DOI: 10.2478/auom-2023-0036
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 31(3),2023, 145–165

Similarity relations and exponential of dual-generalized complex matrices

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Abstract

In this study, taking into account the fundamental properties of dual-generalized complex (DGC) matrices, various types of similarity relations are introduced considering coneigenvalues/coneigenvectors via different conjugates. The exponential version of DGC matrices are identified and then their theoretical characteristic theorems are obtained. Finally, examples for DGC matrix exponential are given.

1 Introduction

Matrix applications are broadly utilized in various branches of science and engineering such as applied mathematics, data analysis, scientific computing, graphic software, optimization, electronics networks, airplane and spacecraft, robotics and automation etc. (see detailed information in [15], [36], [38]). Also, a comprehensive study about matrix theory is presented by Zhang [12] considering quaternions [41]. Inspired by Zhang, many studies considering different types of quaternions are conducted over matrix theory in [18], [30], [42]. Various types of similarity relations considering numbers and quaternions are introduced in [27], [37] and [19], [28], [39], respectively. Several properties of matrices and matrix exponential are discussed in [4], [5], [20]. In these studies, non-commutativity of quaternions leads to numerous challenges in applications of quaternions. Unlike quaternions, dual-generalized complex (DGC)

Key Words: Dual-generalized complex matrices, Fundamental matrix, Eigenvalues and eigenvectors, Similarity, Matrix exponential
2010 Mathematics Subject Classification: Primary 15B33; Secondary 15A18, 15A09.
Received:25.09.2022
Accepted: 29.12.2022

numbers ([32]) are constructed via the Cayley-Dickson doubling process with the combination of generalized complex numbers ([1], [21]) and dual numbers. Generalized complex numbers includes complex numbers ([22]), hyperbolic numbers ([14], [23], [35]) and dual numbers ([8], [10], [23]) under special cases. Additionally, over the years, many approaches have been taken by researchers while studying these number systems. It is to mention that complex-dual numbers ([7], [16], [17], [40]), hyperbolic-dual numbers ([29], [40]) and hyper-dual numbers ([2], [24], [25]).

It should be noted that DGC numbers are commutative. Hence, the selection of these numbers among the others gives a convenient and a functional way for application. From this regard, the generalization of the classical matrix theory is discussed in [33]. It proves that well-known results in matrix theory hold for DGC matrices as well.

This paper is organized as follows: Section 1 presents a general information and discussion about the existing studies in the literature. Section 2 includes fundamental concepts about DGC numbers, DGC vectors and DGC matrices. Section 3 is devoted to various similarity relations for DGC matrices considering coneigenvalues and coneigenvectors. Section 4 deals with the exponential of DGC matrices and their characteristic theorems. Especially, Theorem 6 gives the answer for the question: When does the matrix exponential satisfies the same properties as the usual number exponential? The answer depends on the commutativity of matrices. In this section, examples for matrix exponential are also presented. In the last section, the conclusion is given.

2 Preliminary Information

In this section, some information about DGC numbers, DGC vectors and DGC matrices are given to make sense of our work. In this respect, our main reference sources are [32], [33] and [34]. Note that here and elsewhere $\mathfrak{p} \neq 0$ and $k = 1, 2, 3$.

2.1 DGC Numbers and DGC Vectors

Let us consider the set of DGC numbers as ([32])

$$\mathbb{DC}_{\mathfrak{p}} := \left\{ \tilde{a} = z_1 + z_2\varepsilon : \ z_1, z_2 \in \mathbb{C}_{\mathfrak{p}}, \ \varepsilon^2 = 0, \, \varepsilon \neq 0, \varepsilon \notin \mathbb{R} \right\}.$$

The aforementioned set $\mathbb{C}_{\mathfrak{p}}$ is the set of generalized complex numbers* ([1], [21])

$$\mathbb{C}_{\mathfrak{p}} := \left\{ z = a + bJ : \ a, b \in \mathbb{R}, \ J^2 = \mathfrak{p}, \, \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R} \right\}.$$

Let $\tilde{a}_1 = z_{11} + z_{12}\varepsilon$, $\tilde{a}_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $\lambda \in \mathbb{R}$. Then, the algebraic operations on DGC numbers are given as follows: $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow z_{11} = z_{21}$, $z_{12} = z_{22}$, $\tilde{a}_1 + \tilde{a}_2 = (z_{11} + z_{21}) + (z_{12} + z_{22})\,\varepsilon$, $\lambda\tilde{a}_1 = \lambda\,(z_{11} + z_{12}\varepsilon)$ and $\tilde{a}_1\tilde{a}_2 = (z_{11}z_{21}) + (z_{11}z_{22} + z_{12}z_{21})\,\varepsilon$. Here the real part of $\tilde{a}_1$ is characterized by $\mathrm{Re}(\tilde{a}_1) = \mathrm{Re}(\tilde{z}_{11})$.

The notation ^{†_k} represents the different conjugates and these conjugates are defined as follows: $\tilde{a}^{\dagger_1} = \bar{z}_1 + \bar{z}_2\varepsilon$, $\tilde{a}^{\dagger_2} = z_1 - z_2\varepsilon$ and $\tilde{a}^{\dagger_3} = \bar{z}_1 - \bar{z}_2\varepsilon$. Here $\bar{z}_1$ and $\bar{z}_2$ represent the usual conjugate of z_1, $z_2 \in \mathbb{C}_{\mathfrak{p}}$. Hence, the norms^{†_k} are identified by $|\tilde{a}|^2_{\dagger_k} = \tilde{a}\tilde{a}^{\dagger_k}$, $k = 1, 2, 3$. The multiplication of the base elements are $J\varepsilon = \varepsilon J$ and $(J\varepsilon)^2 = 0$. Every DGC number $\tilde{a}$ can be written as $\tilde{a} = z_1 + z_2\varepsilon = a_1 + a_2 J$ where a_1, a_2 are dual numbers, (see details in [32]). Null (isotropic) DGC numbers are the numbers with zero norm and they identified by the following forms for $a, b, c, d \in \mathbb{R}$:

- ➢ $0, c\varepsilon$, $dJ\varepsilon$, and $c\varepsilon + dJ\varepsilon$ with respect to ^{†_k};
- ➢ $\pm\sqrt{\mathfrak{p}}a + aJ$ with respect to ^{†_1} and ^{†_3} where $\mathfrak{p} > 0$;
- ➢ $\pm\sqrt{\mathfrak{p}}a + aJ \pm \sqrt{\mathfrak{p}}c\varepsilon + cJ\varepsilon$ with respect to ^{†_1} and ^{†_3} where $\mathfrak{p} > 0$ and $a \neq 0$.

Hence, the non-null DGC number has an inverse^{†_k} where $\tilde{a}^{-1}_{\dagger_k} = \dfrac{\tilde{a}^{\dagger_k}}{|\tilde{a}|^2_{\dagger_k}}$. Besides, the set consists of DGC vectors is denoted by $\mathtt{V}^n$ and defined as

$$\mathtt{V}^n := \left\{ V = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) : \tilde{a}_t \in \mathbb{DC}_{\mathfrak{p}}, t = 1, 2 ..., n \right\}.$$

$\mathtt{V}^n$ is a module over $\mathbb{DC}_{\mathfrak{p}}$. The conjugate of $V \in \mathtt{V}^n$ is the conjugate of its components. For $V = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)\,, U = \left(\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n\right) \in \mathtt{V}^n$ and, the standard scalar product and Hermitian^{†_k} scalar product over $\mathtt{V}^n$ are defined by $\langle V, U \rangle = \sum\limits_{r=1}^{n} \tilde{a}_r\tilde{b}_r = V^T U$ and $\langle V, U \rangle_{\dagger_k} = \sum\limits_{r=1}^{n} \tilde{a}_r\tilde{b}_r^{\dagger_k} = V^T U^{\dagger_k}$, respectively. The standard norm and norm^{†_k} of DGC vector V in $\mathtt{V}^n$ are defined as follows: $\|V\|^2 = \langle V, V \rangle$ and $\|V\|^2_{\dagger_k} = \langle V, V \rangle_{\dagger_k}$, respectively. If the norm of a vector $V \in \mathtt{V}^n$ equals 1, then it is called unit vector. There exists an analogy between

*$\mathbb{C}_{\mathfrak{p}}$ is a vector space over $\mathbb{R}$. It is analogue to complex numbers ([22]) $\mathbb{C}$ for $\mathfrak{p} = -1$, hyperbolic numbers ([14], [23], [35]) $\mathbb{H}$ for $\mathfrak{p} = 1$ and dual numbers ([8], [10], [23]) $\mathbb{D}$ for $\mathfrak{p} = 0$. Additionally, the function theory over these numbers can be seen in the studies [11], [31].

$V = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \in \mathtt{V}^n$ and $V = \mathsf{V}_1 + \mathsf{V}_2 J + \mathsf{V}_3 \varepsilon + \mathsf{V}_4 J\varepsilon$ where $\mathsf{V}_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$. Also, null (isotropic) DGC vector in $\mathtt{V}^n$ is a vector which has zero norm.

2.2 DGC Matrices

The matrix with DGC number entries is called DGC matrix (see details and main results in [33], [34]). The DGC matrix $\tilde{A}$ of the order $m \times n$ is of the form $\tilde{A} = [\tilde{a}_{ij}] = [a_{0ij} + a_{1ij} J + a_{2ij}\varepsilon + a_{3ij} J\varepsilon]$, where $\tilde{a}_{ij} \in \mathbb{DC}_{\mathfrak{p}}$, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. The set of all $m \times n$ matrices with DGC number entries is denoted by

$$\mathbb{M}_{m\times n}(\mathbb{DC}_{\mathfrak{p}}) := \left\{ \tilde{A} = [\tilde{a}_{ij}]_{m\times n} : \tilde{a}_{ij} \in \mathbb{DC}_{\mathfrak{p}}, i = 1, 2, ..., m, j = 1, 2, ..., n \right\}.$$

A DGC matrix with all of the entries are zero is called a DGC zero matrix and denoted by $\tilde{0}$. If $m = n$, then $\tilde{A}$ is called DGC square matrix. Every DGC matrix of the order $m \times n$ can be written as $\tilde{A} = A_0 + A_1 J + A_2 \varepsilon + A_3 J\varepsilon$, where A_0, A_1, A_2, A_3 are real matrices of the same order.

Standard elementary matrix operations establish the following operations on DGC matrices. Let $\tilde{A} = [\tilde{a}_{ij}]$, $\tilde{B} = [\tilde{b}_{ij}] \in \mathbb{M}_{m\times n}(\mathbb{DC}_{\mathfrak{p}})$, $\tilde{C} = [\tilde{c}_{js}] \in \mathbb{M}_{n\times r}(\mathbb{DC}_{\mathfrak{p}})$ and $c \in \mathbb{R}$. $\tilde{A}$ and $\tilde{B}$ are equal if $\tilde{a}_{ij} = \tilde{b}_{ij}$. The addition (and hence subtraction) is defined as $\tilde{A} + \tilde{B} = [\tilde{a}_{ij}] + [\tilde{b}_{ij}] = [\tilde{a}_{ij} + \tilde{b}_{ij}] = \tilde{D} \in \mathbb{M}_{m\times n}(\mathbb{DC}_{\mathfrak{p}})$. The scalar multiplication of $\tilde{A}$ by c is defined as $c\tilde{A} = [c\,\tilde{a}_{ij}] \in \mathbb{M}_{m\times n}(\mathbb{DC}_{\mathfrak{p}})$. The product $\tilde{A}\tilde{C}$ is defined as

$$\tilde{A}\tilde{C} = \left[\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{c}_{js} \right] = [\tilde{e}_{is}] = \tilde{E} \in \mathbb{M}_{m\times r}(\mathbb{DC}_{\mathfrak{p}}).$$

The DGC square matrix $\tilde{A}$ of the order n is said to be an invertible if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \tilde{I}_n$ for DGC square matrix $\tilde{B}$ of the same order. The transpose of $\tilde{A}$ is denoted by $\tilde{A}^T$ and defined as $\tilde{A}^T = [\tilde{a}_{ji}] \in \mathbb{M}_{n\times m}(\mathbb{DC}_{\mathfrak{p}})$ or $\tilde{A}^T = A_0^T + A_1^T J + A_2^T \varepsilon + A_3^T J\varepsilon$. The trace of square matrix $\tilde{A}$, denoted by $\operatorname{tr}(\tilde{A})$, is defined as $\operatorname{tr}(\tilde{A}) = \sum_{i=1}^{n} \tilde{a}_{ii} = \operatorname{tr}(A_0) + \operatorname{tr}(A_1)J + \operatorname{tr}(A_2)\varepsilon + \operatorname{tr}(A_3)J\varepsilon$. The conjugations of $\tilde{A}$ are defined as follows:

$$\begin{aligned} \tilde{A}^{\dagger_1} &= A_0 - A_1 J + A_2\varepsilon - A_3 J\varepsilon = (A_0 + A_2\varepsilon) + (-A_1 - A_3\varepsilon)J, \\ \tilde{A}^{\dagger_2} &= A_0 + A_1 J - A_2\varepsilon - A_3 J\varepsilon = (A_0 - A_2\varepsilon) + (A_1 - A_3\varepsilon)J, \\ \tilde{A}^{\dagger_3} &= A_0 - A_1 J - A_2\varepsilon + A_3 J\varepsilon = (A_0 - A_2\varepsilon) + (-A_1 + A_3\varepsilon)J. \end{aligned}$$

Moreover, for any DGC square matrix $\tilde{A}$ of the order n, $\tilde{A}$ is symmetric (skew-symmetric) if and only if $\tilde{A}=\tilde{A}^T$ $(\tilde{A}=-\tilde{A}^T)$, $\tilde{A}$ is Hermitian^{†_k}(skew-Hermitian^{†_k}) if and only if $\tilde{A}^T=\tilde{A}^{\dagger_k}(\tilde{A}^T=-\tilde{A}^{\dagger_k})$, $\tilde{A}$ is orthogonal if and only if $\tilde{A}\tilde{A}^T=\tilde{A}^T\tilde{A}=\tilde{I}_n$, $\tilde{A}$ is unitary^{†_k} if and only if $\tilde{A}\tilde{A}^{\star_k}=\tilde{A}^{\star_k}\tilde{A}=\tilde{I}_n$, where $\left(\tilde{A}^T\right)^{\dagger_k}=\left(\tilde{A}^{\dagger_k}\right)^T=\tilde{A}^{\star_k}$. Also, the DGC matrix $\tilde{A}$ is said to be normal^{†_k} if $\tilde{A}\tilde{A}^{\star_k}=\tilde{A}^{\star_k}\tilde{A}$. The identity, diagonal, scalar, upper/lower triangular and triangular DGC matrices are defined by in a familiar way.

The determinant of $\tilde{A}$ is defined by a familiar way and it exhibits the features of the standard determinant. Namely, it act same as real matrices. If $\det(\tilde{A})$ is a non-null DGC number, then $\tilde{A}$ is invertible and its inverse can be obtained by the formula

$$\tilde{A}^{-1}=\frac{1}{\det(\tilde{A})}\operatorname{adj}(\tilde{\mathrm{A}}), \tag{1}$$

where $\operatorname{adj}(\tilde{\mathrm{A}})$ is the classical adjoint of a matrix. $\tilde{A}$ is not invertible when $\det(\tilde{A})$ is a null DGC number. In other respects, it is also possible to calculate $\tilde{A}^{-1}$ considering conjugate^{†_k} denoting as $\tilde{A}^{-1}_{\dagger_k}$, where the notation $_{\dagger_k}$ represents $\det(\tilde{A})$ is non-null for conjugate^{†_k}. For instance, if $\det(\tilde{A})$ is null^{†_1} but non-null^{†_2} for a DGC matrix $\tilde{A}$, then $\tilde{A}^{-1}_{\dagger_2}$ can be obtained but $\tilde{A}^{-1}_{\dagger_1}$ does not exist ([33], [34]).

For dual matrices $\mathtt{A}_1$ and $\mathtt{A}_2$, every $\tilde{A}=\mathtt{A}_1+\mathtt{A}_2J\in\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ has a dual matrix representation

$$\chi(\tilde{A})=\begin{bmatrix}\mathtt{A}_1 & \mathfrak{p}\mathtt{A}_2\\ \mathtt{A}_2 & \mathtt{A}_1\end{bmatrix}\in D^* \tag{2}$$

where

$$D^*:=\left\{\begin{bmatrix}\mathtt{A}_1 & \mathfrak{p}\mathtt{A}_2\\ \mathtt{A}_2 & \mathtt{A}_1\end{bmatrix}:\mathtt{A}_1,\mathtt{A}_2\in\mathbb{M}_n\left(\mathbb{D}\right)\right\}\subset\mathbb{M}_{2n}\left(\mathbb{D}\right).$$

Here $\chi(\tilde{A})$ is called dual fundamental matrix of $\tilde{A}$.

Any DGC square matrix $\tilde{A}$ can be rewritten as $\tilde{A}\cong\begin{bmatrix}\mathtt{A}_1\\ \mathtt{A}_2\end{bmatrix}$. Then, for DGC square matrices $\tilde{A}$ and $\tilde{B}=\mathtt{B}_1+\mathtt{B}_2J$, we have $\tilde{A}\tilde{B}=\begin{bmatrix}\mathtt{A}_1 & \mathfrak{p}\mathtt{A}_2\\ \mathtt{A}_2 & \mathtt{A}_1\end{bmatrix}\begin{bmatrix}\mathtt{B}_1\\ \mathtt{B}_2\end{bmatrix}=\chi(\tilde{A})\tilde{B}$.

Let $\tilde{A}\in\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ and $V\in\mathtt{V}^n$ be a non-null vector. If $\tilde{A}V=\lambda V$ for some $\lambda\in\mathbb{DC}_{\mathfrak{p}}$, then λ is called an eigenvalue of $\tilde{A}$ and V is called an eigenvector of $\tilde{A}$ associated with λ. Those eigenvalues are said to be the standard eigenvalues of $\tilde{A}$. Besides, if λ is an eigenvalue of $\tilde{A}$ corresponding to the eigenvector V, then $c\lambda$ is an eigenvalue of $c\tilde{A}$ corresponding to the same eigenvector V, where c is a non-zero real scalar.

3 Similarity Relations for DGC Matrices

In this section, the similarity relations for DGC matrices considering coneigenvalues and coneigenvectors are discussed. It is known that the eigenvalue/ eigenvector theory is one of the key stage for different disciplines of sciences. They are widely used in engineering applications to utilize efficient and accurate computation way. Moreover, they have been further applied in differential equation theory, information system design, nonlinear optimization, economics and etc.

The studies [27], [37] motivate the rest of this paper. It is interesting to express a DGC matrix as a product of DGC matrices with special nature.

The following definition is an analog of the standard eigenvalues and eigenvectors. As compared to the standard eigenvalues and eigenvectors, we present their extended versions here.

Definition 1. *Let $\tilde{A} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$ and $V \in \mathtt{V}^n$ be a non-null vector with*

$$\tilde{A}V^{\dagger_k} = \lambda V \tag{3}$$

for some $\lambda \in \mathbb{DC}_{\mathtt{p}}$. Then, λ is called a coneigenvalue^{†_k} of $\tilde{A}$ and V is called a coneigenvector^{†_k} of $\tilde{A}$ associated with λ. The set of all coneigenvalues^{†_k} of $\tilde{A}$ is denoted by $\sigma^{\dagger_k}(\tilde{A})$.

Lemma 1. *If λ is a coneigenvalue^{†_k} of DGC matrix $\tilde{A}$ corresponding to the coneigenvector^{†_k} V, then $c\lambda$ is an coneigenvalue^{†_k} of $c\tilde{A}$ corresponding to the same coneigenvector^{†_k} V, where c is a non-zero real scalar.*

Theorem 1. *For any DGC square matrix $\tilde{A}$, if λ is a coneigenvalue^{†_k} of $\tilde{A}$ then $\gamma^{\dagger_k}\lambda\gamma^{-1} \in \mathbb{DC}_{\mathtt{p}}$ is also a coneigenvalue^{†_k} of $\tilde{A}$ for non-null DGC number γ.*

Proof. Let λ be a coneigenvalue^{†_k} of $\tilde{A} \in \mathbb{M}_n \in (\mathbb{DC}_{\mathtt{p}})$. From equation (3), we can write that

$$\tilde{A}V^{\dagger_k}\gamma^{-1} = \left(\gamma^{\dagger_k}\lambda\gamma^{-1}\right)\left(V\gamma^{-1}_{\dagger_k}\right).$$

Hence $\gamma^{\dagger_k}\lambda\gamma^{-1} \in \mathbb{DC}_{\mathtt{p}}$ is a coneigenvalue^{†_k} of $\tilde{A}$ that corresponds to the vector $V\gamma^{-1}_{\dagger_k}$. □

Definition 2. *Let $\tilde{A}, \tilde{B} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$, the following definitions are given:*

- ➢ *$\tilde{A}$ and $\tilde{B}$ are called similar if there exists an invertible DGC matrix of the same order $\tilde{P}$ such that*

$$\tilde{B} = \tilde{P}^{-1}\tilde{A}\tilde{P}. \tag{4}$$

- $\tilde{A}$ *and* $\tilde{B}$ *are called consimilar*^{†_k} *if there exists an invertible* DGC *matrix of the same order* $\tilde{P}$ *such that*

$$\tilde{B} = \tilde{P}^{\dagger_k} \tilde{A} \tilde{P}^{-1}. \tag{5}$$

Here, we need to remind equation (1) for invertible DGC matrices.

Theorem 2. *For any* DGC *matrices of the same order* $\tilde{A}$ *and* $\tilde{B}$*;*

- *if* $\tilde{A}$ *is similar to* $\tilde{B}$*, then* $\tilde{A}$ *and* $\tilde{B}$ *has same eigenvalues.*
- *if* $\tilde{A}$ *is consimilar*^{†_k} *to* $\tilde{B}$*, then* $\tilde{A}$ *and* $\tilde{B}$ *has same coneigenvalues*^{†_k}*.*

Proof. We sketch the steps for consimilar^{†_k} and leave the details for similar to the reader. For $\tilde{A}, \tilde{B} \in \mathbb{M}_n(\mathbb{DC}_\mathtt{p})$, let $\tilde{A}$ be consimilar^{†_k} to $\tilde{B}$ and $\lambda \in \mathbb{DC}_\mathtt{p}$ be a coneigenvalue^{†_k} of $\tilde{A}$. By supposing $U^{\dagger_k} = \tilde{P} V^{\dagger_k}$ and considering equations (3) and (5), we have

$$\begin{aligned} \tilde{B} U^{\dagger_k} &= \tilde{P}^{\dagger_k} \tilde{A} \tilde{P}^{-1} U^{\dagger_k} \\ &= \tilde{P}^{\dagger_k} \tilde{A} V^{\dagger_k} \\ &= \tilde{P}^{\dagger_k} \lambda V \\ &= \lambda U. \end{aligned}$$

Thus $\lambda \in \mathbb{DC}_\mathtt{p}$ is a coneigenvalue^{†_k} of $\tilde{B}$. □

Definition 3. *If* $\tilde{A}$ *is* DGC *diagonalizable, then there exists an invertible* DGC *matrix* $\tilde{P}$ *such that* $\tilde{A} = \tilde{P} \tilde{D} \tilde{P}^{-1}$ *where* $\tilde{D}$ *is a* DGC *diagonal matrix. Here the column vectors of* $\tilde{P}$ *are eigenvectors of* $\tilde{A}$*, and the diagonal entries of* $\tilde{D}$ *are the corresponding eigenvalues of* $\tilde{A}$*.*

The following definition describes another similarity relations.

Definition 4. *Let* $\tilde{A}, \tilde{B} \in \mathbb{M}_n(\mathbb{DC}_\mathtt{p})$*, the following definitions are given:*

- $\tilde{A}$ *and* $\tilde{B}$ *are called semi-similar if there exist* DGC *matrices of the same order* $\tilde{X}$ *and* $\tilde{Y}$ *such that*

$$\tilde{Y} \tilde{A} \tilde{X} = \tilde{B} \quad \text{and} \quad \tilde{X} \tilde{B} \tilde{Y} = \tilde{A}. \tag{6}$$

- $\tilde{A}$ *and* $\tilde{B}$ *are called semi-consimilar*^{†_k} *if there exist* DGC *matrices of the same order* $\tilde{X}$ *and* $\tilde{Y}$ *such that*

$$\tilde{Y}^{\dagger_k} \tilde{A} \tilde{X} = \tilde{B} \quad \text{and} \quad \tilde{X}^{\dagger_k} \tilde{B} \tilde{Y} = \tilde{A}. \tag{7}$$

Theorem 3. *Let* $\tilde{A}, \tilde{B}, \tilde{X}, \tilde{Y} \in \mathbb{M}_n(\mathbb{DC}_\mathtt{p})$*. If* $\tilde{A}$ *is semi-similar to* $\tilde{B}$*, then the following statements hold:*

i) $\tilde{A}^{2q}\tilde{X} = \tilde{X}\tilde{B}^{2q}$ *and* $\tilde{B}^{2q}\tilde{Y} = \tilde{Y}\tilde{A}^{2q}$,

ii) $(\tilde{X}\tilde{Y})^q\tilde{A}(\tilde{X}\tilde{Y})^q = \tilde{A}$ *and* $(\tilde{Y}\tilde{X})^q\tilde{B}(\tilde{Y}\tilde{X})^q = \tilde{B}$,

where q is positive integer.

Proof. Let $\tilde{A}$ be semi-similar to $\tilde{B}$.

i) According to equations (6) and (7), we can write that

$$\begin{cases} \tilde{A} = & \tilde{X}\tilde{B}\tilde{Y} = & (\tilde{X}\tilde{Y})\tilde{A}(\tilde{X}\tilde{Y}) \\ \tilde{B} = & \tilde{Y}\tilde{A}\tilde{X} = & (\tilde{Y}\tilde{X})\tilde{B}(\tilde{Y}\tilde{X}). \end{cases} \tag{8}$$

With these two facts in mind, one can calculate

$$\begin{cases} \tilde{A}^2\tilde{X} & = (\tilde{X}\tilde{B}\tilde{Y})(\tilde{X}\tilde{B}\tilde{Y})\tilde{X} & = \tilde{X}\tilde{B}^2 \\ \tilde{B}^2\tilde{Y} & = (\tilde{Y}\tilde{A}\tilde{X})(\tilde{Y}\tilde{A}\tilde{X})\tilde{Y} & = \tilde{Y}\tilde{A}^2. \end{cases}$$

Using the same procedure, the proof is completed for $q \in \mathbb{Z}^+$.

ii) Based on equation (8), we have

$$\begin{aligned} \tilde{A} = & \ (\tilde{X}\tilde{Y})\tilde{A}(\tilde{X}\tilde{Y}) \\ = & \ (\tilde{X}\tilde{Y})(\tilde{X}\tilde{B}\tilde{Y})(\tilde{X}\tilde{Y}) \\ = & \ (\tilde{X}\tilde{Y})\tilde{X}(\tilde{Y}\tilde{A}\tilde{X})\tilde{Y}(\tilde{X}\tilde{Y}) \\ = & \ (\tilde{X}\tilde{Y})^2\tilde{A}(\tilde{X}\tilde{Y})^2. \end{aligned}$$

Similarly we obtain $\tilde{B} = (\tilde{Y}\tilde{X})^2\tilde{B}(\tilde{Y}\tilde{X})^2$. Upon these, we complete the proof by applying the same process for $q \in \mathbb{Z}^+$.

□

Theorem 4. *Let* $\tilde{A}, \tilde{B}, \tilde{X}, \tilde{Y} \in \mathbb{M}_n(\mathbb{DC}_\mathfrak{p})$. *If* $\tilde{A}$ *is semi-consimilar*^{†_k} *to* $\tilde{B}$, *then the following statements hold:*

i) $(\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})^q\tilde{A}(\tilde{X}\tilde{Y})^q = \tilde{A}$ *and* $(\tilde{Y}^{\dagger_k}\tilde{X}^{\dagger_k})^q\tilde{B}(\tilde{Y}\tilde{X})^q = \tilde{B}$,

ii) $(\tilde{A}\tilde{A}^{\dagger_k})^q\tilde{X}^{\dagger_k} = \tilde{X}^{\dagger_k}(\tilde{B}\tilde{B}^{\dagger_k})^q$ *and* $\tilde{Y}^{\dagger_k}(\tilde{A}\tilde{A}^{\dagger_k})^q = (\tilde{B}\tilde{B}^{\dagger_k})^q\tilde{Y}^{\dagger_k}$,

where q is positive integer.

Proof. Let $\tilde{A}$ be semi-consimilar^{†_k} to $\tilde{B}$.

i) From equation (7), we can write that

$$\begin{aligned}\tilde{A} = &\ \tilde{X}^{\dagger_k}\tilde{B}\tilde{Y}\\ = &\ \tilde{X}^{\dagger_k}(\tilde{Y}^{\dagger_k}\tilde{A}\tilde{X})\tilde{Y}\\ = &\ (\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})\tilde{A}(\tilde{X}\tilde{Y})\\ = &\ (\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})\tilde{X}^{\dagger_k}\tilde{B}\tilde{Y}(\tilde{X}\tilde{Y})\\ = &\ (\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})\tilde{X}^{\dagger_k}(\tilde{Y^{\dagger_k}}\tilde{A}\tilde{X})\tilde{Y}(\tilde{X}\tilde{Y})\\ = &\ (\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})^2\tilde{A}(\tilde{X}\tilde{Y})^2.\end{aligned}$$

In same manner, we have $(\tilde{X}^{\dagger_k}\tilde{Y}^{\dagger_k})^q\tilde{A}(\tilde{X}\tilde{Y})^q = \tilde{A}$ for $q \in \mathbb{Z}^+$. The proof of $(\tilde{Y}^{\dagger_k}\tilde{X}^{\dagger_k})^q\tilde{B}(\tilde{Y}\tilde{X})^q = \tilde{B}$ is completed by repeating these arguments.

ii) Considering equation (7), we have

$$\begin{aligned}\tilde{A}\tilde{A}^{\dagger_k}\tilde{X}^{\dagger_k} = &\ (\tilde{X}^{\dagger_k}\tilde{B}\tilde{Y})(\tilde{X}^{\dagger_k}\tilde{B}\tilde{Y})^{\dagger_k}\tilde{X}^{\dagger_k}\\ = &\ (\tilde{X}^{\dagger_k}\tilde{B}\tilde{Y})(\tilde{X}\tilde{B}^{\dagger_k}\tilde{Y}^{\dagger_k})\tilde{X}^{\dagger_k}\\ = &\ \tilde{X}^{\dagger_k}\tilde{B}(\tilde{Y}\tilde{X}\tilde{B}^{\dagger_k}\tilde{Y}^{\dagger_k}\tilde{X}^{\dagger_k}).\end{aligned}$$

We conclude from equation (7) that $\tilde{B}^{\dagger_k} = (\tilde{Y}^{\dagger_k}\tilde{A}\tilde{X})^{\dagger_k} = \tilde{Y}\tilde{A}^{\dagger_k}\tilde{X}^{\dagger_k}$ and $\tilde{A}^{\dagger_k} = (\tilde{X}^{\dagger_k}\tilde{B}\tilde{Y})^{\dagger_k} = \tilde{X}\tilde{B}^{\dagger_k}\tilde{Y}^{\dagger_k}$, hence that

$$\tilde{B}^{\dagger_k} = \tilde{Y}\tilde{A}^{\dagger_k}\tilde{X}^{\dagger_k} = \tilde{Y}\tilde{X}\tilde{B}^{\dagger_k}\tilde{Y}^{\dagger_k}\tilde{X}^{\dagger_k}$$

and finally that

$$\tilde{A}\tilde{A}^{\dagger_k}\tilde{X}^{\dagger_k} = \tilde{X}^{\dagger_k}\tilde{B}\tilde{B}^{\dagger_k}.$$

We leave the details to the reader for $q \in \mathbb{Z}^+$. A similar reasoning allows us to prove that the second one.

□

The general inverse of DGC square matrix which one is different from the standard one establish another similarity relations.

Definition 5. *Let $\tilde{A} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$. If there exists a* DGC *square matrix of the same order $\tilde{X}$ such that $\tilde{A}\tilde{X}\tilde{A} = A$, then $\tilde{X}$ is called generalized inverse of $\tilde{A}$ and denoted by $\tilde{A}^-$.*

Definition 6. *For any $\tilde{A}, \tilde{B} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$, the following definitions are given:*

➢ *$\tilde{A}$ is pseudo-similar to $\tilde{B}$ if there exist* DGC *square matrices $\tilde{X}$ and $\tilde{X}^-$ such that*

$$\tilde{X}^-\tilde{A}\tilde{X} = \tilde{B},\ \tilde{X}\tilde{B}\tilde{X}^- = \tilde{A} \ \textit{and}\ \tilde{X}\tilde{X}^-\tilde{X} = \tilde{X}.$$

➢ *$\tilde{A}$ is pseudo-consimilar^{†_k} to $\tilde{B}$ if there exist* DGC *square matrices $\tilde{X}$ and $\tilde{X}^-$ such that*

$$\left(\tilde{X}^-\right)^{\dagger_k}\tilde{A}\tilde{X} = \tilde{B},\ \tilde{X}^{\dagger_k}\tilde{B}\tilde{X}^- = \tilde{A} \ \textit{and}\ \tilde{X}\tilde{X}^-\tilde{X} = \tilde{X}.$$

4 Exponential of DGC Matrices

Exponential function which has applications in many real-world situations, such as finding exponential decay or exponential growth, can be extended to the complex numbers or generalized to other mathematical objects like matrices. It is also commonly used in many areas such as biological sciences for modelling. Besides, the exponential notion is an efficient for Lie algebra since it enables to determine many structures.

In this section, we will introduce the definition of DGC matrix exponential function through a series of DGC square matrix of the order n and obtain some related results. It is worth to note that the definition of the matrix exponential transform into the usual definition of the exponential for DGC numbers for $n = 1$.

For each DGC square matrix $\tilde{A}$ of the order n, let us define the exponential of $\tilde{A}$ to be the DGC matrix

$$e^{\tilde{A}} = \sum_{s=0}^{\infty} \frac{1}{s!}\tilde{A}^s = \tilde{I}_n + \tilde{A} + \frac{1}{2!}\tilde{A}^2 + ... + \frac{1}{s!}\tilde{A}^s + ... \tag{9}$$

Similar to the ordinary exponential function, the above matrix exponential is a matrix function on square DGC matrices.

Lemma 2. *If $\tilde{D} = \operatorname{diag}\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n\}$ is a DGC diagonal matrix with $\tilde{d}_i \in \mathbb{DC}_{\mathfrak{p}}$, $1 \le i \le n$, then*

$$e^{\tilde{D}} = \operatorname{diag}\left\{e^{\tilde{d}_1}, e^{\tilde{d}_2}, ..., e^{\tilde{d}_n}\right\}.$$

Proof. If $\tilde{D} = \operatorname{diag}\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n\}$ is DGC diagonal, then according to equation (9) the following is written:

$$\begin{aligned} e^{\tilde{D}} &= \tilde{I}_n + \tilde{D} + \tfrac{1}{2!}\tilde{D}^2 + \tfrac{1}{3!}\tilde{D}^3 + ... \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \tilde{d}_1 & 0 & \cdots & 0 \\ 0 & \tilde{d}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{d}_n \end{bmatrix} + \begin{bmatrix} \frac{\tilde{d}_1^2}{2!} & 0 & \cdots & 0 \\ 0 & \frac{\tilde{d}_2^2}{2!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\tilde{d}_n^2}{2!} \end{bmatrix} + ... \end{aligned}$$

We thus get

$$e^{\tilde{D}} = \begin{bmatrix} 1 + \tilde{d}_1 + \frac{\tilde{d}_1^2}{2!} + \frac{\tilde{d}_1^3}{3!} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + \tilde{d}_2 + \frac{\tilde{d}_2^2}{2!} + \frac{\tilde{d}_2^3}{3!} + \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \tilde{d}_n + \frac{\tilde{d}_n^2}{2!} + \frac{\tilde{d}_n^3}{3!} + \cdots \end{bmatrix}.$$

This final matrix gives $e^{\tilde{D}} = \begin{bmatrix} e^{\tilde{d}_1} & 0 & \cdots & 0 \\ 0 & e^{\tilde{d}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\tilde{d}_n} \end{bmatrix}$. So the proof is completed. □

Lemma 3. *Let* $\tilde{A} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{bmatrix} \in \mathbb{M}_2(\mathbb{DC}_{\mathfrak{p}})$. *Then,*

- ➢ *If* $\tilde{a} = \tilde{c}$, $e^{\tilde{A}} = e^{\tilde{a}} \begin{bmatrix} 1 & \tilde{b} \\ 0 & 1 \end{bmatrix}$,
- ➢ *If* $\tilde{a} \neq \tilde{c}$ *and* $\tilde{a} - \tilde{c}$ *is non-null,* $e^{\tilde{A}} = \begin{bmatrix} e^{\tilde{a}} & \frac{\tilde{b}(e^{\tilde{a}} - e^{\tilde{c}})}{\tilde{a} - \tilde{c}} \\ 0 & e^{\tilde{c}} \end{bmatrix}$.

Proof. Using induction on $n = 1, 2, \ldots$, the proof can be easily completed. Here equation (9) is considered with the powers of the upper triangular matrix $\tilde{A} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{bmatrix} \in \mathbb{M}_2(\mathbb{DC}_{\mathfrak{p}})$. □

Proposition 1. *For any* DGC *square matrices of the same order* $\tilde{A}$ *and* $\tilde{B}$, *the followings hold:*

i) $e^{\tilde{0}} = I_n$ *where* $\tilde{0}$ *represents* DGC *zero matrix,*

ii) $\tilde{A}^m e^{\tilde{A}} = e^{\tilde{A}} \tilde{A}^m$ *for every integer* m,

iii) $\left(e^{\tilde{A}}\right)^T = e^{\left(\tilde{A}^T\right)}$,

iv) *If* $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ *then* $\tilde{A}e^{\tilde{B}} = e^{\tilde{B}}\tilde{A}$ *and* $e^{\tilde{A}}e^{\tilde{B}} = e^{\tilde{B}}e^{\tilde{A}}$.

Proof. i) Using equation (9), we have $e^{\tilde{0}} = \tilde{I}_n + \tilde{0} + \frac{1}{2!}\tilde{0}^2 + \ldots + \frac{1}{s!}\tilde{0}^s + \ldots = I_n$.

ii) Considering equation (9), we obtain

$$\begin{aligned} \tilde{A}^m e^{\tilde{A}} &= \tilde{A}^m \left(\tilde{I}_n + \tilde{A} + \tfrac{1}{2!}\tilde{A}^2 + \tfrac{1}{3!}\tilde{A}^3 + \ldots \right) \\ &= \tilde{A}^m + \tilde{A}^{m+1} + \tfrac{1}{2!}\tilde{A}^{m+2} + \tfrac{1}{3!}\tilde{A}^{m+3} + \ldots \\ &= \left(\tilde{I}_n + \tilde{A} + \tfrac{1}{2!}\tilde{A}^2 + \tfrac{1}{3!}\tilde{A}^3 + \ldots \right) \tilde{A}^m \\ &= e^{\tilde{A}} \tilde{A}^m \end{aligned}$$

for every integer m.

iii) It is obvious from equation (9) by applying $(\tilde{A}^s)^T = (\tilde{A}^T)^s$, $s = 0, 1, \ldots$

iv) Supposing $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$ and using equation (9), we get

$$\begin{aligned} \tilde{A}e^{\tilde{B}} &= \tilde{A}\left(\tilde{I}_n + \tilde{B} + \tfrac{1}{2!}\tilde{B}^2 + \tfrac{1}{3!}\tilde{B}^3 + ...\right) \\ &= \tilde{A} + \tilde{A}\tilde{B} + \tfrac{1}{2!}\tilde{A}\tilde{B} + \tfrac{1}{3!}\tilde{A}\tilde{B} + ... \\ &= \tilde{A} + \tilde{B}\tilde{A} + \tfrac{1}{2!}\tilde{B}\tilde{A} + \tfrac{1}{3!}\tilde{B}\tilde{A} + ... \\ &= e^{\tilde{B}}\tilde{A}. \end{aligned}$$

With similar thought, $e^{\tilde{A}}e^{\tilde{B}} = e^{\tilde{B}}e^{\tilde{A}}$ can be proved† quickly.

□

Proposition 2. *For any* DGC *square matrix* $\tilde{A}$ *and real numbers* x_1, x_2*, the following equality hold:*

$$e^{\tilde{A}(x_1+x_2)} = e^{\tilde{A}x_1}e^{\tilde{A}x_2}.$$

Proof. Considering equation (9), an easy calculation gives that

$$\begin{aligned} e^{\tilde{A}x_1}e^{\tilde{A}x_2} = & \left(\tilde{I}_n + \tilde{A}x_1 + \tfrac{1}{2!}\tilde{A}^2x_1^2 + ...\right)\left(\tilde{I}_n + \tilde{A}x_2 + \tfrac{1}{2!}\tilde{A}^2x_2^2 + ...\right) \\ = & \left(\sum_{i=0}^{\infty}\tfrac{1}{i!}\tilde{A}^ix_1^i\right)\left(\sum_{j=0}^{\infty}\tfrac{1}{j!}\tilde{A}^jx_2^j\right) \\ = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tfrac{1}{i!j!}\tilde{A}^{i+j}x_1^ix_2^j. \end{aligned}$$

Taking $n = i + j$, we conclude that

$$\begin{aligned} e^{\tilde{A}x_1}e^{\tilde{A}x_2} = & \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\tfrac{1}{(n-j)!j!}\tilde{A}^nx_1^{n-j}x_2^j \\ = & \sum_{n=0}^{\infty}\tfrac{1}{n!}\tilde{A}^n\sum_{j=0}^{\infty}\tfrac{n!}{(n-j)!j!}x_1^{n-j}x_2^j \\ = & \sum_{n=0}^{\infty}\tfrac{1}{n!}\tilde{A}^n\left(x_1+x_2\right)^n. \end{aligned}$$

This completes the proof. □

Lemma 4. *The* DGC *matrix exponential is always invertible.*

Proof. Writing $x_1 = 1$ and $x_2 = -1$ in Proposition 2 yields

$$e^{\tilde{A}}e^{-\tilde{A}} = e^{\tilde{A}-\tilde{A}} = e^0 = \tilde{I}_n.$$

□

†It can also be proved by using Theorem 5.

Extending classical identities of matrix exponential to DGC matrices, we obtain the following theorems in light of dual fundamental matrix (see equation (2)).

Theorem 5. *For any* DGC *square matrix $\tilde{A}$, the exponential of dual fundamental matrix of $\tilde{A}$ and dual fundamental matrix of exponential of $\tilde{A}$ is equal, that is*

$$e^{\chi(\tilde{A})} = \chi\left(e^{\tilde{A}}\right). \tag{10}$$

Proof. Let $\tilde{A} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$ be given and $\chi\left(\tilde{A}\right)$ be dual fundamental matrix of $\tilde{A}$. Then using equation (9) and properties[‡] of $\chi\left(\tilde{A}\right)$, we obtain

$$\begin{aligned} e^{\chi(\tilde{A})} &= \tilde{I}_{2n} + \chi\left(\tilde{A}\right) + \tfrac{1}{2!}\left(\left(\chi(\tilde{A})\right)\right)^2 + \tfrac{1}{3!}\chi\left(\left(\tilde{A}\right)\right)^3 + \ldots \\ &= \chi\Big(\tilde{I}_n + \tilde{A} + \tfrac{1}{2!}\tilde{A}^2 + \tfrac{1}{3!}\tilde{A}^3 + \ldots\Big). \end{aligned}$$

It is clear that $e^{\chi(\tilde{A})} = \chi\left(e^{\tilde{A}}\right)$. □

The previous auxiliary relation will be used in the proof of next theorems.

Theorem 6. *For any* DGC *square matrices of the same order $\tilde{A}$ and $\tilde{B}$, if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, then we have*

$$e^{\tilde{A}+\tilde{B}} = e^{\tilde{A}}e^{\tilde{B}}. \tag{11}$$

Proof. Let $\tilde{A}$ and $\tilde{B}$ be any commute DGC matrices of same order. Then $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} \Leftrightarrow \chi\left(\tilde{A}\tilde{B}\right) = \chi\left(\tilde{B}\tilde{A}\right)$ (see in [33]). Hence the properties $\chi(\tilde{A}\tilde{B}) = \chi(\tilde{A})\chi(\tilde{B})$ and $\chi(\tilde{B}\tilde{A}) = \chi(\tilde{B})\chi(\tilde{A})$ (see in [33]) clearly forces $\chi\left(\tilde{A}\right)$ and $\chi\left(\tilde{B}\right)$ are also commute. As the statement true for dual matrices[§], we can write that

$$e^{\chi(\tilde{A}+\tilde{B})} = e^{\chi(\tilde{A})+\chi(\tilde{B})} = e^{\chi(\tilde{A})}e^{\chi(\tilde{B})}.$$

Equation (10) and now leads to

$$\chi(e^{\tilde{A}+\tilde{B}}) = \chi(e^{\tilde{A}}e^{\tilde{B}}).$$

We thus get $e^{\tilde{A}+\tilde{B}} = e^{\tilde{A}}e^{\tilde{B}}$. □

[‡]see properties in [33]: for $\tilde{A}, \tilde{B} \in \mathbb{M}_n(\mathbb{DC}_{\mathtt{p}})$ and $c \in \mathbb{R}$, $\chi\left(\tilde{A}+\tilde{B}\right) = \chi\left(\tilde{A}\right) + \chi\left(\tilde{B}\right)$, $\chi\left(\tilde{I}_n\right) = I_{2n}$, $\chi\left(c\tilde{A}\right) = c\chi\left(\tilde{A}\right)$, $\chi\left(\tilde{A}\tilde{B}\right) = \chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right)$.

[§]For commutative dual matrices $\mathtt{A}$ and $\mathtt{B}$ of the same order, we have $e^{\mathtt{A}+\mathtt{B}} = e^{\mathtt{A}}e^{\mathtt{B}}$ considering their real fundamental matrices.

This means that it makes no difference for exponential whether $\tilde{A}$ and $\tilde{B}$ are numbers or matrices under the condition that $\tilde{A}$ and $\tilde{B}$ commute.

Theorem 7. *For any* DGC *square matrix $\tilde{A}$, if the* DGC *matrix $\tilde{P}$ is invertible, then we have*

$$e^{\tilde{P}^{-1}\tilde{A}\tilde{P}} = \tilde{P}^{-1}e^{\tilde{A}}\tilde{P}.$$

Proof. Let $\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathtt{p}}\right)$ and assume that $\tilde{P}$ is invertible. Considering properties $\chi\left(\tilde{A}\tilde{B}\right) = \chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right)$ and $\chi\left(\tilde{A}^{-1}\right) = \chi\left(\tilde{A}\right)^{-1}$ (see in [33]), we write that

$$\chi(\tilde{P}^{-1}\tilde{A}\tilde{P}) = \chi(\tilde{P})^{-1}\chi(\tilde{A})\chi(\tilde{P}). \tag{12}$$

As the statement true for dual matrices[¶], we have

$$e^{\chi(\tilde{P})^{-1}\chi(\tilde{A})\chi(\tilde{P})} = \chi(\tilde{P})^{-1}e^{\chi(\tilde{A})}\chi(\tilde{P}). \tag{13}$$

Equations (12) and (13) gives

$$e^{\chi(\tilde{P}^{-1}\tilde{A}\tilde{P})} = \chi(\tilde{P})^{-1}e^{\chi(\tilde{A})}\chi(\tilde{P}). \tag{14}$$

Then by applying equations (10) and (12) to equation (14) yields

$$\chi\left(e^{\tilde{P}^{-1}\tilde{A}\tilde{P}}\right) = \chi\left(\tilde{P}^{-1}e^{\tilde{A}}\tilde{P}\right).$$

According to the property $\chi\left(\tilde{A}\right) = \chi\left(\tilde{B}\right) \Leftrightarrow \tilde{A} = \tilde{B}$ (see in [33]), $e^{\tilde{P}^{-1}\tilde{A}\tilde{P}} = \tilde{P}^{-1}e^{\tilde{A}}\tilde{P}$ is obtained. □

Corollary 1. *If $\tilde{A}$ is* DGC *diagonalizable, then from Definition 3, there exists an invertible* DGC *matrix $\tilde{P}$ such that $\tilde{A} = \tilde{P}\tilde{D}\tilde{P}^{-1}$ where $\tilde{D}$ is a* DGC *diagonal matrix. Hence considering Theorem 7, we can write $e^{\tilde{A}} = \tilde{P}e^{\tilde{D}}\tilde{P}^{-1}$.*

Theorem 8. *For a* DGC *square matrix $\tilde{A}$ and $\lambda \in \mathbb{R}$, if λ is an eigenvalue of $\tilde{A}$, then e^{λ} is an eigenvalue of the* DGC *matrix $e^{\tilde{A}}$.*

Proof. Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of $\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathtt{p}}\right)$. Then we can write $\tilde{A}V = \lambda V$ where V is non-null eigenvector associated with λ. Bearing in mind this fact and using equation (9), we have

$$\begin{aligned} e^{\tilde{A}}V = &\quad V + \tilde{A}V + \tfrac{\tilde{A}^2}{2!}V + \tfrac{\tilde{A}^3}{3!}V + \ldots \\ = &\quad V + \lambda V + \tfrac{\lambda^2}{2!}V + \tfrac{\lambda^3}{3!}V + \ldots \\ = &\quad e^{\lambda}V. \end{aligned}$$

[¶]For dual matrix $\mathtt{A}$, if $\mathtt{P}$ is invertible dual matrix then we have $e^{\mathtt{P}^{-1}\mathtt{A}\mathtt{P}} = \mathtt{P}^{-1}e^{\mathtt{A}}\mathtt{P}$ considering real fundamental matrix.

We thus get e^{λ} is an eigenvalue of the DGC matrix $e^{\tilde{A}}$ with the same corresponding eigenvector V. □

Corollary 2. *For any* DGC *square matrix* $\tilde{A}$*, if* $\lambda \in \mathbb{R}$ *is an eigenvalue of* $\chi\left(\tilde{A}\right)$*, then* e^{λ} *is also an eigenvalue of the* DGC *matrix* $e^{\tilde{A}}$.

Theorem 9. *For any* DGC *square matrix* $\tilde{A}$ *of order* n*, we have*

i) $\tilde{A} = \tilde{B} + xI_n$,

ii) $e^{\tilde{A}} = e^{x}e^{\tilde{B}}$,

where x *is a real number and* $\tilde{B}$ *is a* DGC *matrix of order* n *satisfied that* $\mathrm{Re}\left(\mathrm{tr}\left(\tilde{B}\right)\right) = 0$.

Proof. Let $\tilde{A} = [\tilde{a}_{ij}] \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$.

i) We get
$$\mathrm{Re}\left(\mathrm{tr}(\tilde{A})\right) = \mathrm{Re}\left(\sum_{i=1}^{n} \tilde{a}_{ii}\right) = \sum_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii}\right).$$

Suppose that $x = \frac{1}{n}\sum\limits_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii}\right)$. Thus, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii} - x\right) = & \sum_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii}\right) - nx \\ = & \sum_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii}\right) - n\frac{1}{n}\sum_{i=1}^{n} \mathrm{Re}\left(\tilde{a}_{ii}\right) = 0. \end{aligned}$$

Taking
$$\tilde{B} = \left[\tilde{b}_{ij}\right] = \begin{cases} \tilde{b}_{ij} = \tilde{a}_{ij} - x, & i = j \\ \tilde{b}_{ij} = \tilde{a}_{ij}, & i \neq j, \end{cases}$$

we can assert that $\tilde{A} = \tilde{B} + xI_n$, which satisfied that $\mathrm{Re}\left(\mathrm{tr}\left(\tilde{B}\right)\right) = 0$.

ii) By the above, we have $\tilde{A} = \tilde{B} + xI_n$. Then considering equation (11), it happens that
$$e^{\tilde{A}} = e^{xI_n}e^{\tilde{B}}$$

since $(xI_n)\tilde{B} = \tilde{B}(xI_n)$. Furthermore, using equation (9), we can write $e^{xI_n} = e^{x}I_n$. Finally, we have $e^{\tilde{A}} = e^{x}e^{\tilde{B}}$.

□

We end this section presenting examples about exponential of DGC square matrices. For the convenience of the study, we prefer to use the same coefficients for the DGC square matrices $\tilde{A}$ in [33] to compute their exponential.

Example 1. *Let us take* $\tilde{A}=\begin{bmatrix}1 & 0\\ 0 & J+\varepsilon\end{bmatrix}$ *for* $\mathfrak{p}>0$. *Then* $\chi\left(\tilde{A}\right)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \varepsilon & 0 & \mathfrak{p}\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & \varepsilon\end{bmatrix}$ *with eigenvalues* $\lambda_1=\lambda_2=1$, $\lambda_3=-\sqrt{\mathfrak{p}}+\varepsilon$, $\lambda_4=\sqrt{\mathfrak{p}}+\varepsilon$ *and corresponding eigenvectors* $\alpha_1=\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^T$, $\alpha_2=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$, $\alpha_3=\begin{bmatrix}0 & -\sqrt{\mathfrak{p}} & 0 & 1\end{bmatrix}^T$, $\alpha_4=\begin{bmatrix}0 & \sqrt{\mathfrak{p}} & 0 & 1\end{bmatrix}^T$, *respectively. We thus get*

$$\chi\left(\tilde{A}\right)=\mathtt{P}\operatorname{diag}\{1,1,-\sqrt{\mathfrak{p}}+\varepsilon,\sqrt{\mathfrak{p}}+\varepsilon\}\mathtt{P}^{-1},$$

where $\mathtt{P}$ *is the regular matrix whose columns are formed from the eigenvectors of* $\chi\left(\tilde{A}\right)$ *such that* $\mathtt{P}=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}$ *(see [33]).*

Using the property of the exponential of dual matrices[‖], *we can write* $e^{\chi(\tilde{A})}=\mathtt{P}e^{\mathtt{D}}\mathtt{P}^{-1}$ *where* $\mathtt{D}$ *is the diagonal matrix whose main diagonal entries are eigenvalues of* $\chi\left(\tilde{A}\right)$. *Then we have*

$$\begin{aligned}
e^{\chi(\tilde{A})} &= \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\sqrt{\mathfrak{p}} & \sqrt{\mathfrak{p}}\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}e & 0 & 0 & 0\\ 0 & e & 0 & 0\\ 0 & 0 & e^{-\sqrt{\mathfrak{p}}+\varepsilon} & 0\\ 0 & 0 & 0 & e^{\sqrt{\mathfrak{p}}+\varepsilon}\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & -\frac{1}{2\sqrt{\mathfrak{p}}} & 0 & \frac{1}{2}\\ 0 & \frac{1}{2\sqrt{\mathfrak{p}}} & 0 & \frac{1}{2}\end{bmatrix}\\
&= \begin{bmatrix}e & 0 & 0 & 0\\ 0 & \frac{e^{-\sqrt{\mathfrak{p}}+\varepsilon}+e^{\sqrt{\mathfrak{p}}+\varepsilon}}{2} & 0 & \frac{-\sqrt{\mathfrak{p}}e^{-\sqrt{\mathfrak{p}}+\varepsilon}+\sqrt{\mathfrak{p}}e^{\sqrt{\mathfrak{p}}+\varepsilon}}{2}\\ 0 & 0 & e & 0\\ 0 & \frac{-e^{-\sqrt{\mathfrak{p}}+\varepsilon}+e^{\sqrt{\mathfrak{p}}+\varepsilon}}{2\sqrt{\mathfrak{p}}} & 0 & \frac{e^{-\sqrt{\mathfrak{p}}+\varepsilon}+e^{\sqrt{\mathfrak{p}}+\varepsilon}}{2}\end{bmatrix}.
\end{aligned}$$

By using equations (2) *and* (10), *we get*

$$e^{\tilde{A}}=\begin{bmatrix}e & 0\\ 0 & \frac{(\sqrt{\mathfrak{p}}-J)e^{-\sqrt{\mathfrak{p}}+\varepsilon}+(\sqrt{\mathfrak{p}}+J)e^{\sqrt{\mathfrak{p}}+\varepsilon}}{2\sqrt{\mathfrak{p}}}\end{bmatrix}. \tag{15}$$

Alternatively, Lemma 2 is applicable since $\tilde{A}$ *is a* DGC *diagonal matrix. Considering Lemma 2, one can see the result as*

$$e^{\tilde{A}}=\begin{bmatrix}e & 0\\ 0 & e^{J+\varepsilon}\end{bmatrix}. \tag{16}$$

[‖]For further information of dual numbered matrices, we refer to study [9].

It is worth to note that equations (15) *and* (16) *are same bearing in mind the expansion of* $e^{J+\varepsilon}$.

Example 2. *Let us consider* $\tilde{A} = \begin{bmatrix} 1 & 0 \\ J+\varepsilon & J \end{bmatrix}$ *for* $\mathtt{p} = 9$. *Then* $\chi\left(\tilde{A}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 0 & 9 & 9 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & \varepsilon & 0 \end{bmatrix}$ *with eigenvalues* $\lambda_1 = -3, \lambda_2 = 3, \lambda_3 = \lambda_4 = 1$ *and corresponding eigenvectors* $\alpha_1 = \begin{bmatrix} 0 & -3 & 0 & 1 \end{bmatrix}^T$, $\alpha_2 = \begin{bmatrix} 0 & 3 & 0 & 1 \end{bmatrix}^T$, $\alpha_3 = \begin{bmatrix} -\frac{8}{1+\varepsilon} & \frac{9+\varepsilon}{1+\varepsilon} & 0 & 1 \end{bmatrix}^T$, $\alpha_3 = \begin{bmatrix} -\frac{9+\varepsilon}{1+\varepsilon} & \frac{9}{1+\varepsilon} & 1 & 0 \end{bmatrix}^T$, *respectively. Thus we have* $\chi\left(\tilde{A}\right) = \mathtt{P}\,\mathrm{diag}\{-3,3,1,1\}\mathtt{P}^{-1}$, *where* $\mathtt{P} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}$ *(see [33]). Writing* $e^{\chi(\tilde{A})} = \mathtt{P}\,\mathrm{diag}\{e^{-3}, e^3, e, e\}\mathtt{P}^{-1}$, $e^{\chi(\tilde{A})}$ *is computed as follows:*

$$e^{\chi(\tilde{A})} = \begin{bmatrix} e & 0 & 0 & 0 \\ -\frac{-3+\varepsilon-2e^6(3+\varepsilon)+e^4(9+\varepsilon)}{8e^3} & \frac{1+e^6}{2e^3} & \frac{3\left(-3+\varepsilon-3e^4(1+\varepsilon)+2e^6(3+\varepsilon)\right)}{8e^3} & \frac{3\left(-1+e^6\right)}{2e^3} \\ 0 & 0 & e & 0 \\ \frac{-3+\varepsilon-3e^4(1+\varepsilon)+2e^6(3+\varepsilon)}{24e^3} & \frac{-1+e^6}{6e^3} & -\frac{-3+\varepsilon-2e^6(3+\varepsilon)+e^4(9+\varepsilon)}{8e^3} & \frac{1+e^6}{2e^3} \end{bmatrix}.$$

Finally, by using equations (2) *and* (10), $e^{\tilde{A}}$ *can be written such that*

$$e^{\tilde{A}} = \begin{bmatrix} e & 0 \\ -\frac{-3+\varepsilon-2e^6(3+\varepsilon)+e^4(9+\varepsilon)}{8e^3} & \frac{1+e^6}{2e^3} \end{bmatrix} + J \begin{bmatrix} 0 & 0 \\ \frac{-3+\varepsilon-3e^4(1+\varepsilon)+2e^6(3+\varepsilon)}{24e^3} & \frac{-1+e^6}{6e^3} \end{bmatrix}. \tag{17}$$

Additionaly, by writing $\tilde{A} = \tilde{B} + \frac{1}{2}I_n$, *where* $\tilde{B} = \begin{bmatrix} \frac{1}{2} & 0 \\ J+\varepsilon & J-\frac{1}{2} \end{bmatrix}$, $\mathrm{Re}\left(\mathrm{tr}\left(\tilde{B}\right)\right) = 0$ *(see Theorem 9), equation* (17) *can also be obtained. By calculating* $e^{\chi(\tilde{B})} = \mathtt{P}\,\mathrm{diag}\{e^{-\frac{7}{5}}, e^{\frac{5}{2}}, e^{\frac{1}{2}}, e^{\frac{1}{2}}\}\mathtt{P}^{-1}$ *and considering equations* (2) *and* (10), $e^{\tilde{A}} = e^{\frac{1}{2}}e^{\tilde{B}}$ *is clear.*

The given examples provide an application of how the matrix exponential is applied to a DGC matrix. It is worth to point that the computing the exponential of a DGC matrix is more elaborated than computing the exponential of a real, complex or dual matrices. Hence, in this applications, the role of the dual fundamental matrix is quite considerable for computing.

5 Conclusion

This paper develops various types of similarity relations and the exponential form of DGC matrices by proving several characteristic theorems. It is surely beyond doubt that, considering the dual fundamental matrix of any DGC square matrix is the main advantage of performing this construction. The sparked interest over dual matrix theory in science and engineering is one of the primary motivation of considering DGC matrices with their dual fundamental matrices. Dual matrices have a wide application area in fields of science and engineering, such as the kinematic analysis and synthesis of spatial mechanisms, robot manipulators [3], [6], [9], [13], [26].

Another point worth mentioning is that one can establish the results and discussions of this study for dual-complex matrices for $\mathfrak{p} = -1$ and dual-hyperbolic matrices for $\mathfrak{p} = 1$ matrices. We hope that the results obtained in this study will become an important guide in many scientific fields and enables a more accurate way for computing. Also, for future work, we are planning to study on many different DGC matrix decompositions and discuss the solution methods of linear equation systems with DGC matrices taking into account the commutativity advantage of DGC numbers.

Acknowledgement

This work has been supported by Yildiz Technical University Scientific Research Projects Coordination Unit under project number FBA-2020-4015.

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