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Geometry of coupled dispersionless equations with Mannheim curves

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Abstract

In this paper, a special kind of curve pair associated with each other by the linear dependency between the principal normal vector of the first curve (called Mannheim curve) and the binormal vector of the second curve (called Mannheim partner curve) is considered. A connection with the coupled dispersionless equation and Mannheim curve pair is established. Also, the Lax pair of the obtained coupled dispersionless equation from the motions of any Mannheim curve pair is given. This gives us a significant condition based on the curvature and torsion of any Mannheim curve for its integrability since it is well-known that the Lax pair provides the integrability of differential equations.

1 Introduction

One of the most important study topics in the differential geometry of curves is curve pairs. One of these curve pairs is the Mannheim curve pair. Even though Mannheim curves were first introduced by Mannheim in 1978, it was actually analyzed theoretically by Blum in 1966 [1]. In 2008 Mannheim curves were studied again in [2] presented by Liu and Wang. They proved that $\kappa = \lambda\left(\kappa^2+\tau^2\right),\ \lambda \neq 0 = \text{constant}$ (κ and τ denote the curvature and the torsion of the curve, receptively) is the necessary and sufficient condition for any curve to be a Mannheim curve. Since then, many comprehensive studies concerning Mannheim curves have been produced in different spaces [3, 4, 5, 6, 7, 8, 9, 10].

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Differential equations, studied since the 18th century, have been the most important tools for modeling physical phenomena in nature. The connection of these obtained differential equations with curves is inevitable. Mathematicians and physicists have obtained some particular differential equations for moving curves in most of their studies [11, 12, 13, 14, 15, 16, 17]. One of the most important of these differential equations is the coupled dispersionless equation (CD) which was first presented by Konno, Oono, and Kakuhata [18, 19]. The generalized version of the coupled dispersionless equation and the complex version were given by [20, 21]. The real and complex coupled dispersionless equations were obtained for the space curve family, respectively, according to the Frenet and Darboux frames in Euclidean space [22]. Also, in Minkowski space, the complex coupled dispersionless equation obtained from the timelike curve was given in [23]. The integrability of these equations was found utilizing Lax equations. In 1968 P.D. Lax introduced the Lax equations providing the integrability of nonlinear differential equations [24].
This study aims to obtain the coupled dispersionless equation from a family of Mannheim curve pairs in light of recent results related to Mannheim curves and coupled dispersionless equations.

2 Preliminaries

2.1 Mannheim Curves

Let $\gamma = \gamma(y)$ be a regular unit speed curve in Euclidean 3-space. If $T(y)$, $N(y)$, and $B(y)$, respectively, denote the tangent, principal normal, and binormal unit vectors at any point $\gamma(y)$ of the curve γ then the Frenet formulas are given

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_y = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where $\kappa = \langle T_y, N \rangle$ and $\tau = -\langle N, B_y \rangle$ are the curvature and the torsion of the curve γ, respectively [25].

Let us give the definitions and theorems related to Mannheim curves which are the subject of many studies in differential geometry.

Definition 1. Let γ and $\tilde{\gamma}$ be any two curves parameterized by their arc length y and $\tilde{y}$ in Euclidean 3-space. Also, the Frenet frames of the curves γ and $\tilde{\gamma}$ are given as $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively. If the principal normal vectors of the curve γ and the binormal vector of the curve $\tilde{\gamma}$ are linearly dependent, then γ is called the Mannheim curve, $\tilde{\gamma}$ is called the Mannheim partner of the curve γ, and $(\gamma, \tilde{\gamma})$ is called the Mannheim pair [1, 2, 3].

Theorem 1. *Let $(\gamma, \tilde{\gamma})$ be a Mannheim pair, then the relation between them is given by*

$$\gamma(y) = \tilde{\gamma}(\tilde{y}) + \lambda(\tilde{y})\tilde{B}(\tilde{y}) \quad or \;\; \tilde{\gamma}(\tilde{y}) = \gamma(y) - \lambda(y)N(y), \tag{2}$$

where $\lambda(\tilde{y})$ and $\lambda(y)$ are real-valued functions [1, 2, 3].

Theorem 2. *Let $(\gamma, \tilde{\gamma})$ be a Mannheim pair, then the distance between the Mannheim pair $(\gamma, \tilde{\gamma})$ is constant [3].*

Theorem 3. *Let $(\gamma, \tilde{\gamma})$ be a Mannheim pair and the Frenet frames of the curves γ and $\tilde{\gamma}$ be $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively. Then the relationship between the Frenet frames of the curves γ and $\tilde{\gamma}$ is*

$$\begin{array}{ll} \tilde{T} = \cos\theta T - \sin\theta B, & T = \cos\theta\tilde{T} + \sin\theta\tilde{N}, \\ \tilde{N} = \sin\theta T + \cos\theta B, \;\; or & N = \tilde{B}, \\ \tilde{B} = N & B = -\sin\theta\tilde{T} + \cos\theta\tilde{N}, \end{array} \tag{3}$$

where θ is the angle between the tangent vectors T and $\tilde{T}$ of the curves γ and $\tilde{\gamma}$ [1, 2, 3].

Theorem 4. *Let γ be a Mannheim curve, then the relation between the curvature κ and the torsion τ of the curve γ is*

$$\mu\tau - \lambda\kappa = 1, \tag{4}$$

where λ and $\mu = \lambda\cot\theta$ are nonzero real numbers [1, 2, 3].

Theorem 5. *Let γ be a Mannheim curve and $\tilde{\gamma}$ be the Mannheim partner curve of γ, then the torsion of the curve $\tilde{\gamma}$ is given by*

$$\tilde{\tau} = \frac{\kappa}{\lambda\tau} \tag{5}$$

[1, 2, 3].

Theorem 6. *Let γ be a Mannheim curve and $\tilde{\gamma}$ be the Mannheim partner curve of γ and the curvature and the torsion of the curves γ and $\tilde{\gamma}$ be κ, τ and $\tilde{\kappa}, \tilde{\tau}$, respectively. Then the relations for the curvatures and the torsions of γ and $\tilde{\gamma}$ are*

$$\begin{array}{l} \tilde{\kappa} = -\theta', \;\; \tilde{\tau} = \sin\theta\kappa\dfrac{dy}{d\tilde{y}} - \cos\theta\tau\dfrac{dy}{d\tilde{y}}, \\ \kappa = \sin\theta\tilde{\tau}\dfrac{d\tilde{y}}{dy}, \;\; \tau = \cos\theta\tilde{\tau}\dfrac{d\tilde{y}}{dy}, \end{array} \tag{6}$$

where $\frac{d\theta}{d\tilde{y}} = \theta'$ [1, 2, 3].

2.2 Coupled Dispersionless (CD) Equations

The coupled dispersionless (CD) equations, which are nonlinear differential equations, were first expressed by Konno, Oono, and Kakuhata as

$$\begin{aligned} u_{ys} &= \rho u, \\ \rho_s + u u_y &= 0. \end{aligned} \tag{7}$$

where u is a real-valued function and, also the subscripts y and s specify which variable to differentiate with respect to [18, 19]. The Lax pair provides the integrability of differential equations. Therefore, the Lax pair of these CD equations are expressed as

$$\psi_y = U\psi, \quad \psi_s = V\psi, \tag{8}$$

$$U = -i\lambda_1 \begin{pmatrix} \rho & u_y \\ u_y & -\rho \end{pmatrix}, \quad V = \begin{pmatrix} \frac{i}{4\lambda_1} & -\frac{1}{2}u \\ \frac{1}{2}u & -\frac{i}{4\lambda_1} \end{pmatrix}. \tag{9}$$

Here, $U_s - V_y + [U, V] = 0$ is provided [18, 19].

3 The link of the coupled dispersionless equation with Mannheim curves

In this part of the study, the correlation between the Mannheim curves and equations of CD is investigated. First, let us assume that

$$\gamma(y, s) : [0, l] \times [0, S] \to E^3$$

is a family of Mannheim curves, where $y \in [0, l]$ is the arc-length parameter and s represents the time, see an example given by figure 1.

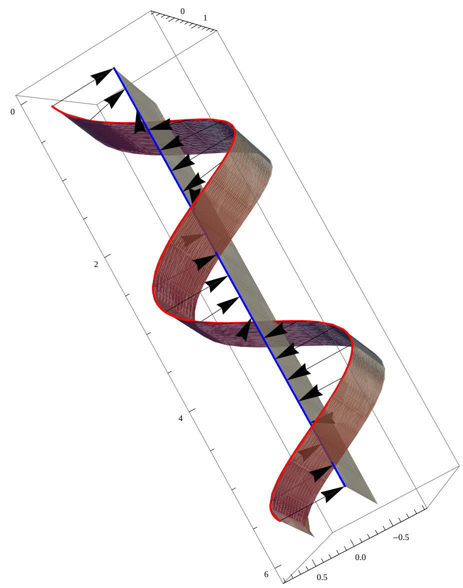

Figure 1: A family of Mannheim curves $\gamma\left(y,s\right)=\left\{\frac{2}{\sqrt{5}}\cos y, s+\frac{2}{\sqrt{5}}\sin y, \frac{y+s}{\sqrt{5}}\right\}$ for $y\in[0,4\pi]$ and $s\in[0,1]$

The time evolution of the family of orthogonal frames $\{T,N,B\}$ is given in matrix form as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{10}$$

where $\alpha,\ \beta$ and δ are functions of y and s.

Theorem 7. *Let $\gamma(y,s)$ be a curves family, then the coupled dispersionless equation corresponds to the set $\{\kappa,\tau,\alpha,\beta,\delta\}=\left\{c\rho, cu_y, -c^{-1}, u, 0\right\}$ where c is nonzero constant [22].*

This also is valid for a Mannheim curves family and the following corollary is a direct consequence of this theorem.

Corollary 1. *Let $\gamma(y,s)$ be a Mannheim curves family. For the curvatures $\kappa(y,s)=c\rho(y,s)$ and the torsions $\tau(y,s)=cu_y(y,s)$ of the Mannheim curves $\gamma(y,s)$, we can write*

$$c\left(\lambda\rho(y,s)-\mu u_y(y,s)\right)=1, \tag{11}$$

where λ and $\mu = \lambda \cot \theta$ are nonzero real numbers.

The family of Frenet frames of the Mannheim partner curves $\tilde{\gamma}(y,s)$ satisfies

$$\begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix}_y = \begin{bmatrix} 0 & \tilde{\kappa} & 0 \\ -\tilde{\kappa} & 0 & \tilde{\tau} \\ 0 & -\tilde{\tau} & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix} \tag{12}$$

and also, the time evolution of the orthogonal frames $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$ of the Mannheim curves family $\gamma(y,s)$ can be written

$$\begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix}_s = \begin{bmatrix} 0 & \varepsilon & \eta \\ -\varepsilon & 0 & \xi \\ -\eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix}. \tag{13}$$

Here, ε, η, and ξ are the functions of y and s.

Theorem 8. *Let $(\gamma(y,s), \tilde{\gamma}(y,s))$ be a family of Mannheim pairs and the Frenet frames of the curves $\gamma(y,s)$ and $\tilde{\gamma}(y,s)$ be $\{T, N, B\}$ and $\left\{\tilde{T}, \tilde{N}, \tilde{B}\right\}$, respectively. Then the relationship between the Frenet frames of the curves $\tilde{\gamma}$ and γ is*

$$\begin{aligned} \tilde{T} &= \frac{1}{\sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}} \left((1+\lambda c\rho)\, T - (\lambda c u_y)\, B\right), \\ \tilde{N} &= \frac{1}{\sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}} \left((\lambda c u_y)\, T + (1+\lambda c\rho)\, B\right), \\ \tilde{B} &= N. \end{aligned} \tag{14}$$

Proof. Considering equations $\kappa = c\rho$ and $\tau = cu_y$, taking the derivate of equation (2), we get

$$\tilde{\gamma}' \tfrac{d\tilde{y}}{dy} = (1+\lambda c\rho)\, T - (\lambda c u_y)\, B.$$

The norm of this equation is

$$\|\tilde{\gamma}'\| \tfrac{d\tilde{y}}{dy} = \sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}.$$

So, the tangent vectors of the curves $\tilde{\gamma}(y,s)$ are obtained as

$$\tilde{T} = \tfrac{1}{\sqrt{(1+\lambda c\rho)^2+(\lambda c u_y)^2}} \left((1+\lambda c\rho)\, T - (\lambda c u_y)\, B\right).$$

If this last equation is compared to equation $\tilde{T} = \cos\theta T - \sin\theta B$, it can be obtained that

$$\cos\theta = \frac{1+\lambda c\rho}{\sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}} \quad \text{and} \quad \sin\theta = \frac{-\lambda c u_y}{\sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}}. \tag{15}$$

So, considering equation (3), we have

$$\tilde{N} = \frac{1}{\sqrt{(1+\lambda c\rho)^2 + (\lambda c u_y)^2}} \left((\lambda c u_y) T + (1+\lambda c\rho) B \right)$$

and

$$\tilde{B} = N.$$

□

Theorem 9. *Let $(\gamma(y,s), \tilde{\gamma}(y,s))$ be a family of Mannheim pairs and the curvature and the torsion of the curves $\gamma(y,s)$ and $\tilde{\gamma}(y,s)$ be κ, τ and $\tilde{\kappa}, \tilde{\tau}$, respectively. Then the relations for the curvatures and the torsions of the curves γ and $\tilde{\gamma}$ are*

$$\begin{aligned} \tilde{\kappa} &= -\theta', \\ \tilde{\tau} &= \frac{\rho}{\lambda u_y}, \end{aligned} \tag{16}$$

where $\frac{d\theta}{d\tilde{y}} = \theta'$.

Proof. Considering equation (3), it can be written

$$\left\langle T, \tilde{T} \right\rangle = \cos\theta.$$

Taking derivate this last equation according to the parameter $\tilde{y}$, it can be got

$$\left\langle c\rho N \tfrac{dy}{d\tilde{y}}, \tilde{T} \right\rangle + \left\langle T, \tilde{\kappa}\tilde{N} \right\rangle = -\sin\theta \tfrac{d\theta}{d\tilde{y}}.$$

Also, since N and $\tilde{B}$ are linearly dependent, it is found

$$\tilde{\kappa} \left\langle T, \tilde{N} \right\rangle = -\sin\theta \tfrac{d\theta}{d\tilde{y}}.$$

So, considering equation (3), the curvatures of the curve $\tilde{\gamma}$ is

$$\tilde{\kappa} = -\tfrac{d\theta}{d\tilde{y}}.$$

Now lets find the torsions of the curves $\tilde{\gamma}(y,s)$. Taking the derivate of equation (2) according to the parameter $\tilde{y}$ and it is found

$$T=\left(\tilde{T}-\lambda\tilde{\tau}\tilde{N}\right)\tfrac{d\tilde{y}}{dy}.$$

If this last equation is compared with equation (3), it is obtained

$$\cos\theta=\frac{d\tilde{y}}{dy}\ \text{ and }\ \sin\theta=-\lambda\tilde{\tau}\frac{d\tilde{y}}{dy}. \tag{17}$$

On the other hand, if taking the derivate of equation (2) according to the parameter y, it can be written

$$\tilde{T}=\left((1+\lambda c\rho)\,T-\lambda cu_yB\right)\tfrac{dy}{d\tilde{y}}.$$

If this last equation is compared to equation (3), It is represented as

$$\cos\theta=(1+\lambda c\rho)\frac{dy}{d\tilde{y}}\ \text{ and }\ \sin\theta=\lambda cu_y\frac{dy}{d\tilde{y}}. \tag{18}$$

From equations (17) and (18), it is found as

$$1=(1+\lambda c\rho)-\lambda^2cu_y\tilde{\tau}.$$

This completes the proof. □

Theorem 10. *Let $(\gamma(y,s),\tilde{\gamma}(y,s))$ be a family of Mannheim pairs and the Frenet frames of the curves $\gamma(y,s)$ and $\tilde{\gamma}(y,s)$ be $\{T,N,B\}$ and $\left\{\tilde{T},\tilde{N},\tilde{B}\right\}$, respectively. Then the following statement provides the CD equation:*

$$\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}_s=\begin{bmatrix}0 & -\omega^{-1} & q\\ \omega^{-1} & 0 & 0\\ -q & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{T}\\ \tilde{N}\\ \tilde{B}\end{bmatrix}, \tag{19}$$

where $\left\{\tilde{T},\tilde{N},\tilde{B}\right\}$ is the Frenet frame family of the curves $\tilde{\gamma}(y,s)$, $\omega\neq 0$ is constant, q is a real valued function.

Proof. From the Frenet formulas and the time evolutions for the Frenet frame given by (12) and (13), respectively, it can be written

$$\begin{aligned}\tilde{T}_y&=-\theta'\tilde{N},\\ \tilde{N}_y&=\theta'\tilde{T}+\frac{\rho}{\lambda u_y}\tilde{B},\\ \tilde{B}_y&=-\frac{\rho}{\lambda u_y}\tilde{N}.\end{aligned}\quad \text{and}\quad \begin{aligned}\tilde{T}_s&=\varepsilon\tilde{N}+\eta\tilde{B},\\ \tilde{N}_s&=-\varepsilon\tilde{T}+\xi\tilde{B},\\ \tilde{B}_s&=-\eta\tilde{T}-\xi\tilde{N}.\end{aligned}$$

Thus, using equations $\tilde{T}_{sy}=\tilde{T}_{ys},\ \tilde{N}_{sy}=\tilde{N}_{ys},\ \tilde{B}_{sy}=\tilde{B}_{ys}$, the following equations are calculated

$$\varepsilon_y=(-\theta')_s+\frac{\rho}{\lambda u_y}\eta, \tag{20}$$

$$\eta_y=-\theta'\xi-\frac{\rho}{\lambda u_y}\varepsilon, \tag{21}$$

$$\xi_y=\left(\frac{\rho}{\lambda u_y}\right)_s+\theta'\eta. \tag{22}$$

By the hypothesis $\varepsilon=-\omega^{-1},\eta=q,$ and $\xi=0$ are satisfied, and then equations (20)-(22) become

$$(-\theta')_s=-q\frac{\rho}{\lambda u_y}, \tag{23}$$

$$q_y=\omega^{-1}\frac{\rho}{\lambda u_y}, \tag{24}$$

$$\left(\frac{\rho}{\lambda u_y}\right)_s=-\theta' q, \tag{25}$$

respectively. Taking derivate of equation (24), it is obtanied $\left(\frac{\rho}{\lambda u_y}\right)_s=\omega q_{ys}$. By substituting equations $\left(\frac{\rho}{\lambda u_y}\right)_s=\omega q_{ys}$ into equation (25), it is found

$$q_{ys}=pq, \tag{26}$$

where

$$p=-\frac{\theta'}{\omega}. \tag{27}$$

By substituting equations (24) and (27) into equations (23), the following equation is expressed

$$p_s+qq_y=0. \tag{28}$$

As a result, equations (26) and (28) express the CD equation, completing the proof. □

Now, lets give the Lax pair, which provides the integrability of the CD equation by the following theorem.

Theorem 11. *Let $(\gamma(y,s),\tilde{\gamma}(y,s))$ be a family of Mannheim pairs and $\psi=\psi(y,s)$ be a function with value $so(3)$, then the Lax pair of the CD equation is*

$$\psi_y=P\psi,\ \ \psi_s=Q\psi \tag{29}$$

such that $P=-\tilde{\kappa}e_3-\tilde{\tau}e_1,\ Q=-\varepsilon e_3+\eta e_2,$ *where* $\tilde{\kappa}=\left\langle \tilde{T}_y,\tilde{N}\right\rangle,\ \tilde{\tau}=-\left\langle \tilde{B}_y,\tilde{N}\right\rangle,\ \varepsilon=\left\langle \tilde{T}_s,\tilde{N}\right\rangle,\ \eta=\left\langle \tilde{T}_s,\tilde{B}\right\rangle,\ \left\{\tilde{T},\tilde{N},\tilde{B}\right\}$ *is the Frenet frames of the curve* $\tilde{\gamma}(y,s)$.

Proof. The Lax pair of the CD equation is

$$P=-i\lambda_2\begin{pmatrix} p & q_y \\ q_y & -p\end{pmatrix},\ Q=\begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-q}{2} \\ \frac{q}{2} & -\frac{i}{4\lambda_2}\end{pmatrix}. \tag{30}$$

Also, the compatibility condition $P_y-Q_s+PQ-QP=0$ satisfies the CD equation. The basis of su(2) and so(3) are

$$e_1=\frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ \mathrm{e}_2=\frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},\ \mathrm{e}_3=\frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$$

and

$$L_1=\begin{pmatrix} 0&0&0\\ 0&0&-1\\ 0&1&0\end{pmatrix},\ L_2=\begin{pmatrix} 0&0&1\\ 0&0&0\\ -1&0&0\end{pmatrix},\ L_3=\begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&0\end{pmatrix},$$

respectively and there is an isomorphism $L_j\to e_j,\ (j=1,2,3)$ between the Lie algebras su(2) and so(3). Under this isomorphism, the CD equation provides equation (29). The functions P and Q are found as

$$\begin{aligned} P=-\tilde{\kappa}e_3-\tilde{\tau}e_1 &= -\omega p e_3-\omega q_y e_2\\ &= -i\lambda_2\begin{pmatrix} p & q_y\\ q_y & -p\end{pmatrix}\\ &= -i\lambda_2\begin{pmatrix} \frac{-\theta'}{\omega} & \frac{\rho}{\omega\lambda u_y}\\ \frac{\rho}{\omega\lambda u_y} & \frac{\theta'}{\omega}\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned} Q=-\varepsilon e_3+\eta e_2 &= \omega^{-1}e_3+qe_2\\ &= \begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-q}{2}\\ \frac{q}{2} & -\frac{i}{4\lambda_2}\end{pmatrix}\\ &= \begin{pmatrix} \frac{i}{4\lambda_2} & \frac{-1}{2}\int\frac{\rho}{\omega\lambda u_y}dy\\ \frac{1}{2}\int\frac{\rho}{\omega\lambda u_y}dy & -\frac{i}{4\lambda_2}\end{pmatrix}.\end{aligned}$$

where $\omega=-2\lambda_2$. These provide a Lax pair of the CD equation as it is desired. □

Let us give the geometric interpretation of the conserved quantity of the CD equation by the following theorem.

Theorem 12. *Let $(\gamma(y,s),\tilde{\gamma}(y,s))$ be a family of Mannheim pairs, then the conserved quantity of the CD equation is constant,*

$$I = p^2 + q_y^2$$

where $p = \frac{-\theta'}{\omega} = \frac{\tilde{\kappa}}{\omega}$, $q_y = \frac{1}{\omega}\left(\frac{\rho}{\lambda u_y}\right) = \frac{\tilde{\tau}}{\omega}$, $\tilde{\kappa}$ and $\tilde{\tau}$ are the Frenet frame curvatures of the space curves $\tilde{\gamma}(y,s)$.

Proof. The conserved quantity of the CD equation with the curves γ is $I = \rho^2 + u_y^2$. From equations (16), (24) and (27), it can be written

$$\frac{d}{ds}\left(\tilde{\kappa}^2 + \tilde{\tau}^2\right) = \frac{d}{ds}\left((-\theta')^2 + \left(\frac{\rho}{\lambda u_y}\right)^2\right) = \omega^2 \frac{d}{ds}\left(p^2 + q_y^2\right).$$

On the other hand, from equation (23) and (25), it can be obtained

$$\begin{aligned}\frac{d}{ds}\left(\tilde{\kappa}^2 + \tilde{\tau}^2\right) = 2\tilde{\kappa}\tilde{\kappa}_s + 2\tilde{\tau}\tilde{\tau}_s &= -2\theta'(-\theta')_s + 2\left(\frac{\rho}{\lambda u_y}\right)\left(\frac{\rho}{\lambda u_y}\right)_s \\ &= -2\theta'\left(-q\frac{\rho}{\lambda u_y}\right) + 2\left(\frac{\rho}{\lambda u_y}\right)(-\theta' q) \\ &= 0.\end{aligned}$$

As a result, it is represented

$$\frac{d}{ds}\left(p^2 + q_y^2\right) = 0.$$

where $p = \frac{-\theta'}{\omega}$ and $q_y = \frac{\rho}{\omega\lambda u_y}$. Hence, it can be easily seen that the conserved quantity of the CD equation with the curve $\tilde{\gamma}(y,s)$ is constant. □

Corollary 2. *Let $(\gamma(y,s),\tilde{\gamma}(y,s))$ be a family of Mannheim pairs, then the conserved quantity of the CD equation with the curves γ and the curves $\tilde{\gamma}$ is constant.*

4 Conclusions

In this study, the coupled dispersionless equation from the motion of the Mannheim pairs, which provides a favorable interpretation for the integrability conditions is derived. This nonlinear differential equation allows us to introduce the Lax pair providing the integrability of this equation. Finally, it is proven that the conserved quantity of this equation is constant.

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