclic code over $\mathbb{F}_4$ for $\iota \in \{1, \ldots, 4\}$.*

*Proof.* Suppose that $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ is a linear code over $R$ and $C_\iota$ is a linear code over $\mathbb{F}_4$ for $\iota \in \{1, \ldots, 4\}$. Let $x = (x_1, x_2, \ldots, x_n)$ be any codeword in $C$, where $x_\iota = \mu_1 a_\iota + \mu_2 b_\iota + \mu_3 d_\iota + \mu_4 e_\iota \in R$, $a_\iota$, $b_\iota$, $c_\iota$, and $d_\iota$ belongs to $\mathbb{F}_4$ for $1 \leq \iota \leq n$. Let $a = (a_1, a_2, \ldots, a_n) \in C_1$, $b = (b_1, b_2, \ldots, b_n) \in C_2$, $d = (d_1, d_2, \ldots, d_n) \in C_3$ and $e = (e_1, e_2, \ldots, e_n) \in C_4$. Then, we have $x = \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e$. If $C_\iota$ for $\iota \in \{1, \ldots, 4\}$ is a skew cyclic code over $\mathbb{F}_4$, then skew cyclic shifts $T_\theta(a) = (\theta(a_n), \theta(a_1), \ldots, \theta(a_{n-1})) \in C_1$, $T_\theta(b) = (\theta(b_n), \theta(b_1), \ldots, \theta(b_{n-1})) \in C_2$, $T_\theta(d) = (\theta(d_n), \theta(d_1), \ldots, \theta(d_{n-1})) \in C_3$ and $T_\theta(e) = (\theta(e_n), \theta(e_1), \ldots, \theta(e_{n-1})) \in C_4$. Therefore, we have $T_\theta(x) = (\theta(x_n), \theta(x_1), \ldots, \theta(x_{n-1})) = \mu_1 T_\theta(a) + \mu_2 T_\theta(b) + \mu_3 T_\theta(d) + \mu_4 T_\theta(e) \in C$. Hence, $C$ is a skew cyclic code over $R$.

Conversely, assume that $C$ is a skew cyclic code, then for any codeword $x = (x_1, x_2, \ldots x_n)$ in $C$, its skew cyclic shift is $T_\theta(x) = (\theta(x_n), \theta(x_1), \ldots, \theta(x_{n-1})) = \mu_1 T_\theta(a) + \mu_2 T_\theta(b) + \mu_3 T_\theta(d) + \mu_4 T_\theta(e) \in C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$. This implies that, $T_\theta(a) \in C_1$, $T_\theta(b) \in C_2$, $T_\theta(d) \in C_3$ and $T_\theta(e) \in C_4$. Hence, $C_\iota$ is a skew cyclic code over $\mathbb{F}_4$ for $\iota \in \{1, \ldots, 4\}$. □

**Theorem 3.4.** *[20] Let $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ be a skew cyclic code of length $n$ over $R$. If $g_\iota(x)$ is a generator polynomial of skew cyclic code $C_\iota$ for $\iota \in \{1, \ldots, 4\}$ over $\mathbb{F}_4$, respectively. Then, $C = \langle \mu_1 g_1(x), \mu_2 g_2(x), \mu_3 g_3(x), \mu_4 g_4(x) \rangle$ and $|C| = 4^{4n - \sum_{\iota=1}^{4} \deg(g_\iota(x))}$. Furthermore, $C = \langle g(x) \rangle$, where $g(x) = \sum_{\iota=1}^{4} \mu_\iota g_\iota(x) \in R[x, \theta]$ is unique and $g(x) | (x^n - 1)$.*

**Theorem 3.5.** *Let $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ be a skew cyclic code of length $n$ over $R$, where $C_\iota$ is a skew cyclic code with parameters $[n, k_\iota, d_\iota]$ for $\iota \in \{1, \ldots, 4\}$, respectively. Then, $\Phi(C) = (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$. Moreover, $\Phi(C)$ is a code with parameters $[4n, k_1 + k_2 + k_3 + k_4, min\{2d_1, d_2, 2d_3, d_4\}]$.*

*Proof.* Assume that $C = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ is a skew cyclic code over $R$. Additionally, let $\Phi : R \longrightarrow \mathbb{F}_4^4$ be a Gray map defined as $\Phi(\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4) = (a_1, a_1 + a_2, a_3, a_3 + a_4)$. To establish the result, consider $x \in \Phi(C)$. Then $x = \Phi(y)$ for some $y = \mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4 \in C$, where $a_\iota \in C_\iota$ for $\iota \in \{1, \ldots, 4\}$. Thus, we have $x = (a_1, a_1 + a_2, a_3, a_3 + a_4) \in (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$. Consequently, $\Phi(C) \subseteq (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$.

Conversely, assume that $x = (b_1, b_1 + b_2, b_3, b_3 + b_4)$ is an element in $(C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$, where $b_\iota \in C_\iota$ for $\iota \in \{1, \ldots, 4\}$. Then, there exist a $y = \mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3 + \mu_4 b_4 \in C$ such that $\Phi(y) = x$. Thus, we have $(C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4) \subseteq \Phi(C)$. Moreover, by the definition of direct product and the Plotkin sum if $C_\iota$ is a code with parameters $[n, k_\iota, d_\iota]$ for $\iota \in \{1, \ldots, 4\}$ over $\mathbb{F}_4$, respectively. Then $\Phi(C)$ is a code with parameters $[4n, k_1 + k_2 + k_3 + k_4, min\{2d_1, d_2, 2d_3, d_4\}]$. □

**Example 3.1.** Suppose that $n = 6$ then $x^6 - 1 = (x^2 + 1)(x^2 + \hbar)(x^2 + \hbar^2) \in \mathbb{F}_4[x, \theta]$. Let $C_1 = \langle x^2 + 1 \rangle$, and $C_2 = C_3 = C_4 = \langle x^2 + \hbar^2 \rangle$ be skew cyclic codes with parameters $[6, 4, 2]$ over $\mathbb{F}_4$. Assume that $g(x) = \mu_1 g_1(x) + \mu_2 g_2(x) + \mu_3 g_3(x) + \mu_4 g_4(x) = x^2 + 1 + \hbar(u + v + w)$, then $C = \langle g(x) \rangle$ is a skew cyclic code, and the Gray image $\Phi(C)$ is a code with parameters $[24, 16, 2]$ over $\mathbb{F}_4$.

## 4 Generator polynomials of skew cyclic codes over $\mathbb{F}_4R$

The polynomial representation of an element $p = (a_0, a_1, \ldots, a_{\gamma-1}, b_0, b_1, \ldots, b_{\delta-1}) \in \mathbb{F}_4^\gamma R^\delta$ is $p(x) = (a(x), b(x))$, also denoted as $(a(x)|b(x))$, where $a(x) = a_0 + a_1 x + \cdots + a_{\gamma-1} x^{\gamma-1} \in \frac{\mathbb{F}_4[x]}{(x^\gamma - 1)}$, and $b(x) = b_0 + b_1 x + \cdots + b_{\delta-1} x^{\delta-1} \in \frac{R[x,\theta]}{(x^\delta - 1)}$. Consequently, there is a one-to-one correspondence between $\mathbb{F}_4^\gamma R^\delta$ and $R_{\gamma,\delta} = \frac{\mathbb{F}_4[x]}{(x^\gamma - 1)} \times \frac{R[x,\theta]}{(x^\delta - 1)}$.

Let $\mathbb{F}_4R = \{(a, b) \ : \ a \in \mathbb{F}_4, b \in R\}$. Define a ring homomorphism

$$\eta : R \longrightarrow \mathbb{F}_4$$

$$a + ub + vc + wd \longmapsto a \tag{5}$$

Under the multiplication operation defined as $r \cdot (a, b) = (\eta(r)a, rb)$, the set $\mathbb{F}_4R$ is an $R$-module, where $r \in R$, $\eta(r)a$ represents multiplication in $\mathbb{F}_4$ and $rb$ signifies multiplication in $R$.

Consider the set $\mathbb{F}_4^\gamma R^\delta = \{(a_1, a_2, \ldots, a_\gamma | b_1, b_2, \ldots, b_\delta) \ : \ a_\iota \in \mathbb{F}_4, \ b_\jmath \in R, \ 1 \le \iota \le \gamma, \ 1 \le \jmath \le \delta\}$. Then, for any $r \in R$ and $p = (a_1, a_2, \ldots, a_\gamma | b_1, b_2, \ldots, b_\delta) \in \mathbb{F}_4^\gamma R^\delta$, we can extend the multiplication operation as follows:

$$r \cdot p = (\eta(r)a_1, \eta(r)a_2, \ldots, \eta(r)a_\gamma | rb_1, rb_2, \ldots, rb_\delta). \tag{6}$$

With this operation, the set $\mathbb{F}_4^\gamma R^\delta$ is an $R$-module. The $\gamma\delta$-cyclic shift of an element $p \in \mathbb{F}_4^\gamma R^\delta$ is defined as ${}^{\gamma\delta}T(p) = (a_\gamma, a_1, \ldots, a_{\gamma-1} | b_\delta, b_1, \ldots, b_{\delta-1})$. The $\gamma\delta$-skew cyclic shift of an element $p \in \mathbb{F}_4^\gamma R^\delta$ is defined as ${}^{\gamma\delta}T_\theta(p) = (a_\gamma, a_1, \ldots, a_{\gamma-1} | \theta(b_\delta), \theta(b_1), \ldots, \theta(b_{\delta-1}))$.

**Definition 4.1.** Let $C \subseteq \mathbb{F}_4^\gamma R^\delta$. Then (*i*) $C$ is said to be an $\mathbb{F}_4R$-linear code with a block length $(\gamma,\delta)$, if it is an $R$-submodule of $\mathbb{F}_4^{\gamma}R^{\delta}$.

(*ii*) $C$ is said to be an $\mathbb{F}_4R$-cyclic code with a block length $(\gamma,\delta)$, if ${}^{\gamma\delta}T(C)=C$, where ${}^{\gamma\delta}T$ is a $\gamma\delta$-cyclic shift.

(*iii*) $C$ is said to be an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma,\delta)$, if ${}^{\gamma\delta}T_\theta(C)=C$, where ${}^{\gamma\delta}T_\theta$ is a $\gamma\delta$ skew cyclic shift.

**Theorem 4.2.** *An $\mathbb{F}_4R$-linear code $C$ with a block length $(\gamma,\delta)$ is an $\mathbb{F}_4R$-skew cyclic code, if and only if it is a left $R[x,\theta]$-submodule of $\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}\times\frac{R[x,\theta]}{(x^\delta-1)}$.*

*Proof.* Suppose that $C$ is an $\mathbb{F}_4R$-skew cyclic code. Assume that $p(x)=(p_1(x)|p_2(x))$ is an element in $C$, where $p_1(x)=a_0+a_1x+\cdots+a_{\gamma-1}x^{\gamma-1}\in\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$, and $p_2(x)=b_0+b_1x+\cdots+b_{\delta-1}x^{\delta-1}\in\frac{R[x,\theta]}{(x^\delta-1)}$. Here $p(x)$ is identified with the codeword $p=(a_0,a_1,\dots,a_{\gamma-1}|b_0,b_1,\dots,b_{\delta-1})\in C$. Now, for any positive integer $\jmath$, the polynomial $x^\jmath p(x)=(a_{\gamma-\jmath}+a_{\gamma-\jmath+1}x+\cdots+a_{\gamma-\jmath-1}x^{\gamma-1}|\theta^\jmath(b_{\delta-\jmath})+\theta^\jmath(b_{\delta-\jmath+1})x+\cdots+\theta^\jmath(b_{\delta-\jmath-1})x^{\delta-1})$ belongs to $C$, which can be identified with the vector $(a_{\gamma-\jmath},a_{\gamma-\jmath+1},\dots,a_{\gamma-\jmath-1}|\theta^\jmath(b_{\delta-\jmath}),\theta^\jmath(b_{\delta-\jmath+1}),\dots,\theta^\jmath(b_{\delta-\jmath-1}))\in C$. Let $r(x)$ be any polynomial in $R[x,\theta]$ and $p(x)$ be any codeword in $C$. Then, by the $\mathbb{F}_4R$-linearity of $C$, we have $r(x)\cdot p(x)\in C$. Thus, $C$ is a left $R[x,\theta]$-submodule of $\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}\times\frac{R[x,\theta]}{(x^\delta-1)}$.

Conversely, assume that $C$ is a left $R[x,\theta]$-submodule of $R_{\gamma,\delta}$. Then $r(x)\cdot p(x)\in C$ for any polynomial $r(x)\in R[x,\theta]$ and a codeword $p(x)\in C$. In particular, $x\cdot p(x)\in C$, where $x\cdot p(x)=(a_{\gamma-1}+a_0x+\cdots+a_{\gamma-2}x^{\gamma-1}|\theta(b_{\delta-1})+\theta(b_0)x+\cdots+\theta(b_{\delta-2})x^{\delta-1})$, can be identified with the codeword $(a_{\gamma-1},a_0,\dots,a_{\gamma-2}|\theta(b_{\delta-1}),\theta(b_0),\dots,\theta(b_{\delta-2}))\in C$. Hence, $C$ is an $\mathbb{F}_4R$-skew cyclic code. □

Assume that $C$ is an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma,\delta)$ and let $p(x)=(p_1(x)|p_2(x))$ represent any codeword within $C$. Consequently, we proceed to define the projection maps $\Pi_1$ and $\Pi_2$ on $R_{\gamma,\delta}$ as follows:

$$\Pi_1: R_{\gamma,\delta}\longrightarrow\frac{\mathbb{F}_4[x]}{(x^\gamma-1)},$$

$$(p_1(x)|p_2(x))\longmapsto p_1(x)\text{ and}$$

$$\Pi_2: R_{\gamma,\delta}\longrightarrow\frac{R[x,\theta]}{(x^\delta-1)},$$

$$(p_1(x)|p_2(x))\longmapsto p_2(x).$$

The set $C_\gamma=\{a(x)\in\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}\mid(a(x),0)\in C\}$ is an ideal of $\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$. Therefore, a cyclic code of length $\gamma$ over $\mathbb{F}_4$ is generated by $f(x)$ (say) such that $f(x)|(x^\gamma-1)$. Similarly, the set $C_\delta = \{b(x) \in \frac{R[x,\theta]}{(x^\delta-1)} :$ there exists $h(x) \in \frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$, $(h(x), b(x)) \in C\}$ is a left $R[x,\theta]$-submodule of $\frac{R[x,\theta]}{(x^\delta-1)}$ is generated by $g(x)$ (say) such that $g(x)|(x^\delta-1)$. Therefore, $C_\delta$ is a skew cyclic code over $R$. By Theorem 3.4, $g(x) = \sum_{\iota=1}^{4} \mu_\iota g_\iota(x)$. Thus, we have the following result.

**Lemma 4.1.** *[11] Let $C$ be an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$. Then, $\Pi_1(C)$ is a cyclic code of length $\gamma$ over $\mathbb{F}_4$ and $\Pi_2(C)$ is a skew cyclic code of length $\delta$ over $R$.*

**Theorem 4.3.** *Let $C$ be an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$ and $C_\delta$ has a non-zero polynomial $g(x)$ of the lowest degree with a unit leading coefficient. Then $C = \langle (f(x), 0), (h(x), g(x)) \rangle$, where $h(x) \in \frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$, $C_\gamma = \langle f(x) \rangle$, where $f(x)|(x^\gamma-1)$ and $C_\delta = \langle g(x) \rangle$, where $g(x)|(x^\delta-1)$.*

*Proof.* Suppose that $C$ is an $\mathbb{F}_4R$-skew cyclic code with a block length of $(\gamma, \delta)$, such that $C_\gamma = \langle f(x) \rangle$, where $f(x)|(x^\gamma-1)$ and $C_\delta = \langle g(x) \rangle$, where $g(x)|(x^\delta-1)$ and $g(x)$ is a non-zero polynomial of the lowest degree with a unit leading coefficient. Now, consider an arbitrary codeword $p(x) = (p_1(x)|p_2(x)) \in C$. It can be expressed as

$$p(x) = (p_1(x), 0) + (0, p_2(x)) = (q(x)f(x), 0) + (0, r(x)g(x)),$$

for some $q(x) \in \frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$ and $r(x) \in \frac{R[x,\theta]}{(x^\delta-1)}$. Let $h(x)$ be a member of $\frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$ such that $(\eta(r(x))h(x)|r(x)g(x)) \in C$, then

$$\begin{aligned} p(x) &= (q(x)f(x), 0) + (\eta(r(x))h(x)|r(x)g(x)) + (\eta(r(x))h(x), 0) \\ &= (q(x)f(x) + \eta(r(x))h(x), 0) + (\eta(r(x))h(x)|r(x)g(x)) \\ &= t(x)(f(x), 0) + r(x)(h(x)|g(x)), \end{aligned}$$

where $t(x) \in \frac{\mathbb{F}_4[x]}{(x^\gamma-1)}$ and $q(x)f(x) + \eta(r(x))h(x)$ is a member of $C_\gamma$. Therefore, $C \subseteq \big\langle (f(x), 0), (h(x)|b(x)) \big\rangle$. Conversely, as $(f(x), 0)$ and $(h(x)|b(x))$ belongs to $C$. So, we have $\langle (f(x), 0), (h(x)|b(x)) \rangle \subseteq C$. Hence, $C = \langle (f(x), 0), (h(x)|b(x)) \rangle$. ☐

Two outcomes concerning skew cyclic codes over the ring $\mathbb{F}_4(\mathbb{F}_4+u\mathbb{F}_4+v\mathbb{F}_4)$ hold valid in the expanded ring $\mathbb{F}_4(\mathbb{F}_4+u\mathbb{F}_4+v\mathbb{F}_4+w\mathbb{F}_4)$ as well.

**Theorem 4.4.** *An $\mathbb{F}_4R$-skew cyclic code $C$ with a block length $(\gamma, \delta)$ is equivalent to an $\mathbb{F}_4R$-cyclic code, provided both $\gamma$ and $\delta$ are odd integers.*

**Theorem 4.5.** *An $\mathbb{F}_4R$-skew cyclic code $C$ with a block length $(\gamma, \delta)$ is equivalent to an $\mathbb{F}_4R$-quasi-cyclic code of index 2 provided both $\gamma$ and $\delta$ are even integers.*

**Theorem 4.6.** *An $\mathbb{F}_4R$-skew cyclic code $C$ with a block length $(\gamma, \delta)$, where $\gamma$ and $\delta$ are multiple of some positive integer $k$ is equivalent to an $\mathbb{F}_4R$-quasi-cyclic code with index $k$.*

*Proof.* Suppose that $C$ is an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$, where $\gamma = km$ and $\delta = kn$ for $k, m, n \in \mathbb{Z}^+$. Assume that $\alpha = lcm(\gamma, \delta)$, then $\alpha$ is a multiple of positive integer $k$ with $gcd(\alpha, k) = k$. Consequently, there exist integers $l_1$ and $l_2$, such that $\alpha l_1 + kl_2 = k \implies kl_2 = k + \alpha D$ for some $D \geq 0$ and $D \equiv -l_1 (mod \alpha)$. Let $c = (a_{1,1}, \ldots, a_{1,k}, \ldots, a_{n,1}, \ldots, a_{n,k} | b_{1,1}, \ldots, b_{1,k}, \ \ldots, b_{m,1}, \ldots, b_{m,k})$ be any codeword in $C$. If ${}^{\gamma\delta}T_\theta(c)$ represents the $\gamma\delta$-skew cyclic shift of $c$, then ${}^{\gamma\delta}T_{\theta^\alpha}(c) = c$ and ${}^{\gamma\delta}T_{\theta^{\alpha D}}(c) = c$ for any $c \in C$. Consider

$$\begin{aligned} {}^{\gamma\delta}T_{\theta^{k+\alpha D}}(c) =& {}^{\gamma\delta}T_{\theta^{\alpha D}}(a_{n,1}, \ldots, a_{n,k}, a_{1,1}, \ldots, a_{1,k}, \ldots, a_{n-1,1}, \ldots, a_{n-1,k} | \\ & b_{m,1}, \ldots, b_{m,k}, b_{1,1}, \ldots, b_{1,k}, \ldots, b_{m-1,1}, \ldots, b_{m-1,k}) \\ =& (a_{n,1}, \ldots, a_{n,k}, a_{1,1}, \ldots, a_{1,k}, \ldots, a_{n-1,1}, \ldots, a_{n-1,k} | \\ & b_{m,1}, \ldots, b_{m,k}, b_{1,1}, \ldots, b_{1,k}, \ldots, b_{m-1,1}, \ldots, b_{m-1,k}). \end{aligned}$$

Since ${}^{\gamma\delta}T_{\theta^{k+\alpha D}}(c) = {}^{\gamma\delta}T_{\theta^k}(c)$ for arbitrary $c \in \mathbb{F}_4^\gamma R^\delta$. Consequently, $C$ is equivalent to an $\mathbb{F}_4R$-quasi-cyclic code with a block length $(\gamma, \delta)$ and index $k$. ☐

**Example 4.1.** For $n = 4$, we have $x^4 - 1 = (x+1)^4 \in \mathbb{F}_4[x, \theta]$. Assume that $f(x) = (x+1)$ and $C_0 = \langle f(x) \rangle$ be the skew cyclic code with parameter $[4, 3, 2]$ over $\mathbb{F}_4$. Also, for $n = 6$, we have $x^6 - 1 = (x+1)^2(x+\hbar)^2(x+\hbar^2)^2 \in \mathbb{F}_4[x, \theta]$. Let $C_1 = \langle x+1 \rangle$, $C_2 = C_3 = C_4 = \langle x + \hbar^2 \rangle$ be skew cyclic codes with parameters $[6, 5, 2]$ over $\mathbb{F}_4$. Let $g(x) = \mu_1 g_1(x) + \mu_2 g_2(x) + \mu_3 g_3(x) + \mu_4 g_4(x) = x + 1 + \hbar(u + v + w)$, then the code $C = \langle g(x) \rangle$ is a skew cyclic code of length 6 over $R$. Therefore, the code $C = \langle (f(x, 0)), (0, g(x)) \rangle$ is an $\mathbb{F}_4R$-skew cyclic code with a block length $(4, 6)$, equivalent to an $\mathbb{F}_4R$-quasi-cyclic code of block length $(4, 6)$ with index 2. Moreover, the Gray image $\Phi(C)$ is a code with parameters $[28, 23, 2]$.

## 5 The Gray Map

The map $\phi : R \longrightarrow \mathbb{F}_4^4$ defined as $\phi(a + ub + vd + we) = (a, b, a + d, d + e)$ can be extended to a map $\phi^* : \mathbb{F}_4R \longrightarrow \mathbb{F}_4^5$, where $\phi^*(x, y) = (x, \phi(y)) = (x, a, b, a + d, d + e)$. Here, $x \in \mathbb{F}_4$ and $y = a + ub + vd + we \in R$. This extended map $\phi^*$ further can be expanded to $\mathbb{F}_4^\gamma R^\delta$ as follows:

$$\Phi : \mathbb{F}_4^\gamma R^\delta \longrightarrow \mathbb{F}_4^{\gamma + 4\delta}$$

$$(X, Y) \longmapsto (X, \phi(Y)),$$

where $X = (x_0, x_1, \ldots, x_{\gamma-1}) \in \mathbb{F}_4^\gamma$ and $Y = \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e = (\mu_1 a_0 + \mu_2 b_0 + \mu_3 d_0 + \mu_4 e_0, \ldots, \mu_1 a_{\delta-1} + \mu_2 b_{\delta-1} + \mu_3 d_{\delta-1} + \mu_4 e_{\delta-1}) \in R^\delta$. For any $(X, Y) \in \mathbb{F}_4^\gamma R^\delta$, its Gray weight is defined as $wt_G(X, Y) = wt_H(X) + wt_L(Y)$, where $wt_H(X)$ represents the Hamming weight of $X$ and $wt_L(Y)$ represents the Lee weight of $Y$.

Assume that $C$ is an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$. Consider

$$\begin{aligned}
C_0 &= \{X \in \mathbb{F}_4^\gamma \mid (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C \mid a, b, d, e \in \mathbb{F}_4^\delta\},\\
C_1 &= \{a \in \mathbb{F}_4^\delta \mid (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C \mid X \in \mathbb{F}_4^\gamma,\ b, d, e \in \mathbb{F}_4^\delta\},\\
C_2 &= \{b \in \mathbb{F}_4^\delta \mid (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C \mid X \in \mathbb{F}_4^\gamma,\ a, d, e \in \mathbb{F}_4^\delta\},\\
C_3 &= \{d \in \mathbb{F}_4^\delta \mid (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C \mid X \in \mathbb{F}_4^\gamma,\ a, b, e \in \mathbb{F}_4^\delta\},\\
C_4 &= \{e \in \mathbb{F}_4^\delta \mid (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C \mid X \in \mathbb{F}_4^\gamma,\ a, b, d \in \mathbb{F}_4^\delta\}.
\end{aligned}$$

**Lemma 5.1.** *Let $C$ be an $\mathbb{F}_4R$-skew cyclic code of block length $(\gamma, \delta)$. Then, $\Phi(C) = C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$ and $|\Phi(C)| = \prod_{\iota=0}^4 |C_\iota|$.*

*Proof.* Suppose that $C$ is an $\mathbb{F}_4R$-skew cyclic code of block length $(\gamma, \delta)$ and the Gray map $\Phi : \mathbb{F}_4^\gamma R^\delta \longrightarrow \mathbb{F}_4^{\gamma+4\delta}$ as defined above. Let $u \in \Phi(C)$, then $u = \Phi(v)$ for some $v = (X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e) \in C$. So $u = (X, a, a+b, d, d+e)$, which implies that $u \in C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$. Therefore, $\Phi(C) \subseteq C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$.

Conversely, for any $u \in C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$, we have $u = (X, a, a+b, d, d+e) = \Phi(X, \mu_1 a + \mu_2 b + \mu_3 d + \mu_4 e)$, where $X \in C_0$, $a \in C_1$, $b \in C_2$, $d \in C_3$, $e \in C_4$. Hence, $u \in \Phi(C)$ implies that $C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4) \subseteq \Phi(C)$. Finally, we conclude that $\Phi(C) = C_0 \otimes (C_1 \oplus_p C_2) \otimes (C_3 \oplus_p C_4)$ and $|\Phi(C)| = \prod_{\iota=0}^4 |C_\iota|$. □

**Theorem 5.1.** *Let $C$ be an $\mathbb{F}_4R$-skew cyclic code of block length $(\gamma, \delta)$ over $R$. Then, $C_0$ is a cyclic code of length $\gamma$ over $\mathbb{F}_4$ and $C_\iota$ for $\iota \in \{1, \ldots, 4\}$ is a skew cyclic code of length $\delta$ over $\mathbb{F}_4$.*

*Proof.* Suppose that $C$ is an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$ and $\Pi_\iota (\iota = 1, 2)$ are projection maps as defined above. Then, by Lemma 4.1, $\Pi_1(C) = C_0$ is a cyclic code over $\mathbb{F}_4$ and $\Pi_2(C) = \mu_1 C_1 \oplus \mu_2 C_2 \oplus \mu_3 C_3 \oplus \mu_4 C_4$ is a skew cyclic code over $R$. So by Theorem 3.3 $C_\iota$ for $\iota \in \{1, 2, 3, 4\}$ is a skew cyclic code over $\mathbb{F}_4$. □

## 6 DNA (Deoxyribonucleic acid) Codes Over $\mathbb{F}_4R$

DNA has emerged as a potential medium for data storage and computation in recent years due to its remarkable properties, such as high storage capacity, longevity, and data density. These properties have sparked interest in developing DNA code encoding schemes that allow digital data representation using DNA sequences. Moreover, the concept of DNA codes is not limited to data storage but extends to error-correction coding and cryptography. Beyond its role in biology, DNA has also inspired researchers in various fields, including computer science and information theory.

Here, skew cyclic codes over $R$ and $\mathbb{F}_4R$ are provided with necessary and sufficient conditions to be a reversible complement. Let $C$ be a DNA code and $x = (x_1, x_2, \ldots, x_n)$ be any codeword in $C$. Then, $x^r = (x_n, x_{n-1}, \ldots, x_1)$, is the reverse of $x$, $x^c = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ is the complement of $x$ and $x^{rc} = (\overline{x_n}, \overline{x_{n-1}}, \ldots, \overline{x_1})$ is the reverse complement of $x$. The fundamental building blocks of DNA structure are the set of nucleotides $\Sigma = \{A, T, C, G\}$, which satisfies the Watson-Crick complement rule ($\overline{A} = T$, $\overline{C} = G$) and vice-versa. For example, $ACCTAG$ is connected with $TGGATC$.

Let $C$ be a DNA code with parameters $[n, M, d]$, then the constraints on the Hamming distance $wt_H(x, y) \geq d$ and $wt_H(x^r, y^c) \geq d$ for all $x, y \in C$ are put in place. When constructing DNA codes using algebraic techniques, rings and fields of order 4 and $4^k$ are utilised because the DNA alphabet has a size of 4. Abualrub et al. [1] examined the $\mathbb{F}_4$-DNA codes by employing the bijection between the set of DNA alphabets $\Sigma$ and $\mathbb{F}_4$, such as $A, T, C$ and $G$ are mapped to $0, 1, \hbar$ and $\hbar^2$, respectively. Benbelkacem et al. [5] extended this bijection to a bijection from $\mathbb{F}_4 + v\mathbb{F}_4$ to the DNA codons in $\Sigma^2$ and Dertli et al. [11] from $\mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4$ to the DNA codons in $\Sigma^3$.

Now we define a bijection between the elements of $R = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4 + w\mathbb{F}_4$ to the DNA codons in $\Sigma^4 = \{A, T, C, G\}^4$ by $\phi(a + ub + vd + we) = (a, b, a + d, d + e)$. This bijection is defined in the table below.

| $r \in R$ | codon | $r \in R$ | codon | $r \in R$ | codon | $r \in R$ | codon |
|---|---|---|---|---|---|---|---|
| $0$ | AAAA | v | AATT | $v\hbar$ | AACC | $v\hbar^2$ | AAGG |
| $1$ | TATA | $1+v$ | TAAT | $1+v\hbar$ | TAGC | $1+v\hbar^2$ | TACG |
| $\hbar$ | CACA | $\hbar+v$ | CAGT | $\hbar+v\hbar$ | CAAC | $\hbar+v\hbar^2$ | CATG |
| $\hbar^2$ | GAGA | $\hbar^2+v$ | GACT | $\hbar^2+v\hbar$ | GATC | $\hbar^2+v\hbar^2$ | GAAG |
| $u$ | ATAA | $u+v$ | ATTT | $u+v\hbar$ | ATCC | $u+v\hbar^2$ | ATGG |
| $1+u$ | TTTA | $1+u+v$ | TTAT | $1+u+v\hbar$ | TTGC | $1+u+v\hbar^2$ | TTCG |
| $\hbar+u$ | CTCA | $\hbar+u+v$ | CTGT | $\hbar+u+v\hbar$ | CTAC | $\hbar+u+v\hbar^2$ | CTTG |
| $\hbar^2+u$ | GTGA | $\hbar^2+u+v$ | GTCT | $\hbar^2+u+v\hbar$ | GTTC | $\hbar^2+u+v\hbar^2$ | GTAG |
| $u\hbar$ | ACAA | $u\hbar+v$ | ACTT | $u\hbar+v\hbar$ | ACCC | $u\hbar+v\hbar^2$ | ACGG |
| $1+u\hbar$ | TCTA | $1+u\hbar+v$ | TCAT | $1+u\hbar+v\hbar$ | TCGC | $1+u\hbar+v\hbar^2$ | TCCG |
| $\hbar+u\hbar$ | CCCA | $\hbar+u\hbar+v$ | CCGT | $\hbar+u\hbar+v\hbar$ | CCAC | $\hbar+u\hbar+v\hbar^2$ | CCTG |
| $\hbar^2+u\hbar$ | GCGA | $\hbar^2+u\hbar+v$ | GCCT | $\hbar^2+u\hbar+v\hbar$ | GCTC | $\hbar^2+u\hbar+v\hbar^2$ | GCAG |
| $u\hbar^2$ | AGAA | $u\hbar^2+v$ | AGTT | $u\hbar^2+v\hbar$ | AGCC | $u\hbar^2+v\hbar^2$ | AGGG |
| $1+u\hbar^2$ | TGTA | $1+u\hbar^2+v$ | TGAT | $1+u\hbar^2+v\hbar$ | TGGC | $1+u\hbar^2+v\hbar^2$ | TGCG |
| $\hbar+u\hbar^2$ | CGCA | $\hbar+u\hbar^2+v$ | CGGT | $\hbar+u\hbar^2+v\hbar$ | CGAC | $\hbar+u\hbar^2+v\hbar^2$ | CGTG |
| $\hbar^2+u\hbar^2$ | GGGA | $\hbar^2+u\hbar^2+v$ | GGCT | $\hbar^2+u\hbar^2+v\hbar$ | GGTC | $\hbar^2+u\hbar^2+v\hbar^2$ | GGAG |
| $w$ | AAAT | $v+w$ | AATA | $v\hbar+w$ | AACG | $v\hbar^2+w$ | AAGC |
| $1+w$ | TATT | $1+v+w$ | TAAA | $1+v\hbar+w$ | TAGG | $1+v\hbar^2+w$ | TACC |
| $\hbar+w$ | CACT | $\hbar+v+w$ | CAGA | $\hbar+v\hbar+w$ | CAAG | $\hbar+v\hbar^2+w$ | CATC |
| $\hbar^2+w$ | GAGT | $\hbar^2+v+w$ | GACA | $\hbar^2+v\hbar+w$ | GATG | $\hbar^2+v\hbar^2+w$ | GAAC |
| $u+w$ | ATAT | $u+v+w$ | ATTA | $u+v\hbar+w$ | ATCG | $u+v\hbar^2+w$ | ATGC |
| $1+u+w$ | TTTT | $1+u+v+w$ | TTAA | $1+u+v\hbar+w$ | TTGG | $1+u+v\hbar^2+w$ | TTCC |
| $\hbar+u+w$ | CTCT | $\hbar+u+v+w$ | CTGA | $\hbar+u+v\hbar+w$ | CTAG | $\hbar+u+v\hbar^2+w$ | CTTC |
| $\hbar^2+u+w$ | GTGT | $\hbar^2+u+v+w$ | GTCA | $\hbar^2+u+v\hbar+w$ | GTTG | $\hbar^2+u+v\hbar^2+w$ | GTAC |
| $u\hbar+w$ | ACAT | $u\hbar+v+w$ | ACTA | $u\hbar+v\hbar+w$ | ACCG | $u\hbar+v\hbar^2+w$ | ACGC |
| $1+u\hbar+w$ | TCTT | $1+u\hbar+v+w$ | TCAA | $1+u\hbar+v\hbar+w$ | TCGG | $1+u\hbar+v\hbar^2+w$ | TCCC |
| $\hbar+u\hbar+w$ | CCCT | $\hbar+u\hbar+v+w$ | CCGA | $\hbar+u\hbar+v\hbar+w$ | CCAG | $\hbar+u\hbar+v\hbar^2+w$ | CCTC |
| $\hbar^2+u\hbar+w$ | GCGT | $\hbar^2+u\hbar+v+w$ | GCCA | $\hbar^2+u\hbar+v\hbar+w$ | GCTG | $\hbar^2+u\hbar+v\hbar^2+w$ | GCAC |
| $u\hbar^2+w$ | AGAT | $u\hbar^2+v+w$ | AGTA | $u\hbar^2+v\hbar+w$ | AGCG | $u\hbar^2+v\hbar^2+w$ | AGGC |
| $1+u\hbar^2+w$ | TGTT | $1+u\hbar^2+v+w$ | TGAA | $1+u\hbar^2+v\hbar+w$ | TGGG | $1+u\hbar^2+v\hbar^2+w$ | TGCC |
| $\hbar+u\hbar^2+w$ | CGCT | $\hbar+u\hbar^2+v+w$ | CGGA | $\hbar+u\hbar^2+v\hbar+w$ | CGAG | $\hbar+u\hbar^2+v\hbar^2+w$ | CGTC |
| $\hbar^2+u\hbar^2+w$ | GGGT | $\hbar^2+u\hbar^2+v+w$ | GGCA | $\hbar^2+u\hbar^2+v\hbar+w$ | GGTG | $\hbar^2+u\hbar^2+v\hbar^2+w$ | GGAC |
| $w\hbar$ | AAAC | $v+w\hbar$ | AATG | $v\hbar+w\hbar$ | AACA | $v\hbar^2+w\hbar$ | AAGT |
| $1+w\hbar$ | TATC | $1+v+w\hbar$ | TAAG | $1+v\hbar+w\hbar$ | TAGA | $1+v\hbar^2+w\hbar$ | TACT |
| $\hbar+w\hbar$ | CACC | $\hbar+v+w\hbar$ | CAGG | $\hbar+v\hbar+w\hbar$ | CAAA | $\hbar+v\hbar^2+w\hbar$ | CATT |
| $\hbar^2+w\hbar$ | GAGC | $\hbar^2+v+w\hbar$ | GACG | $\hbar^2+v\hbar+w\hbar$ | GATA | $\hbar^2+v\hbar^2+w\hbar$ | GAAT |
| $u+w\hbar$ | ATAC | $u+v+w\hbar$ | ATTG | $u+v\hbar+w\hbar$ | ATCA | $u+v\hbar^2+w\hbar$ | ATGT |
| $1+u+w\hbar$ | TTTC | $1+u+v+w\hbar$ | TTAG | $1+u+v\hbar+w\hbar$ | TTGA | $1+u+v\hbar^2+w\hbar$ | TTCT |
| $\hbar+u+w\hbar$ | CTCC | $\hbar+u+v+w\hbar$ | CTGG | $\hbar+u+v\hbar+w\hbar$ | CTAA | $\hbar+u+v\hbar^2+w\hbar$ | CTTT |
| $\hbar^2+u+w\hbar$ | GTGC | $\hbar^2+u+v+w\hbar$ | GTCG | $\hbar^2+u+v\hbar+w\hbar$ | GTTA | $\hbar^2+u+v\hbar^2+w\hbar$ | GTAT |
| $u\hbar+w\hbar$ | ACAC | $u\hbar+v+w\hbar$ | ACTG | $u\hbar+v\hbar+w\hbar$ | ACCA | $u\hbar+v\hbar^2+w\hbar$ | ACGT |
| $1+u\hbar+w\hbar$ | TCTC | $1+u\hbar+v+w\hbar$ | TCAG | $1+u\hbar+v\hbar+w\hbar$ | TCGA | $1+u\hbar+v\hbar^2+w\hbar$ | TCCT |
| $\hbar+u\hbar+w\hbar$ | CCCC | $\hbar+u\hbar+v+w\hbar$ | CCGG | $\hbar+u\hbar+v\hbar+w\hbar$ | CCAA | $\hbar+u\hbar+v\hbar^2+w\hbar$ | CCTT |
| $\hbar^2+u\hbar+w\hbar$ | GCGC | $\hbar^2+u\hbar+v+w\hbar$ | GCCG | $\hbar^2+u\hbar+v\hbar+w\hbar$ | GCTA | $\hbar^2+u\hbar+v\hbar^2+w\hbar$ | GCAT |
| $u\hbar^2+w\hbar$ | AGAC | $u\hbar^2+v+w\hbar$ | AGTG | $u\hbar^2+v\hbar+w\hbar$ | AGCA | $u\hbar^2+v\hbar^2+w\hbar$ | AGGT |
| $1+u\hbar^2+w\hbar$ | TGTC | $1+u\hbar^2+v+w\hbar$ | TGAG | $1+u\hbar^2+v\hbar+w\hbar$ | TGGA | $1+u\hbar^2+v\hbar^2+w\hbar$ | TGCT |
| $\hbar+u\hbar^2+w\hbar$ | CGCC | $\hbar+u\hbar^2+v+w\hbar$ | CGGG | $\hbar+u\hbar^2+v\hbar+w\hbar$ | CGAA | $\hbar+u\hbar^2+v\hbar^2+w\hbar$ | CGTT |
| $\hbar^2+u\hbar^2+w\hbar$ | GGGC | $\hbar^2+u\hbar^2+v+w\hbar$ | GGCG | $\hbar^2+u\hbar^2+v\hbar+w\hbar$ | GGTA | $\hbar^2+u\hbar^2+v\hbar^2+w\hbar$ | GGAT |
| $w\hbar^2$ | AAAG | $v+w\hbar^2$ | AATC | $v\hbar+w\hbar^2$ | AACT | $v\hbar^2+w\hbar^2$ | AAGA |
| $1+w\hbar^2$ | TATG | $1+v+w\hbar^2$ | TAAC | $1+v\hbar+w\hbar^2$ | TAGT | $1+v\hbar^2+w\hbar^2$ | TACA |
| $\hbar+w\hbar^2$ | CACG | $\hbar+v+w\hbar^2$ | CAGC | $\hbar+v\hbar+w\hbar^2$ | CAAT | $\hbar+v\hbar^2+w\hbar^2$ | CATA |
| $\hbar^2+w\hbar^2$ | GAGG | $\hbar^2+v+w\hbar^2$ | GACC | $\hbar^2+v\hbar+w\hbar^2$ | GATT | $\hbar^2+v\hbar^2+w\hbar^2$ | GAAA |
| $u+w\hbar^2$ | ATAG | $u+v+w\hbar^2$ | ATTC | $u+v\hbar+w\hbar^2$ | ATCT | $u+v\hbar^2+w\hbar^2$ | ATGA |
| $1+u+w\hbar^2$ | TTTG | $1+u+v+w\hbar^2$ | TTAC | $1+u+v\hbar+w\hbar^2$ | TTGT | $1+u+v\hbar^2+w\hbar^2$ | TTCA |
| $\hbar+u+w\hbar^2$ | CTCG | $\hbar+u+v+w\hbar^2$ | CTGC | $\hbar+u+v\hbar+w\hbar^2$ | CTAT | $\hbar+u+v\hbar^2+w\hbar^2$ | CTTA |
| $\hbar^2+u+w\hbar^2$ | GTGG | $\hbar^2+u+v+w\hbar^2$ | GTCC | $\hbar^2+u+v\hbar+w\hbar^2$ | GTTT | $\hbar^2+u+v\hbar^2+w\hbar^2$ | GTAA |
| $u\hbar+w\hbar^2$ | ACAG | $u\hbar+v+w\hbar^2$ | ACTC | $u\hbar+v\hbar+w\hbar^2$ | ACCT | $u\hbar+v\hbar^2+w\hbar^2$ | ACGA |
| $1+u\hbar+w\hbar^2$ | TCTG | $1+u\hbar+v+w\hbar^2$ | TCAC | $1+u\hbar+v\hbar+w\hbar^2$ | TCGT | $1+u\hbar+v\hbar^2+w\hbar^2$ | TCCA |
| $\hbar+u\hbar+w\hbar^2$ | CCCG | $\hbar+u\hbar+v+w\hbar^2$ | CCGC | $\hbar+u\hbar+v\hbar+w\hbar^2$ | CCAT | $\hbar+u\hbar+v\hbar^2+w\hbar^2$ | CCTA |
| $\hbar^2+u\hbar+w\hbar^2$ | GCGG | $\hbar^2+u\hbar+v+w\hbar^2$ | GCCC | $\hbar^2+u\hbar+v\hbar+w\hbar^2$ | GCTT | $\hbar^2+u\hbar+v\hbar^2+w\hbar^2$ | GCAA |
| $u\hbar^2+w\hbar^2$ | AGAG | $u\hbar^2+v+w\hbar^2$ | AGTC | $u\hbar^2+v\hbar+w\hbar^2$ | AGCT | $u\hbar^2+v\hbar^2+w\hbar^2$ | AGGA |
| $1+u\hbar^2+w\hbar^2$ | TGTG | $1+u\hbar^2+v+w\hbar^2$ | TGAC | $1+u\hbar^2+v\hbar+w\hbar^2$ | TGGT | $1+u\hbar^2+v\hbar^2+w\hbar^2$ | TGCA |
| $\hbar+u\hbar^2+w\hbar^2$ | CGCG | $\hbar+u\hbar^2+v+w\hbar^2$ | CGGC | $\hbar+u\hbar^2+v\hbar+w\hbar^2$ | CGAT | $\hbar+u\hbar^2+v\hbar^2+w\hbar^2$ | CGTA |
| $\hbar^2+u\hbar^2+w\hbar^2$ | GGGG | $\hbar^2+u\hbar^2+v+w\hbar^2$ | GGCC | $\hbar^2+u\hbar^2+v\hbar+w\hbar^2$ | GGTT | $\hbar^2+u\hbar^2+v\hbar^2+w\hbar^2$ | GGAA |

An element $y \in R$ is referred to as the complement of $x \in R$ if $\phi(y)$ is the complement of $\phi(x)$ in $\mathbb{F}_4^4$. Let $x = a + ub + vd + we \in R$ with $a, b, d, e \in \mathbb{F}_4$. Then $x^c$ is given by

$$\overline{x} = x + 1 + u + w = a + 1 + u(b + 1) + vd + w(e + 1).$$

**Lemma 6.1.** *If $r, r_1, r_2 \in R$, then the following results hold:*

1. $\overline{r_1 + r_2} = r_1 + r_2 + 1 + u + w = \overline{r_1} + \overline{r_2} + 1 + u + w,$
2. $\overline{ru} = ru + 1 + u + w = \overline{r}u + 1 + u + w,$
3. $\overline{rv} = rv + 1 + u + w = \overline{r}v + 1 + u + w,$
4. $\overline{rw} = rw + 1 + u + w = \overline{r}w + 1 + u + w,$
5. $\overline{r(1 + u + v + w)} = r(1 + u + v + w) + 1 + u + w = \overline{r}(1 + u + v + w) + v.$

**Definition 6.1.** An $R$-linear code $C$ of length $\delta$ over $R$ is said to be a DNA-skew cyclic code if $C$ is an $R$ skew cyclic code of length $\delta$, and for any codeword $x \in C$, $x \neq x^{rc}$ with the reverse complement $x^{rc} \in C$. A code $C$ is called a reversible complement code if $x^{rc} \in C$, for any codeword $x \in C$.

For any polynomial $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ with non-zero leading coefficient, its reciprocal is defined as $f^*(x) = x^{n-1}f(1/x) = a_{n-1} + a_{n-2}x + \cdots + a_1x^{n-2} + a_0x^{n-1}$. Note that, $\deg(f^*(x)) \leq \deg(f(x))$ depend on the constant term of $f(x)$. The polynomial $f(x)$ is referred to as self-reciprocal provided $f^*(x) = f(x)$.

**Lemma 6.2.** *Let $p_1(x)$ and $p_2(x)$ be any two polynomials over $R$ satisfying the condition $\deg(p_1(x)) \geq \deg(p_2(x))$. Then,*

1. $(p_1(x) \cdot p_2(x))^* = p_1^*(x) \cdot p_2^*(x),$
2. $(p_1(x) + p_2(x))^* = p_1^*(x) + x^{\deg(p_1(x)) - \deg(p_2(x))}p_2^*(x).$

**Theorem 6.2.** *Let $C = \langle g(x) \rangle$ be an $R$-skew cyclic code of length $\delta$. Then, $C$ is reversible complement if and only if $(1 + u + w)(\frac{x^\delta - 1}{x - 1}) \in C$ and $g(x)$ is a self-reciprocal polynomial.*

*Proof.* Suppose that $C = \langle g(x) \rangle$ is an $R$-skew cyclic code of length $\delta$, where $g(x) = ug_1(x) + vg_2(x) + wg_3(x) + (1 + u + v + w)g_4(x)$. The monic polynomial $g_\iota(x)$ divides $(x^\delta - 1)$ in $\mathbb{F}_4[x]$ for $\iota \in \{1, \ldots, 4\}$. Assume that $C$ is a reversible complement code, then $\mathbf{0} = (0, 0, \ldots, 0) \in C$ implies that, its complement $\overline{\mathbf{0}} = (\overline{0}, \overline{0}, \ldots, \overline{0}) \in C$. Thus, we have the corresponding polynomial

$$\begin{aligned}\overline{\mathbf{0}} &= (1+u+w, 1+u+w, \ldots, 1+u+w)\\ &= (1+u+w)(1, 1, \ldots, 1)\\ &\equiv (1+u+w)(1+x+x^2+\cdots+x^{\delta-1})\\ &\equiv (1+u+w)(\frac{x^\delta - 1}{x-1}) \in C.\end{aligned}$$

Let $g_1(x) = a_0+a_1x+\cdots+a_{r-1}x^{r-1}+x^r$, $g_2(x) = b_0+b_1x+\cdots+b_{s-1}x^{s-1}+x^s$, $g_3(x) = c_0+c_1x+\cdots+c_{t-1}x^{t-1}+x^t$, and $g_4(x) = d_0+d_1x+\cdots+d_{k-1}x^{k-1}+x^k$, where $r \leq s \leq t \leq k$. Assume that $A_\iota = ua_\iota+vb_\iota+wc_\iota+(1+u+v+w)d_\iota$ for $0 \leq \iota \leq r$, $B_\iota = vb_\iota + wc_\iota + (1+u+v+w)d_\iota$ for $r+1 \leq \iota \leq s$, $C_\iota = wc_\iota + (1+u+v+w)d_\iota$ for $s+1 \leq \iota \leq t$ and $D_\iota = (1+u+v+w)d_\iota$ for $t+1 \leq \iota \leq k$. Then

$$\begin{aligned}g(x) &= ug_1(x) + vg_2(x) + wg_3(x) + (1+u+v+w)g_4(x)\\ &= \sum_{\iota=0}^{r} A_\iota x^\iota + \sum_{\iota=r+1}^{s} B_\iota x^\iota + \sum_{\iota=s+1}^{t} C_\iota x^\iota + \sum_{\iota=t+1}^{k} D_\iota x^\iota + 0x^{k+1} + \ldots + 0x^{\delta-1}.\end{aligned}$$

Since $C$ is a reversible complement code and $g(x) \in C$. Thus the reverse complement $g(x)^{rc}$ becomes a member of $C$, where

$$\begin{aligned}g(x)^{rc} =&(1+u+w)(1+x+\cdots+x^{\delta-k-2}) + \sum_{\iota=t+1}^{k} \overline{D_\iota} x^{\delta-\iota-1} + \sum_{\iota=s+1}^{t} \overline{C_\iota} x^{\delta-\iota-1}\\ &+ \sum_{\iota=r+1}^{s} \overline{B_\iota} x^{\delta-\iota-1} + \sum_{\iota=0}^{r} \overline{A_\iota} x^{\delta-\iota-1}\\ =&(1+u+w)(1+x+\cdots+x^{\delta-k-2}) + \sum_{\iota=t+1}^{k} (D_\iota + 1+u+w) x^{\delta-\iota-1}\\ &+ \sum_{\iota=s+1}^{t} (C_\iota + 1+u+w) x^{\delta-\iota-1} + \sum_{\iota=r+1}^{s} (B_\iota + 1+u+w) x^{\delta-\iota-1}\\ &+ \sum_{\iota=0}^{r} (A_\iota + 1+u+w) x^{\delta-\iota-1}.\end{aligned}$$

Since $C$ is a linear code over $R$, $g(x)^{rc}$ and $(1+u+w)(\frac{x^\delta-1}{x-1})$ are members of

$C$. Therefore, we can deduce that $g(x)^{rc}+(1+u+w)(\frac{x^\delta-1}{x-1})\in C$, where

$$\begin{aligned} g(x)^{rc}+(1+u+w)(\frac{x^\delta-1}{x-1}) =& \sum_{\iota=0}^{r} A_\iota x^{\delta-\iota-1} + \sum_{\iota=r+1}^{s} B_\iota x^{\delta-\iota-1} \\ &+ \sum_{\iota=s+1}^{t} C_\iota x^{\delta-\iota-1} + \sum_{\iota=t+1}^{k} D_\iota x^{\delta-\iota-1}. \end{aligned}$$

Since $C$ is an $R$-skew cyclic code, the result of multiplying on the right by $x^{k+1-\delta}$ is

$$\begin{aligned} (g(x)^{rc}+(1+u+w)(\frac{x^\delta-1}{x-1})(x^{k+1-\delta}) =& \sum_{\iota=0}^{r} A_\iota x^{k-\iota} + \sum_{\iota=r+1}^{s} B_\iota x^{k-\iota} \\ &+ \sum_{\iota=s+1}^{t} C_\iota x^{k-\iota} + \sum_{\iota=t+1}^{k} D_\iota x^{k-\iota} \\ =& g^*(x). \end{aligned}$$

Thus, $g^*(x)$ is an element of $C$ and given that $C=\langle g(x)\rangle$, there exists a polynomial $p(x)\in R[x,\theta]$ such that $g^*(x)=p(x)g(x)$. However, since $\deg(g^*(x))\leq \deg(g(x))$, we conclude that $p(x)=1$ leading to $g^*(x)=g(x)$. Consequently, $g(x)$ demonstrates a self-reciprocal property.

Conversely, assume that $C$ is an $R$-skew cyclic code of length $\delta$ generated by a self-reciprocal polynomial $g(x)$ and $(1+u+w)(\frac{x^\delta-1}{x-1})\in C$. Then we show that $C$ is a reversible complement code. For this, suppose that $c(x)=c_0+c_1x+\cdots+c_kx^k$ is an arbitrary codeword in $C$. Then the reciprocal $c^*(x)=c_k+c_{k-1}x+\cdots+c_0x^k\in C$. Now, we have

$$\begin{aligned} (c^*(x))^{rc} &= (1+u+w)(1+x+\cdots+x^{\delta-k-2})+\bar{c_0}x^{\delta-k-1}+\bar{c_1}x^{\delta-k}+\cdots+\bar{c_k}x^{\delta-1} \\ &= (1+u+w)(1+x+\cdots+x^{\delta-1})+c_0x^{\delta-k-1}+c_1x^{\delta-k}+\cdots+c_kx^{\delta-1} \\ &= (1+u+w)(\frac{x^\delta-1}{x-1})+c(x)x^{\delta-k-1}. \end{aligned}$$

Since $c^*(x)=p^*(x)g(x)\in C$ for some polynomial $p(x)\in R[x,\theta]$ and given that $C$ is a linear code, it follows that $(c^*(x))^{rc}=(1+u+w)(\frac{x^\delta-1}{x-1})+c(x)x^{\delta-k-1}\in C$. Thus, we conclude that $C$ is a reversible complement code. □

**Example 6.1.** Suppose that $\delta=5$, then we have $x^\delta-1=(x+1)(x^2+\hbar x+1)(x^2+\hbar^2x+1)\in\mathbb{F}_4[x,\theta]$. Let $g_1(x)=g_2(x)=g_3(x)=g_4(x)=x^2+\hbar x+1$

and define $g(x) = \mu_1 g_1(x) + \mu_2 g_2(x) + \mu_3 g_3(x) + \mu_4 g_4(x) = x^2 + \hbar x + 1$. Then, $C = \langle g(x) \rangle$ is skew cyclic over $R$. Since $g(x)$ exhibits self-reciprocal characteristics and $(1+u+w)(\frac{x^\delta - 1}{x-1}) \in C$ leads $C$ to be a reversible complement code over $R$.

**Example 6.2.** Suppose that $\delta = 7$ then, we have $x^\delta - 1 = (x+1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \in \mathbb{F}_4[x, \theta]$. Now, let $g_1(x) = g_2(x) = g_3(x) = g_4(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ and define $g(x) = \mu_1 g_1(x) + \mu_2 g_2(x) + \mu_3 g_3(x) + \mu_4 g_4(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. Then, $C = \langle g(x) \rangle$ is a skew cyclic code over $R$. Since $g(x)$ exhibits self-reciprocal characteristics, and $(1 + u + w)(\frac{x^\delta - 1}{x-1}) \in C$, leads $C$ to be a reversible complement code over $R$.

**Definition 6.3.** An $\mathbb{F}_4R$-linear code $C$ is a DNA-skew cyclic code if it satisfies the following conditions:

1. $C$ is an $\mathbb{F}_4R$-skew cyclic code, and
2. If $c = (c_1, c_2)$ be any codeword in $C$, then the reverse complement $c^{rc} = (c_1^{rc}, c_2^{rc}) \in C$ and $c \neq c^{rc}$.

**Theorem 6.4.** *Let $C = \langle (f(x), 0),\ (h(x)|g(x)) \rangle = C_\gamma \otimes C_\delta$ be an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$, where $h(x)) = 0$. Then, $C$ is reversible complement if and only if $f(x)$ and $g(x)$ are both self-reciprocal polynomials, $(\frac{x^\gamma - 1}{x-1}) \in C_\gamma$ and $(1 + u + w)(\frac{x^\delta - 1}{x-1}) \in C_\delta$.*

*Proof.* Suppose that $C = \langle (f(x), 0), (0, g(x)) \rangle = C_\gamma \otimes C_\delta$ be an $\mathbb{F}_4R$-skew cyclic code with a block length $(\gamma, \delta)$, where $f(x) \in \frac{\mathbb{F}_4[x]}{(x^\gamma - 1)}$ and $g(x) \in \frac{R[x,\theta]}{(x^\delta - 1)}$. Then, by Lemma 4.1 $\Pi_1(C) = C_\gamma$ is cyclic over $\mathbb{F}_4$ and $\Pi_2(C) = C_\delta$ is skew cyclic over $R$. Assume that $C$ is a reversible complement code and $c = (c_1, c_2) \in C = C_\gamma \otimes C_\delta$ is an arbitrary codeword. Then $c^{rc} = (c_1^{rc}, c_2^{rc}) \in C = C_\gamma \otimes C_\delta$. For any $c_1 \in C_\gamma$, $c_1^{rc} \in C_\gamma$ and for any $c_2 \in C_\delta$, $c_2^{rc} \in C_\delta$. Hence, $C_\gamma$ (resp. $C_\delta$) is a reversible complement code over $\mathbb{F}_4$ (resp. $R$).

Since $C_\gamma = \left\langle f(x) \right\rangle$ is cyclic reversible complement code over $\mathbb{F}_4$, where $f(x) = f_0 + f_1 x + \cdots + f_r x^r$ and $\mathbf{0} = (0, 0, \ldots, 0) \in C$. Complement of $a \in \mathbb{F}_4$ is defined as $\overline{a} = a + 1$. So, we have $\overline{\mathbf{0}} = (1, 1, \ldots, 1) = 1 + x + \cdots + x^{\gamma - 1} = (\frac{x^\gamma - 1}{x-1}) \in C$ and

$$(f(x))^{rc} = 1 + x + \cdots + x^{\gamma - r - 2} + \overline{f_r} x^{\gamma - r - 1} + \overline{f_{r-1}} x^{\gamma - r} + \cdots + \overline{f_0} x^{\gamma - 1} \in C.$$

Since $C_\gamma$ is an $\mathbb{F}_4$-linear code, so $(f(x))^{rc} + (\frac{x^\gamma - 1}{x-1}) \in C$, where

$$\begin{aligned}(f(x))^{rc} + (\frac{x^\gamma - 1}{x-1}) &= (\bar{f}_r + 1) x^{\gamma - r - 1} + (\bar{f}_{r-1} + 1) x^{\gamma - r} + \cdots + (\bar{f}_0 + 1) x^{\gamma - 1} \\ &= f_r x^{\gamma - r - 1} + f_{r-1} x^{\gamma - r} + \cdots + f_0 x^{\gamma - 1}.\end{aligned}$$

Since $C$ is a cyclic code, $((f(x))^{rc} + (\frac{x^\gamma-1}{x-1}))x^{r+1-\gamma} = f_r + f_{r-1}x + \cdots + f_0x^r = f^*(x) \in C$. Thus, we can find a polynomial $p(x) \in \mathbb{F}_4[x]$ that satisfies $f^*(x) = p(x)f(x)$. But $\deg(f^*(x)) \leq \deg(f(x))$ asserted that $p(x) = 1$, leads $f(x) = f^*(x)$. Hence, $f(x)$ is a self-reciprocal polynomial.

Since $C_\delta = \langle g(x)\rangle$ is an $R$-skew cyclic code, which is also a reversible complement code. So, by Theorem 6.2 $g(x)$ is self-reciprocal and $(1 + u + w)(\frac{x^\delta-1}{x-1}) \in C$.

Conversely, suppose that $f(x)$ and $g(x)$ are both self-reciprocal polynomials with $(\frac{x^\gamma-1}{x-1}) \in C_\gamma$ and $(1 + u + w)(\frac{x^\delta-1}{x-1}) \in C_\delta$. Then, by Theorem 6.2, it is evident that $C_\gamma$ (resp. $C_\delta$) is a reversible complement code over $\mathbb{F}_4$ (resp. $R$). Now, assuming $C_\gamma$ and $C_\delta$ are both reversible complement codes. For any $c_1 \in C_\gamma$ and $c_2 \in C_\delta$, we have $c_1^{rc} \in C_\gamma$ and $c_2^{rc} \in C_\delta$. Consequently, for any $c = (c_1, c_2) \in C = C_\gamma \otimes C_\delta$, we can deduce that $c^{rc} = (c_1^{rc}, c_2^{rc}) \in C = C_\gamma \otimes C_\delta$. Thus, it becomes apparent that $C$ is a reversible complement code. □

**Example 6.3.** For $\gamma = 17$, consider the polynomial $f(x) = x^4+x^3+\hbar x^2+x+1$ then $f(x)|(x^{17} - 1)$ over $\mathbb{F}_4[x, \theta]$ and $f(x)$ is self-reciprocal. So, $C_1 = \langle f(x)\rangle$ is a reversible complement code with parameters $[17, 13, 5]$ over $R$. Next, for $\delta = 13$, let $g(x) = x^6 + \hbar^2x^5 + \hbar x^3 + \hbar^2x + 1$, then $g(x)|(x^{13} - 1)$ over $\mathbb{F}_4[x, \theta]$ and $g(x)$ is self-reciprocal. Hence, $C_2 = \langle g(x)\rangle$ is a reversible complement code with parameters $[13, 7, 5]$ over $R$. Therefore, $C = C_1 \otimes C_2$ is an $\mathbb{F}_4R$-reversible complement code with parameters $[30, 20, 5]$.

## 7 Conclusion

The primary objective of this research is to analyze the configuration of $\mathbb{F}_4R$-submodule and establish their connection with DNA codes, where $R = \mathbb{F}_4 + u\mathbb{F}_4 + v\mathbb{F}_4 + w\mathbb{F}_4$ with $u^2 = u,\ v^2 = v,\ w^2 = w,\ uv = vu = 0,\ vw = wv = 0,\ wu = uw = 0$. This is achieved by examining particular subclasses like reversible codes. Ultimately, the aim of this study is to utilize Gray maps to derive codes that possess the characteristics of DNA structures. At the end of this paper, we have provided the condition under which skew cyclic codes are reversible.

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