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On the dual quaternion geometry of screw motions

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Abstract

In this study, the screw motions are studied using dual quaternions with the help of different perspectives. Firstly, orthogonality definition of dual quaternions is given and geometric interpretation of orthogonality condition is made. Then, the definition of dual circle is given using orthogonal dual quaternions and it is proved that this dual circle can represent the set of all screw motions. Also, these given theorems are reinforced with some conclusions. In addition, it is seen that a dual quaternion represents a screw motions as a screw operator therefore, other dual quaternions derived from the same dual quaternion represent the same screw motions. Then, it is seen that a screw motions symbolized by a dual quaternion transforms one dual vector to another, and when the sign of the dual vectors changes, it provides the same screw motions. Consequently, the answer of the question "Which dual circles symbolizing screw motions are dual orthonormal to each other?" is given and an important conclusion is obtained regarding this.

1 Introduction

Quaternions were first obtained in 1843 by Irish mathematician William R. Hamilton by generalizing complex numbers [1]. Quaternions, an alternative way of describing rotation motions play important role not only in mathematics but also in physics, engineering and many application areas of science.

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Quaternions are used in many different areas such as solving the gimbal lock problem in the motions of aircraft, forward and backward kinematics problems in robotics, and smooth transitions in computer games. It is also used in many disciplines such as graphic designs and physical problems [2, 3, 24, 25]. On the other hand, dual numbers were first put forward by mathematician scientist W. Clifford in 1873 [10]. Dual numbers, in short, are pairs that can be expressed as $x + \varepsilon x^*$ where x, x^* are real numbers and the unit ε is a dual unit that satisfies $\varepsilon^2 = 0$. The discovery of dual numbers has led to many studies. One of them is the finding of dual quaternions. Also, dual quaternions which are an extension of real quaternions were proposed by Clifford [10]. The dual quaternions represent both rotations and translations [11]. Majernik emphasized that the set of dual quaternions is a vector space on the field of real numbers and its dimension is eight [4]. In another study, Majernik expressed Galilean space-time transformations using dual quaternions [5]. The dual quaternions are a compact representation that combines dual numbers and real quaternions and offers useful analytical features. Although dual quaternion are eight dimensional, they are the most efficient and useful way to represent screw motions, i.e both translation and rotation transformations. In addition, Kula and Yaylı showed the commutative multiplication of dual numbers by using multiplication in dual quaternions [8]. Later, Yayli and Tütüncü studied generalized Galilean transformations and dual quaternions [6]. In addition to these studies, Yavh et al. have studies with dual quaternions [15, 16, 17, 18, 19]. In addition, Güngör and Sarduvan gave dual quaternion and dual quaternion matrices [20]. Euler's formula, De-Moivre formula and matrix representations of dual quaternions were given by Ercan and Yüce [7]. Later, many scientists presented many studies on quaternions and dual quaternions [12, 13, 14, 15, 21, 22, 23, 26, 27, 28, 29, 30, 31, 32]. Examples of dual quaternion applications are applications such as robotics studies, back and forth kinematics problems, improvements in digital imaging, computer games. In geometry, on the other hand, it has attracted a lot of attention recently by geometers, thanks to the fact that a very short and pleasant representation of motions with dual quaternions is obtained. As is known, dual quaternions are isomorphic to four-dimensional dual space \mathbb{D}^4 . In addition, the points of the unit dual sphere $\mathbb{D} - Module$ match one to one with the directional lines of the real space \mathbb{R}^3 with the aid of E. Study transformation given by the German mathematician E. Study [11]. Therefore, thanks to the E. Study transformation defined in dual space, the motions in space \mathbb{R}^3 can be expressed with dual quaternions. The motions in this study, which is expressed with dual quaternions, are the screw motions. These motions that express a rotation around a dual axis and a translation through same axis are called screw motions. The operator which makes these motions

is called the screw operator. Therefore, dual quaternions can be considered as screw operators. Hence, dual quaternions are the operator that superimposes a line in \mathbb{R}^3 by rotating and translation on another line with the help of the E. Study transform [2]. In addition, since the motions in the space \mathbb{R}^3 can be easily expressed with dual quaternions, these quaternions are used in robotic applications and to obtain kinematic expressions. Dual quaternions offered alternative solutions in robot kinematics. In addition, extensive information is given about the application of dual number theory to kinematic problems in the book of Bottema and Roth in 1979 [9].

2 Preliminaries

In this section, we give some definition and theorems about dual quaternions.

Definition 1. Assume that $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 i + \mathbf{q}_2 j + \mathbf{q}_3 k$ and $\mathbf{q}^* = \mathbf{q}_0^* + \mathbf{q}_1^* i + \mathbf{q}_2^* j + \mathbf{q}_3^* k$ are two real quaternions in quaternion space \mathbb{H} where $i^2 = j^2 = k^2 = -1$, ij = -ji = k, jk = -kj = i, and ki = -ik = j. Therefore, the quaternion

 $Q = \mathfrak{q} + \varepsilon \mathfrak{q}^*$

is called dual quaternion in $\mathbb{H}_{\mathbb{D}}$ where $\varepsilon^2 = 0$ [2].

Moreover, if we take the dual numbers as

$$Q_0 = \mathfrak{q}_{\mathfrak{o}} + \varepsilon \mathfrak{q}_{\mathfrak{o}}^*, \quad Q_1 = \mathfrak{q}_1 + \varepsilon \mathfrak{q}_1^*, \quad Q_2 = \mathfrak{q}_2 + \varepsilon \mathfrak{q}_2^*, \quad Q_3 = \mathfrak{q}_3 + \varepsilon \mathfrak{q}_3^*$$

then, we get the dual quaternion Q in $\mathbb{H}_{\mathbb{D}}$ as

$$Q = Q_0 + Q_1 i + Q_2 j + Q_3 k.$$

Now, we consider $ScQ = Q_0$ and $VecQ = \vec{Q} = Q_1i + Q_2j + Q_3k$. Hence, the dual quaternion Q can be written by Q = ScQ + VecQ where ScQ is called as the scalar part and VecQ is also called the vectorial part of the dual quaternion Q. If Q is chosen by $Q = \vec{Q} \in Vec\mathbb{H}_{\mathbb{D}} \cong \mathbb{D}$ – Module then, Qis called as pure dual quaternion. In addition to that, the set of all scalars (dual numbers) can be denoted $Sc\mathbb{H}_{\mathbb{D}} \cong \mathbb{D}$. Therefore, \mathbb{D} and \mathbb{D} – Module are embedded in $\mathbb{H}_{\mathbb{D}}$ [2].

Now, we suppose that any two dual quaternions are $P = P_0 + P_1 i + P_2 j + P_3 k$, $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k \in \mathbb{H}_{\mathbb{D}}$. Hence, the dual quaternion product of $P, Q \in \mathbb{H}_{\mathbb{D}}$ is given by

$$Q \times P = Q_0 P_0 - \langle \vec{Q}, \vec{P} \rangle + Q_0 \vec{P} + P_0 \vec{Q} + \vec{Q} \wedge \vec{P}$$
⁽¹⁾

where $\langle \ , \ \rangle$ and \wedge are scalar and vectorial product in $\mathbb{D}-Module.$ Therefore, we can write

$$i) \quad Sc(Q \times P) = Q_0 P_0 - \langle \vec{Q}, \vec{P} \rangle$$
$$ii) \quad Vec(Q \times P) = Q_0 \vec{P} + P_0 \vec{Q} + \vec{Q} \wedge \vec{P}.$$

Moreover, if the dual quaternion multiplication in equation (1) reduces for pure dual quaternions $Q, P \in Vec \mathbb{H}_{\mathbb{D}}$ then, we have the equation

$$Q imes P = ec{Q} imes ec{P} = - \langle ec{Q}, ec{P}
angle + ec{Q} \wedge ec{P}$$

where $Sc(Q \times P) = -\langle \vec{Q}, \vec{P} \rangle$ and $Vec(Q \times P) = \vec{Q} \wedge \vec{P}$ for pure dual quaternions $Q, P \in Vec\mathbb{H}_{\mathbb{D}}$ [2].

Definition 2. Let the dual quaternion be $Q = ScQ+VecQ = Q_0+Q_1i+Q_2j+Q_3k$. In this case, there are three definitions of dual quaternion conjugate as follows

- i) Quaternion Conjugate : $\overline{Q} = ScQ VecQ = Q_0 Q_1i Q_2j Q_3k$,
- $ii) \ \ Dual \ Conjugate: Q^* = (ScQ)^* + (VecQ)^* = Q_0^* + Q_1^*i + Q_2^*j + Q_3^*k,$

iii) Total Conjugate :
$$(\overline{Q})^* = (ScQ)^* - (VecQ)^* = Q_0^* - Q_1^*i - Q_2^*j - Q_3^*k$$
,

[2].

Considering Definition 2, we have

$$i) \quad ScQ = Q_0 = \frac{1}{2}(Q + \overline{Q})$$

$$ii) \quad VecQ = \overline{Q} = \frac{1}{2}(Q - \overline{Q})$$

$$iii) \quad Vec\overline{Q} = -VecQ$$

$$(2)$$

for $Q \in \mathbb{H}_{\mathbb{D}}$. Moreover, for pure dual quaternion Q we get

$$\overline{Q} = -VecQ = -\vec{Q} = -Q$$

and for pure dual quaternions $P, Q \in \mathbb{H}_{\mathbb{D}}$, we have

$$i) \quad Sc(P \times \overline{Q}) = Q_0 = -\frac{1}{2}(P \times Q + Q \times P) = \langle \vec{P}, \vec{Q} \rangle,$$
$$ii) \quad Vec(P \times \overline{Q}) = -\frac{1}{2}(P \times Q - Q \times P) = -\vec{P} \wedge \vec{Q}$$

[2].

Definition 3. The norm of any dual quaternion Q in $\mathbb{H}_{\mathbb{D}}$ is defined as

$$\|Q\|^2 = Q \times \overline{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$$

and called the norm of Q where |||| is norm in \mathbb{D}^4 [2].

The norm of dual quaternions coincides with the norm of $\mathbb{H}_{\mathbb{D}}$ regarded as an element of the dual space \mathbb{D}^4 .

If ||Q|| = (1,0) = 1 then, Q is called unit dual quaternion. We know that each non-zero dual quaternion Q has a unique inverse

$$Q^{-1} = \frac{\overline{Q}}{\|Q\|}$$

where $||Q^{-1}|| = ||Q||^{-1}$ [2]. If we assume that the dual quaternion Q is unit hence, $Q^{-1} = \overline{Q}$. If the dual quaternion Q is considered both pure and unit then, one can get

$$Q^{-1} = \overline{Q} = -Q \Longrightarrow Q \times Q = -1.$$

In this case, the scalar part of the dual quaternion multiplication $P \times \overline{Q}$ corresponds to the dual scalar product in \mathbb{D}^4 and we also know

$$Sc(P \times \overline{Q}) = \langle P, Q \rangle$$

where $(\overline{P \times Q}) = \overline{Q} \times \overline{P}$ and $||P \times Q|| = ||P|| ||Q||$ and $\langle , \rangle, ||||$ are scalar product and norm in \mathbb{D}^4 [2].

Now, let Q be an arbitrary dual unit quaternion $Q \neq 0$. Therefore, we can demonstrate this quaternion as

$$Q = \cos\frac{\Theta}{2} + \vec{S}\sin\frac{\Theta}{2} \tag{3}$$

where the unit dual vector $\vec{S} = \frac{\vec{Q}}{\|\vec{Q}\|}$ is axis of the dual quaternion Q. Moreover, $\Theta = \theta + \varepsilon \theta^*$ is dual angle, $\theta = 2 \arccos q_0$ and $\theta^* = \frac{-2q_0^*}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$ where $Q_0 = q_0 + \varepsilon q_0^*$, $Q_1 = q_1 + \varepsilon q_1^*$, $Q_2 = q_2 + \varepsilon q_2^*$, $Q_3 = q_3 + \varepsilon q_3^*$ [2]. Any screw motions $A \in S\hat{O}(3)$ mapping the unit dual vector $\vec{H} \in Vec \mathbb{SD}^2$

onto the unit dual vector $\vec{R} \in Vec \mathbb{S}_{\mathbb{D}}^2$ according to

$$A\vec{H} = \vec{R}$$

can be written in terms of its dual quaternion representation $Q \in \mathbb{H}_{\mathbb{D}}$

$$Q \times \vec{H} \times Q^{-1} = \vec{R}.$$

Moreover, for $Q \in \mathbb{S}_{\mathbb{D}}^3$ the previous expression becomes

$$Q \times \vec{H} \times \bar{Q} = \vec{R} \tag{4}$$

which explicitly reads then

$$\vec{R} = \cos\Theta\vec{H} + \sin\Theta(\vec{S}\wedge\vec{H}) + (1-\cos\Theta)\langle\vec{S},\vec{H}\rangle$$
(5)

where $Q = \cos \frac{\Theta}{2} + \vec{S} \sin \frac{\Theta}{2}$ is unit dual quaternion, $\Theta = \theta + \varepsilon \theta^*$ is dual angle and \langle , \rangle and \wedge are the scalar and vectorial product in $\mathbb{D} - Module$. The unit dual quaternion $Q \in \mathbb{S}_{\mathbb{D}}^3$ represents the screw motions about the unit dual axis $\vec{S} = \frac{\vec{Q}}{\|\vec{Q}\|}$ by the rotation angle θ and the translation θ^* where $Q_0 = \cos \frac{\Theta}{2}$ and $\Theta = 2 \arccos Q_0$. Moreover, since $\Theta = \theta + \varepsilon \theta^*$ one can get $\theta = 2 \arccos q_0$ and $\theta^* = \frac{-2q_0}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$. Therefore, each unit dual quaternion $Q \in \mathbb{S}_{\mathbb{D}}^3$ can be seen as a representation of a screw motions in $\mathbb{D} - Module$. Hence, the unit dual quaternion Q can be considered as a screw operator [2].

3 Main Theorems and Results

In this section, we give the screw motions mapping the dual vectors \vec{H} to \vec{R} with the aid of dual quaternions using different perspectives. Firstly, we express the definition of dual orthogonal quaternions and the geometric interpretation of these quaternions. Then, using the dual orthogonal quaternions we introduce the dual circles and say that these dual circles can represent the set of all screw motions mapping \vec{H} to \vec{R} . Moreover, for the pure dual quaternions H and R we give some conclusions. Also, we express that the screw motions mapping \vec{H} to \vec{R} symbolized by a unit dual quaternion depends on the common sign of H and R. Consequently, we conclude that the dual circles represent the screw motions mapping \vec{H} to \vec{R} and $-\vec{H}$ to \vec{R} are dual orthonormal.

Theorem 4. Consider that $P, Q \in \mathbb{S}_{\mathbb{D}}^3$ are two arbitrary unit dual quaternions where Q represents the rotation about the axis \vec{S} with the angle θ and the translation about the axis \vec{S} by θ^* (i.e. screw motions) and $\mathbb{S}_{\mathbb{D}}^3$ is unit dual sphere in \mathbb{D}^4 . Therefore, $P \times Q \times \bar{P} \in \mathbb{S}_{\mathbb{D}}^3$ represents the same screw motions which have the dual axis is $P \times \vec{S} \times \bar{P}$ where $\Theta = \theta + \varepsilon \theta^*$ is dual angle of screw motions.

Proof. Let $P, Q \in \mathbb{S}_{\mathbb{D}}^3$ be two arbitrary unit dual quaternions. We know that the dual quaternion Q is a screw operator providing the screw motions. In that case, we can write Q as

$$Q = \cos\frac{\Theta}{2} + \vec{S}\sin\frac{\Theta}{2}.$$

Hence, we get

$$P \times Q \times \bar{P} = P \times (\cos \frac{\Theta}{2} + \vec{S} \sin \frac{\Theta}{2}) \times \bar{P}$$
$$= \cos \frac{\Theta}{2} (P \times \bar{P}) + \sin \frac{\Theta}{2} (P \times \vec{S} \times \bar{P})$$

and consequently, we obtain

$$P \times Q \times \bar{P} = \cos \frac{\Theta}{2} + (P \times \vec{S} \times \bar{P}) \sin \frac{\Theta}{2}$$

where $\Theta = \theta + \varepsilon \theta^*$ is dual angle of screw motions. Therefore, we can express that the dual quaternion $P \times Q \times \bar{P}$ is screw operator providing the screw motions with dual angle $\Theta = \theta + \varepsilon \theta^*$ and dual axis $P \times S \times \bar{P}$.

Definition 5. Let P, Q be any two dual quaternions in $\mathbb{H}_{\mathbb{D}}$. In this case, the dual quaternion Q is orthogonal with the dual quaternion P if $P \times \overline{Q}$ is a pure dual quaternion. Moreover, if the dual quaternions P, Q are both orthogonal and unit then, they are called dual orthonormal quaternions.

Now, we analyze the condition of dual orthogonality for dual quaternions. Considering the equation (2), the condition of dual orthogonality is also indicated as

$$Sc(P \times \overline{Q}) = \frac{1}{2}(P \times \overline{Q} + Q \times \overline{P}) = 0.$$

Therefore, we can say that the orthogonality of these quaternions is characterized by the fact that the scalar product of the dual vectors corresponding to this dual quaternions in four dimensional dual space \mathbb{D}^4 is zero. We can also do the same geometric interpretation for pure dual quaternions. The vector parts of pure dual quaternions are dual orthogonal then, they are dual orthogonal. In addition to that, we consider that the dual quaternions $Q, P \in \mathbb{H}_{\mathbb{D}}$ are pure hence, the condition of pure dual orthogonality is

$$P \times Q = \vec{P} \wedge \vec{Q}$$

where $VecP = \vec{P}$ and $VecQ = \vec{Q}$.

Proposition 6. Consider that $\mathbb{S}_{\mathbb{D}}^3$ is unit dual sphere in \mathbb{D}^4 and $P, Q \in \mathbb{S}_{\mathbb{D}}^3$ are any two unit dual orthogonal quaternions. Therefore, there is a pure unit dual quaternion $V \in Vec\mathbb{H}_{\mathbb{D}}$ satisfying the condition $P = V \times Q$.

Proof. Let $P, Q \in \mathbb{S}_{\mathbb{D}}^3$ be two unit dual orthogonal quaternions. In this case, the dual quaternion multiplication $P \times \overline{Q}$ is pure dual quaternion. Hence, one

can take $P \times \overline{Q} = V \in Vec\mathbb{H}_{\mathbb{D}}$. Consequently, we obtain

$$V = P \times \overline{Q} \iff V \times Q = (P \times \overline{Q}) \times Q$$
$$\iff V \times Q = P \times ||Q||^2$$
$$\iff P = V \times Q$$

where Q is unit dual quaternion and ||Q|| = 1.

Definition 7. Suppose that Q_1 and Q_2 are two unit dual orthogonal quaternion in $\mathbb{H}_{\mathbb{D}}$. Therefore, the set of dual quaternions

$$Q(\phi) = Q_1 \cos \phi + Q_2 \sin \phi$$

is defined as a dual circle in the dual quaternionic space $\mathbb{H}_{\mathbb{D}}$ and is denoted by $C_{\mathbb{D}}(Q_1, Q_2)$ where $\phi = \varphi + \varepsilon \varphi^* \in \mathbb{D}$ ($\varphi \in [0, 2\pi)$) is dual angle.

It is clear that, the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ is the intersection of the dual plane $E_{\mathbb{D}}(Q_1, Q_2) \subset \mathbb{H}_{\mathbb{D}}$ spanned by Q_1, Q_2 and passing through the origin with the unit dual sphere $\mathbb{S}_{\mathbb{D}}^3 \subset \mathbb{H}_{\mathbb{D}}$. In this case, we can write

$$C_{\mathbb{D}}(Q_1, Q_2) = E(Q_1, Q_2) \cap \mathbb{S}_{\mathbb{D}}^3.$$
(6)

Proposition 8. Let $(\vec{H}, \vec{R}) \in \mathbb{S}_{\mathbb{D}}^2 \times \mathbb{S}_{\mathbb{D}}^2$ be unit dual vector pair, where $\vec{H} \wedge \vec{R} \neq 0$ (i. e. their axes are not coincident) and $\mathbb{S}_{\mathbb{D}}^2$ is unit dual sphere in \mathbb{D} – Module. Therefore, the set $G_{\mathbb{D}}(\vec{H}, \vec{R}) \subset SO(3)$ of all screw motions $A\vec{H} = \vec{R} \ (A \in SO(3))$ can be represented as the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ of unit dual quaternions Q where

$$Q \times \vec{H} \times \bar{Q} = \vec{R}, \ \forall \ Q \in C_{\mathbb{D}}(Q_1, Q_2)$$

with dual orthogonal quaternions

$$Q_1 := \frac{1}{\|1 - R \times H\|} \cdot (1 - R \times H) = \cos \frac{\Psi}{2} + \frac{H \wedge R}{\|H \wedge R\|} \sin \frac{\Psi}{2}$$

$$Q_2 := \frac{H+R}{\|H+R\|}$$

and

$$||1 - R \times H|| = \sqrt{2(1 + \cos \Psi)} = 2\cos \frac{\Psi}{2},$$

$$|H+R|| = 2\cos\frac{\Psi}{2}$$

where Ψ is dual angle for the unit pure dual quaternion H, R.

Proof. We assume that the dual axis of any screw motions $A\vec{H} = \vec{R}$ in the dual plane spanned by

$$\vec{S}_{1} = \frac{\vec{H} \wedge \vec{R}}{\|\vec{H} \wedge \vec{R}\|} = \frac{1}{\sin \Psi} \vec{H} \wedge \vec{R}$$

$$\vec{S}_{2} = \frac{\vec{H} + \vec{R}}{\|\vec{H} + \vec{R}\|} = \frac{1}{2 \cos \frac{\Psi}{2}} (\vec{H} + \vec{R})$$
(7)

where the dual angle Ψ is $\Psi = \psi + \varepsilon \psi^* \in \mathbb{D}$. Therefore, the dual plane $E_{\mathbb{D}}(\vec{S_1}, \vec{S_2}) = \langle \vec{S_1}, \vec{S_2} \rangle \in \mathbb{D} - Module$ of screw motions axes can be written by the unit normal of this dual plane as

$$\vec{S}_1 \wedge \vec{S}_2 = \frac{1}{2\sin\Psi\cos\frac{\Psi}{2}} \left[(\vec{H} \wedge \vec{R}) \wedge (\vec{H} + \vec{R}) \right]$$
$$= \frac{1}{2\sin\Psi\cos\frac{\Psi}{2}} \left(\begin{array}{c} \langle \vec{H}, \vec{H} \rangle \vec{R} - \langle \vec{H}, \vec{R} \rangle \vec{H} \\ -\langle \vec{R}, \vec{R} \rangle \vec{H} + \langle \vec{H}, \vec{R} \rangle \vec{R} \end{array} \right)$$
$$= \frac{1 + \cos\Psi}{2\sin\Psi\cos\frac{\Psi}{2}} (\vec{R} - \vec{H})$$

and consequently, we get

$$\vec{S}_1 \wedge \vec{S}_2 = \frac{1}{2\sin\frac{\Psi}{2}} (\vec{R} - \vec{H}).$$
(8)

Now, the dual angles of screw motions about the axes $\vec{S_1}$ and $\vec{S_2}$ are $\Psi = \psi + \varepsilon \psi^* \in \mathbb{D}$ and $\pi \in \mathbb{R}$ (only rotation), respectively. Therefore, we can give that the dual quaternions Q_1 and Q_2 are

$$Q_1 = \cos\frac{\Psi}{2} + \vec{S_1}\sin\frac{\Psi}{2}$$
$$Q_2 = \vec{S_2}.$$

Obviously, the dual quaternions Q_1 and Q_2 are dual orthogonal quaternions. Then, we get

$$Q_1 := \cos\frac{\Psi}{2} + \frac{\vec{H} \wedge \vec{R}}{\|\vec{H} \wedge \vec{R}\|} \sin\frac{\Psi}{2},$$

$$Q_2 := \frac{\vec{H} + \vec{R}}{\|\vec{H} + \vec{R}\|}.$$

The proof is completed.

The normalized dual axis \vec{S}_{Ω} of any screw motions, i.e. the normalized dual vector part of a unit dual quaternion Q with $Q \times \vec{H} \times \bar{Q} = \vec{R}$, is an element of dual circle $C_{\mathbb{D}}(\vec{S}_1, \vec{S}_2)$ where $C_{\mathbb{D}}(\vec{S}_1, \vec{S}_2) = E_{\mathbb{D}}(\vec{S}_1, \vec{S}_2) \cap \mathbb{S}_{\mathbb{D}}^2$. Now, we assume that $\Omega = \omega + \varepsilon \omega^* \in \mathbb{D}$ is dual angle and the Euclidean angle ω in this dual angle is in between ψ and $2\pi - \psi$. Hence, the group of screw motions between \vec{H} and \vec{R} can be written by

$$G_{\mathbb{D}} = (\vec{H}, \vec{R}) = \{A \in SO(3) \mid A = A(\Omega, S_{\Omega})\}$$

where the dual axis of screw motions $\vec{S_{\Omega}}$ is

$$\vec{S}_{\Omega} = \vec{S}_1 \cos \Gamma + \vec{S}_2 \sin \Gamma \ (\Gamma = \gamma + \varepsilon \gamma^*, \ \gamma \in [0, 2\pi)).$$
(9)

In addition to that, the dual angle $\Omega = \omega + \varepsilon \omega^*$ is related to the dual axis $\vec{S_{\Omega}}$ and one can write

$$Q = \cos\frac{\Omega}{2} + \vec{S_{\Omega}}\sin\frac{\Omega}{2}$$
$$Q = \cos\frac{\Omega}{2} + \left(\frac{Q_1 - \cos\frac{\Psi}{2}}{\sin\frac{\Psi}{2}}\right)\cos\Gamma\sin\frac{\Omega}{2} + Q_2\sin\Gamma\sin\frac{\Omega}{2}$$

and since $Q \in C_{\mathbb{D}}(Q_1, Q_2)$ one can get $(\cos \frac{\Omega}{2} \sin \frac{\Psi}{2} = \cos \frac{\Psi}{2} \cos \Gamma \sin \frac{\Omega}{2})$ and consequently,

$$\tan\frac{\Omega}{2} = \frac{\sin\frac{\Psi}{2}}{\cos\frac{\Psi}{2}\cos\Gamma}$$

where

$$\sin \frac{\Omega}{2} = \frac{\sin \frac{\Psi}{2}}{(\sin^2 \frac{\Psi}{2} + \cos^2 \frac{\Psi}{2} \cos^2 \Gamma)^{1/2}},$$
(10)
$$\cos \frac{\Omega}{2} = \frac{\cos \frac{\Psi}{2} \cos \Gamma}{(\sin^2 \frac{\Psi}{2} + \cos^2 \frac{\Psi}{2} \cos^2 \Gamma)^{1/2}}.$$

Therefore, the set of unit dual quaternions $Q(\Gamma)$, i.e.

$$Q(\Gamma) = \cos\frac{\Omega}{2} + \vec{S_{\Omega}}\sin\frac{\Omega}{2}$$

$$Q(\Gamma) = \frac{1}{(\sin^2 \frac{\Psi}{2} + \cos^2 \frac{\Psi}{2} \cos^2 \Gamma)^{1/2}} \left[(\cos \frac{\Psi}{2} + \vec{S_1} \sin \frac{\Psi}{2}) \cos \Gamma + \vec{S_2} \sin \frac{\Psi}{2} \sin \Gamma \right].$$

Now, we assume that

$$A_1(\Gamma) = \frac{\cos \Gamma}{(\sin^2 \frac{\Psi}{2} + \cos^2 \frac{\Psi}{2} \cos^2 \Gamma)^{1/2}},$$

$$A_2(\Gamma) = \frac{\sin\frac{\Psi}{2}\sin\Gamma}{(\sin^2\frac{\Psi}{2} + \cos^2\frac{\Psi}{2}\cos^2\Gamma)^{1/2}},$$

hence, we obtain

$$Q(\Gamma) = Q_1 A_1(\Gamma) + Q_2 A_2(\Gamma).$$
(11)

One can easily see that

$$\begin{aligned} A_1^2(\Gamma) + A_2^2(\Gamma) &= \frac{\cos^2 \Gamma + \sin^2 \Gamma \sin^2 \frac{\Psi}{2}}{\sin^2 \frac{\Psi}{2} + \cos^2 \frac{\Psi}{2} \cos^2 \Gamma}, \\ &= \frac{1 - \sin^2 \Gamma + \sin^2 \Gamma \sin^2 \frac{\Psi}{2}}{\sin^2 \frac{\Psi}{2} + (1 - \sin^2 \frac{\Psi}{2})(1 - \sin^2 \Gamma)} \\ &= 1 \end{aligned}$$

and consider that

$$A_1(\Gamma) = \cos \phi, \ A_2(\Gamma) = \sin \phi$$

with a new parameter $\phi = \varphi + \varepsilon \varphi^* \ (\varphi \in [0, 2\pi))$. Therefore the equation (11) can be written as

$$Q(\phi) = Q_1 \cos \phi + Q_2 \sin \phi \tag{12}$$

where $\phi = \varphi + \varepsilon \varphi^*$ and $\varphi \in [0, 2\pi)$. In that case, we write a conclusion that the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ corresponding to the equation (12) represents all screw motions mapping \vec{H} to \vec{R} . Moreover, we can see that for $\varphi = 0$ and $\varphi = \frac{\pi}{2}$

$$Q(0) = Q_1,$$
$$Q(\frac{\pi}{2}) = Q_2$$

where $\varphi^* = 0$.

Now, we obtain some useful equations as follows.

Conclusion 9. We assume that the dual vectors \vec{H} and \vec{R} are two pure unit dual quaternions \vec{H} and \vec{R} . Therefore, we get

$$R \times (1 - R \times H) = (1 - R \times H) \times H = H + R.$$
⁽¹³⁾

Proof. Consider that the dual vectors \vec{H} and \vec{R} as pure unit dual quaternions. Then we get

$$R \times (1 - R \times H) = R - R \times (R \times H)$$

= R + \langle R, R \langle H \rangle + \langle R, H \rangle R - R \langle (R \langle H)
= R + \langle R, H \rangle R - R \langle (R \langle H)
= R + H

and similarly we obtain

$$\begin{split} (1-R\times H)\times H =& H - (R\times H)\times H \\ =& H + \langle R\wedge H, H\rangle + \langle R, H\rangle H - (R\wedge H)\wedge H \\ =& H + R. \end{split}$$

Consequently, we easily obtain

$$R \times (1 - R \times H) = (1 - R \times H) \times H = H + R.$$

Conclusion 10. Assume that the dual vectors \vec{H} and \vec{R} are two pure unit dual quaternions and the dual quaternions Q_1 and Q_2 are dual orthogonal quaternions. Therefore, the equation

$$R \times Q_1 = Q_1 \times H = Q_2 \tag{14}$$

is held.

Proof. Let Q_1, Q_2 be dual orthogonal quaternions and H, R be pure unit dual quaternions. Hence, we take the norm of dual quaternions in equation (13) and we get

$$||H + R|| = ||R \times (1 - R \times H)|| = ||1 - R \times H||.$$
(15)

On the other hand, from the equation (13) and (15) we get

$$R \times Q_1 = \frac{1}{\|1 - R \times H\|} \left(R \times (1 - R \times H) \right)$$
$$= \frac{1}{\|H + R\|} (H + R) = Q_2$$

and

$$Q_1 \times H = \frac{1}{\|1 - R \times H\|} \left((1 - R \times H) \times H \right)$$
$$= \frac{1}{\|H + R\|} (H + R) = Q_2.$$

Consequently, we have

$$R \times Q_1 = Q_1 \times H = Q_2.$$

We conclude that the dual quaternions $R \times Q_1$ and $Q_1 \times H$ also represents screw motions mapping \vec{H} to \vec{R} .

Moreover, we can give the conclusion as the result of the equation (14).

Conclusion 11. Consider that the unit dual quaternions Q_1 and Q_2 are dual orthogonal quaternions and the unit dual vectors are \vec{H} and \vec{R} providing the condition $Q \times \vec{H} \times \overline{Q} = \vec{R}$ where $Q \in C_{\mathbb{D}}(Q_1, Q_2)$. Therefore, the equations

$$\overline{Q_2} \times \vec{R} \times Q_1 = 1, \qquad Q_1 \times \vec{H} \times \overline{Q_2} = 1$$

are held. These equations emphasize a magnificent factorization of number 1. In addition to that, the pure dual quaternions H and R can be written as

$$H = \overline{Q_1} \times Q_2, \qquad R = Q_2 \times \overline{Q_1}.$$

Theorem 12. Let the dual quaternion Q in $\mathbb{H}_{\mathbb{D}}$ be element of the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$. Therefore, the dual quaternions $R \times Q$, $Q \times H$ and $R \times Q \times H$ represent the screw motions mapping the unit dual vectors \vec{H} to \vec{R} .

Proof. We know that the dual quaternion Q represent the screw motions mapping \vec{H} to \vec{R} with $Q \times \vec{H} \times \overline{Q} = \vec{R}$. In that case;

• If the dual quaternion $R \times Q$ is written instead of the dual quaternion Q in the last equation, then the equation

$$(R \times Q) \times H \times (\overline{R \times Q}) = R \times (Q \times H \times \overline{Q})) \times \overline{R} = R$$

is obtained. This means that the dual quaternion $R \times Q$ represents the screw motions mapping \vec{H} to \vec{R} .

• If the dual quaternion $Q \times H$ is considered, the equation

$$(Q \times H) \times H \times (\overline{Q \times H}) = Q \times H \times \overline{Q} = R$$

is held. Hence, the dual quaternion $Q\times H$ also represents the same screw motions.

• A similar proof can be easily made for the dual quaternion $R \times Q \times H$. \Box

Proposition 13. Assume that the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ is defined by the circle with origin centre in dual plane $E_{\mathbb{D}}(Q_1, Q_2) \subseteq \mathbb{H}_{\mathbb{D}}$ spanned by the dual orthonormal quaternions $Q_1, Q_2 \in \mathbb{S}_{\mathbb{D}}^3$. There is the pair of unit dual vectors $(\vec{H}, \vec{R}) \in \mathbb{S}_{\mathbb{D}}^2 \times \mathbb{S}_{\mathbb{D}}^2$ such that

$$Q \times \vec{H} \times \overline{Q} = \vec{R}$$

 $\forall Q \in C_{\mathbb{D}}(Q_1, Q_2)$ and the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ is the set of all screw motions mapping \vec{H} to \vec{R} . Therefore, this pair of unit dual vector depends on the common sign of \vec{H} and \vec{R} i.e

$$Q \times (-\vec{H}) \times \overline{Q} = -\vec{R}.$$

Proof. We know that the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ with origin centre can be written as

$$Q(\phi) = Q_1 \cos \phi + Q_2 \sin \phi \quad (\phi = \varphi + \varepsilon \varphi^*, \ \varphi \in [0, 2\pi))$$

and since the dual quaternions Q_1 and Q_2 are dual orthogonal to each other, we write

$$Sc(Q_1 \times \overline{Q_2}) = Sc(Q_2 \times \overline{Q_1}) = 0.$$

Hence, the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ represent all screw motions mapping the dual vector $\vec{H} = \overline{Q_1} \times Q_2$ to the dual vector $\vec{R} = Q_2 \times \overline{Q_1}$. Consequently, for $\forall Q(\phi) \in C_{\mathbb{D}}(Q_1, Q_2)$

$$\begin{aligned} Q(\phi) \times (-\vec{H}) \times \overline{Q}(\phi) \\ &= (Q_1 \cos \phi + Q_2 \sin \phi) \times (-\overline{Q_1} \times Q_2) \times (\overline{Q_1} \cos \phi + \overline{Q_2} \sin \phi) \\ &= -Q_2 \times \overline{Q_1} - \cos \phi \sin \phi + Q_1 \times (\overline{Q_2} \times Q_2) \times \overline{Q_1} \cos \phi \sin \phi \\ &= -Q_2 \times \overline{Q_1} = -\vec{R}. \end{aligned}$$

Therefore, while the unit dual quaternion Q makes the screw motions mapping the dual vectors \vec{H} to \vec{R} , it makes the screw motions mapping the dual vectors $-\vec{H}$ to $-\vec{R}$.

Theorem 14. The dual circles $C_{\mathbb{D}}(Q_1, Q_2)$ and $C_{\mathbb{D}}(Q_3, Q_4)$ represent the screw motions $G_{\mathbb{D}}(\vec{H}, \vec{R})$ and $G_{\mathbb{D}}(-\vec{H}, \vec{R})$, respectively, are dual orthonormal with each other.

Proof. Consider that while the dual circle $C_{\mathbb{D}}(Q_1, Q_2)$ represents the screw motions $G_{\mathbb{D}}(\vec{H}, \vec{R})$, the dual circle $C_{\mathbb{D}}(Q_3, Q_4)$ represent also the screw motions

 $C_{\mathbb{D}}(Q_3, Q_4)$. Let \vec{S}_3 and \vec{S}_4 be the axes of the screw motions $G_{\mathbb{D}}(-\vec{H}, \vec{R})$ for the dual quaternions Q_3 and Q_4 . Hence, using the equation (7) we get

$$\vec{S}_3 = \frac{-\vec{H} \wedge \vec{R}}{\|-\vec{H} \wedge \vec{R}\|},$$

where $\| - \vec{H} \wedge \vec{R} \| = \| \vec{H} \wedge \vec{R} \|$. In this case, we obtain

$$\vec{S}_3 = -\frac{\vec{H} \wedge \vec{R}}{\|\vec{H} \wedge \vec{R}\|} = -\vec{S}_1.$$

We know that the dual angle between the dual vectors \vec{H} and \vec{R} is Ψ = $\psi + \varepsilon \psi^*$. Therefore, the dual angle between the dual vectors $-\vec{H}$ and \vec{R} is $\Psi' = (\pi - \psi) - \varepsilon \psi^*$. Moreover, we can write for the unit dual quaternion $Q_3 = \cos \frac{\Psi'}{2} + \vec{S}_3 \sin \frac{\Psi'}{2}$ where $\Psi' = (\pi - \psi) - \varepsilon \psi^*$. Similar to this, the dual axis \vec{S}_4 can be written

$$\vec{S_4} = \frac{-\vec{H} + \vec{R}}{\| - \vec{H} + \vec{R} \|}$$

where $\| - \vec{H} + \vec{R} \| = 2 \sin \frac{\Psi}{2}$. In addition to that, using the equation (8) we have

$$\vec{S_4} = \vec{S_1} \wedge \vec{S_2}$$

and for the unit dual quaternion Q_4 we get

$$Q_4 = \vec{S_4}.$$

Therefore, the dual circles $C_{\mathbb{D}}(\vec{S_1}, \vec{S_2})$ and $C_{\mathbb{D}}(\vec{S_3}, \vec{S_4}) \subset \mathbb{S}_{\mathbb{D}}^2$ are dual orthogonal to each other. Now, we assume that $Q \in C_{\mathbb{D}}(Q_1, Q_2)$ and $P \in C_{\mathbb{D}}(Q_3, Q_4)$ are arbitrary unit dual quaternions. In that case, we can write the unit dual quaternion $Q \in C_{\mathbb{D}}(Q_1, Q_2)$ is

$$Q = Q_1 \cos \phi + Q_2 \sin \phi$$
$$= (\cos \frac{\Psi}{2} + \vec{S_1} \sin \frac{\Psi}{2}) \cos \phi + \vec{S_2} \sin \phi$$

and

$$Q = \cos\frac{\Psi}{2}\cos\phi + \vec{S_1}\sin\frac{\Psi}{2}\cos\phi + \vec{S_2}\sin\phi.$$

Similarly, the unit dual quaternion $P \in C_{\mathbb{D}}(Q_3, Q_4)$ can be written as

$$P = Q_3 \cos \phi + Q_4 \sin \phi$$
$$= (\cos \frac{\Psi'}{2} + \vec{S_3} \sin \frac{\Psi}{2}) \cos \phi + \vec{S_4} \sin \phi$$

where $\Psi' = (\pi - \psi) - \varepsilon \psi^*$, $\cos \frac{\Psi'}{2} = \sin \frac{\Psi}{2}$ and $\sin \frac{\Psi'}{2} = \cos \frac{\Psi}{2}$ and

$$P = \sin \frac{\Psi}{2} \cos \phi - \vec{S_1} \cos \frac{\Psi}{2} \cos \phi + \vec{S_1} \wedge \vec{S_2} \sin \phi.$$

Now, we can obtain the dual orthogonality of the unit dual quaternions Q and P. Therefore, we have

$$\begin{aligned} Sc(Q \times \overline{P}) &= \sin \frac{\Psi}{2} \cos \phi \cos \frac{\Psi}{2} \cos \phi \\ &+ \left\langle \vec{S_1} \sin \frac{\Psi}{2} \cos \phi + \vec{S_2} \sin \phi, -\vec{S_1} \cos \frac{\Psi}{2} \cos \phi + \vec{S_1} \wedge \vec{S_2} \sin \phi \right\rangle = 0. \end{aligned}$$

Consequently, the unit dual quaternions $Q \in C_{\mathbb{D}}(Q_1, Q_2)$ and $P \in C_{\mathbb{D}}(Q_3, Q_4)$ are dual orthogonal. Therefore, the dual circles $C_{\mathbb{D}}(Q_1, Q_2) \subset \mathbb{S}_{\mathbb{D}}^3$ and $C_{\mathbb{D}}(Q_3, Q_4) \subset \mathbb{S}_{\mathbb{D}}^3$ are dual orthonormal to each other.

Result: The unit dual quaternions $Q_1, Q_2, Q_3, Q_4 \in \mathbb{H}_{\mathbb{D}}$ also represent a right handed dual orthonormal basis of $\mathbb{H}_{\mathbb{D}}$ while the dual unit vectors $\vec{S_1}, \vec{S_2}, \vec{S_4}$ represent a right handed dual orthonormal basis of the dual space $Vec\mathbb{H}_{\mathbb{D}} \cong \mathbb{D} - Module$.

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