DOI: 10.2478/auom-2023-0033
An. Şt. Univ. Ovidius Constanţa **Vol. 31(3),2023, 87–109**

Spacelike Bertrand curves in Minkowski 3-space revisited

Hatice Altın Erdem and Kazım İlarslan

Abstract

In the geometry of curves in $\mathbb{E}^3$, if the principal normal vector field of a given space curve φ with non-zero curvatures is the principal normal vector field of another space curve φ^*, then the curve φ is called a Bertrand curve and φ^* is called Bertrand partner of φ. These curves have been studied in different space over a long period of time and found wide application in different areas. Therefore, we have a great knowledge of geometric properties of these curves. In this paper, revested results for spacelike Bertrand curves with non-null normal vectors will be given with the previous studies on Bertrand curves in $\mathbb{E}^3_1$. Follow this revested results, the Bertrand curve conditions of a spacelike curve are obtained in $\mathbb{E}^3_1$. In addition, new curve samples that meet the obtained conditions are constructed and the graphs of these curves are given.

1 Introduction

Moving Frenet frames have significant place in the geometry of curves in $\mathbb{E}^3$. As a result of the connections among the Frenet vectors located at the opposite points of the pairs of space curves, we obtain many special classes of curves. For example, Mannheim curves, Combescure related curves and Bertrand curves are some curves in this class. Bertrand curves have a long history. In $\mathbb{E}^3$, the problem of whether the principal normal of a certain space

Key Words: Minkowski 3-space, Bertrand curves, Non-null curves.
2010 Mathematics Subject Classification: 53C50, 53C40.
Received: 23.09.2022
Accepted: 22.12.2022

curve can be the principal normal of distinct space curve, put forward by Saint-Venant in 1845 [34], led to the emergence of Bertrand curves. This problem has been solved by Bertrand in 1850 [5]. In this study, it is proved that for a space curve to be a solution to the above-mentioned problem, there must be a correlation between its curvatures. This relationship is $\lambda\kappa + \mu\tau = 1$, $\lambda \in \mathbb{R}_0$ and $\mu \in \mathbb{R}$, with the curvature function κ and the torsion function τ of the given curve.

We can consider the studies published on Bertrand curves in three main groups. The first group includes generalizations of Bertrand curves in Euclidean space. Some of these studies are as follows. In [28], Pears studied Bertrand curves in $\mathbb{E}^n$ and proved that for $n > 3$, second or third curvature functions have to zero. Thus Bertrand curves in $\mathbb{E}^n$ are degenerate curves. Accordingly, Bertrand curves must lie in 3-dimensional subspace of $\mathbb{E}^n$. In another study, Lucas and Ortega-Yages studied Bertrand curves on a 3-dimensional sphere $\mathbb{S}^3$ in [22], while Izumiya and Takeuchui proved that a Bertrand curve in $\mathbb{E}^3$ is expressed using a spherical curve [17].

In a distinct paper, Matsudo and Yaruzo [24] considered Bertrand curves in $\mathbb{E}^4$. In this paper, they showed that there are no special Bertrand curves (for curves with all nonzero curvature functions) and they defined a new Bertrand curve class that entered the literature as $(1,3)$-Bertrand curves. For Bertrand curves in $\mathbb{E}^n$, the following studies can also be considered [9], [13].

In the second group, we can talk about the studies carried out for the spaces equipped with Riemannian or pseudo-Riemannian metrics instead of Euclidean metrics and studies in three or higher dimensional real and Lorentz space forms [14], [20], [23]. We see that studies for Bertrand curves in Euclidean space (for dimensions 3, 4 and n) are intensely carried out, especially in $\mathbb{E}_1^3$, $\mathbb{E}_1^4$ and $\mathbb{E}_\nu^n$ [3],[4], [11], [12], [16], [18], [31],[30], [32], [33] and Galilean and pseudo-Galilean 3-space[26], [27].

The studies in the third and last group are the applications of Bertrand curves to the theory of surfaces [29], [35], [36] and the studies that bring a new perspective to Bertrand curves. Although there are intensive studies on Bertrand curves, examples of Bertrand curves are very limited. New Bertrand curve examples have been added to the literature, especially thanks to the new perspective brought to Bertrand curves in $\mathbb{E}^3$ by Camcı et al [6]. They also showed that general helix curves (curves with constant curvature ratio) can also be Bertrand curves.

In this study, we give a new revisited results for a spacelike Bertrand curves with spacelike or timelike normals in $\mathbb{E}_1^3$. Follow this revested results, the Bertrand curve conditions of a spacelike curve with non-null normals are obtained in $\mathbb{E}_1^3$. In addition, new curve samples that meet the obtained conditions are constructed and the graphs of these curves are given.

2 Preliminaries

The Minkowski 3- space (or Lorentz-Minkowski 3-space), $\mathbb{E}_1^3$, is the standart real vector space $\mathbb{R}^3$ accoutred with the Minkowski scalar product given by

$$g(\overrightarrow{U},\overrightarrow{V}) = -u_1v_1 + u_2v_2 + u_3v_3,$$

where $\overrightarrow{U} = (u_1, u_2, u_3)$ and $\overrightarrow{V} = (v_1, v_2, v_3)$. Since $g(.,.)$ is an indefinite metric, there are three different cases for the causal character of an arbitrary vector $\overrightarrow{W} \in \mathbb{E}_1^3\backslash\{0\}$. It is a spacelike vector if $g(\overrightarrow{W},\overrightarrow{W}) > 0$, it is a timelike vector if $g(\overrightarrow{W},\overrightarrow{W}) < 0$ and it is a null (lightlike) vector if $g(\overrightarrow{W},\overrightarrow{W}) = 0$. Additionally, the vector $\overrightarrow{W} = (0,0,0)$ is considered a spacelike vector. A real number defined as $||\overrightarrow{W}|| = \sqrt{|g(\overrightarrow{W},\overrightarrow{W})|} \geq 0$, determines the norm of the vector $\overrightarrow{W}$. If $g(\overrightarrow{U},\overrightarrow{V}) = 0$, then the vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ are said to be orthogonal. The velocity vector of the curve determines the causal character of a regular curve. Namely, a random regular curve $\varphi(s)$ in $\mathbb{E}_1^3$, can locally be spacelike if its velocity vector $\varphi'(s)$ is a spacelike vector, can locally be timelike if its velocity vector$\varphi'(s)$ is a timelike vector and can locally be null(lightlike) if its velocity vector $\varphi'(s)$ is a null vector [19], [21], [25].

For a null(or lightlike) curve φ,if $g(\varphi''(s),\varphi''(s)) = 1$ is satisfy then the curve is called parameterized by pseudo-arc s. In this instance, the null curve is called a Cartan null curve. if $g(\varphi'(s),\varphi'(s)) = \pm 1$ is provided then the curve is called a unit speed curve [10], [21], [25].

The set $\{T,N,B\}$ consist of the tangent vector T, the principal normal vector N and the binormal vector B along a curve φ in $\mathbb{E}_1^3$ is called the moving Frenet frame. With respect to the causal character of the given curve, the equations expressing the derivatives of Frenet vectors in terms of these vectors and curvature functions are called Frenet equations. We can give these equations as:

Case A. If φ is a timelike or a spacelike curve with non-null normals, Frenet equations can be expressed as follows [19], [21]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & \varepsilon_2\tau \\ 0 & -\varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

here κ is curvature and τ is torsion functions of the curve. Accordingly, let us put,

$$g(T,T) = \varepsilon_1 = \pm 1, g(N,N) = \varepsilon_2 \ = \pm 1, g(B,B) = \varepsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

Case B. If φ is a Cartan null curve the Frenet equations can be given as [10], [15]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & -\kappa \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2}$$

where $\kappa = 0$, the curve is a straight line and in all other cases $\kappa = 1$. Also, let us put:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, g(N,N) = g(T,B) = 1.$$

We will use the definitions and results given below later.

Definition 1. If the principal normal vector field N of a curve makes a constant angle with a fixed direction in $\mathbb{E}_1^3$, the curve is called a slant helix [2], [17].

Lemma 1. *Let φ be a unit speed spacelike curve with timelike N in $\mathbb{E}_1^3$. Then φ is a slant helix if and only if the geodesic curvature of the spherical image of principal normal indicatrix (n) of φ,*

$$\sigma = \frac{\kappa^2}{(\tau^2 + \kappa^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)',$$

is constant and $\tau^2 + \kappa^2 \neq 0$ [2].

Lemma 2. *Let φ be a unit speed spacelike curve with spacelike N in $\mathbb{E}_1^3$. Then φ is a slant helix if and only if geodesic curvatures of the spherical image of principal normal indicatrix (n) of φ,*

$$\sigma = \frac{\kappa^2}{(\tau^2 - \kappa^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)',$$

is constant and $\tau^2 - \kappa^2 \neq 0$ [2].

Definition 2. If the position vector of a curve always lies in its rectifying plane,then the curve is called a rectifying curve [7], [8].

3 Spacelike Bertrand Curves in Minkowski 3-Space revisited

In this part, spacelike Bertrand curves with non-null normals in $\mathbb{E}_1^3$ will be discussed with a new approach. First, let's start by giving the notion of Bertrand curve in this space.

Definition 3. In Minkowski 3-space, if the principal normal vector fields at the opposite points of a given unit speed curve φ with non-zero curvatures and another space curve $\varphi^\star$ are linearly dependent, then the curve φ is called the Bertrand curve and the other curve $\varphi^\star$ is called the Bertrand partner curve of φ .

We assume that φ is a spacelike Bertrand curve with the Frenet frame $\{T, N, B\}$ and non-zero curvatures κ, τ, and $\varphi^\star$ is a Bertrand partner curve of φ with the Frenet frame $\{T^\star, N^\star, B^\star\}$ and curvatures $\kappa^\star, \tau^\star$ in $\mathbb{E}_1^3$. According to the new approach, $\varphi^\star$ can be given in the form below:

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s) T(s) + \mu_2(s) N(s) + \mu_3(s) B(s)$$

where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable function on I. Since φ is a spacelike curve with spacelike N, one of the following situations is possible for the Bertrand partner curve $\varphi^\star$ of φ:

(1) $\varphi^\star$ is a timelike curve,

(2) $\varphi^\star$ is a spacelike curve with spacelike principal normal vector,

(3) $\varphi^\star$ is a Cartan null curve.

We will assess all situations severally in the following theorem.

Theorem 1. *Let φ be a unit speed spacelike curve with spacelike principal normal with the non-zero curvatures κ, τ in $\mathbb{E}_1^3$. Then the curve φ is a Bertrand curve with Bertrand partner curve φ^* if and only if one of the following conditions are met:*

(i) there are differentiable functions μ_1, μ_2 and μ_3 such as

$$\begin{aligned} \mu_3\tau &= \mu_2' + \mu_1\kappa \\ \mu_3' &= \mu_2\tau \end{aligned} \tag{3}$$

or there are differentiable functions μ_1, μ_2, μ_3 and $h \in \mathbb{R}$ providing

$$\begin{aligned} \mu_3' - \mu_2\tau &\neq 0 \\ \mu_2' + \mu_1\kappa &= \mu_3\tau \\ 1 + \mu_1' - \mu_2\kappa &= h\left(\mu_3' - \mu_2\tau\right) \\ h\tau - \kappa &\neq 0 \\ h\kappa - \tau &\neq 0 \\ 1 - h^2 &> 0 \end{aligned} \tag{4}$$

In this situation, the Bertrand partner curve $\varphi^\star$ is a timelike curve.

(ii) there are differentiable functions μ_1, μ_2 *and* μ_3 *providing*

$$\begin{aligned} \mu_3\tau &= \mu_2^{'} + \mu_1\kappa \\ \mu_3^{'} - \mu_2\tau &= 0 \end{aligned} \tag{5}$$

or there are differentiable functions μ_1, μ_2, μ_3 *and* $h \in \mathbb{R}$ *providing*

$$\begin{aligned} \mu_3^{'} - \mu_2\tau &\neq 0 \\ \mu_2^{'} + \mu_1\kappa &= \mu_3\tau \\ 1 + \mu_1^{'} - \mu_2\kappa &= h\left(\mu_3^{'} - \mu_2\tau\right) \\ h\tau - \kappa &\neq 0 \\ h\kappa - \tau &\neq 0 \\ 1 - h^2 &< 0 \end{aligned} \tag{6}$$

In this situation, the Bertrand partner curve $\varphi^{\star}$ *is a spacelike curve with spacelike* N.

(iii) there are differentiable functions μ_1, μ_2, μ_3 *and real numbers* γ, $h = \pm 1$ *providing*

$$\begin{aligned} \mu_3^{'} - \mu_2\tau &\neq 0 \\ \mu_2^{'} + \mu_1\kappa &= \mu_3\tau \\ 1 + \mu_1^{'} - \mu_2\kappa &= h\left(\mu_3^{'} - \mu_2\tau\right) \\ \left|\mu_3^{'} - \mu_2\tau\right| &= \gamma^2\left|h\kappa - \tau\right| \\ h\kappa - \tau &\neq 0 \\ h\kappa + \tau &\neq 0 \end{aligned} \tag{7}$$

In this situation, the Bertrand partner curve $\varphi^{\star}$ *is a Cartan null curve.*

Proof. Let's admit that φ is a spacelike Bertrand curve given by its arc-length s and non-zero curvatures κ, τ and the curve $\varphi^{\star}$ is the Bertrand partner curve of φ given by its arc-length or pseudo arc $s^{\star}$. So, the curve $\varphi^{\star}$ can be stated by

$$\varphi^{\star}(s^{\star}) = \varphi^{\star}(f(s)) = \varphi(s) + \mu_1(s)T(s) + \mu_2(s)N(s) + \mu_3(s)B(s) \tag{8}$$

for all $s \in I$ where $\mu_1(s), \mu_2(s)$ and $\mu_3(s)$ are differentiable functions on I.

(i) Let $\varphi^{\star}$ be a timelike curve. If the derivative of equation (8) is taken according to the parameter s and using the equations (1), we get

$$f^{'}T^{\star} = \left(1 + \mu_1^{'} - \mu_2\kappa\right)T + \left(\mu_2^{'} + \mu_1\kappa - \mu_3\tau\right)N + \left(\mu_3^{'} - \mu_2\tau\right)B \tag{9}$$

If the equation (9) is multiplied by N, we find

$$\mu_3\tau = \mu_2^{'} + \kappa \tag{10}$$

If we write instead of (10) in (9), we obtain

$$f^{'}T^{\star} = \left(1 + \mu_1^{'} - \mu_2\kappa\right)T + \left(\mu_3^{'} - \mu_2\tau\right)B \tag{11}$$

If the equation (11) is multiplied by itself, we get

$$\left(f^{'}\right)^2 = \left(\mu_3^{'} - \mu_2\tau\right)^2 - \left(1 + \mu_1^{'} - \mu_2\kappa\right)^2 \tag{12}$$

If we take

$$\delta = \frac{1 + \mu_1^{'} - \mu_2\kappa}{f^{'}} \quad , \ \gamma = \frac{\mu_3^{'} - \mu_2\tau}{f^{'}} \tag{13}$$

we get from (11)

$$T^{\star} = \delta T + \gamma B, \tag{14}$$

if the derivative of equation (14) is taken according to the parameter s and using (1), we find

$$f^{'}\kappa^{\star}N = \delta^{'}T + (\delta\kappa - \gamma\tau)N + \gamma^{'}B. \tag{15}$$

If the equation (15) is multiplied by N, we obtain

$$\delta^{'} = 0 \quad \text{and} \quad \gamma^{'} = 0. \tag{16}$$

Firstly, we suppose that $\gamma = 0$. Then we have $\mu_3^{'} - \mu_2\tau = 0$. Now we suppose that $\gamma \neq 0$. So,

$$1 + \mu_1^{'} - \mu_2\kappa = h\left(\mu_3^{'} - \mu_2\tau\right) \tag{17}$$

where $h = \delta/\gamma$. If we put equation (16) in (15), we get

$$f^{'}\kappa^{\star}N = (\delta\kappa - \gamma\tau)N \tag{18}$$

If the equation (18) is multiplied with itself, and handling (10) and (11), we get

$$\left(f^{'}\right)^2(\kappa^{\star})^2 = \frac{(h\kappa - \tau)^2}{1 - h^2}, \tag{19}$$

where $h\kappa - \tau \neq 0$ and $1 - h^2 > 0$. If we write instead

$$\lambda = \frac{\delta\kappa - \gamma\tau}{f^{'}\kappa^{\star}}$$

we have

$$N^{\star} = \lambda N. \tag{20}$$

If the derivative of equation (20) is taken according to the parameter s and using equation (1), we get

$$f'\tau^\star B^\star = -\lambda\kappa T + \lambda^{'} N - \lambda\tau B - f'\kappa^\star T^\star \tag{21}$$

where $\lambda^{'} = 0$. If we reconsider the equation (21) by using (9), we obtain

$$f'\tau^\star B^\star = P\left(s\right)T + Q\left(s\right)B \tag{22}$$

where

$$\begin{aligned} P\left(s\right) &= \frac{\left(h\kappa-\tau\right)\left(\mu_3^{'}-\mu_2\tau\right)\left(h\tau-\kappa\right)}{\left(f'\right)^2\kappa^\star\left(1-h^2\right)} \\ Q\left(s\right) &= \frac{\left(h\kappa-\tau\right)\left(\mu_3^{'}-\mu_2\tau\right)\left(h\tau-\kappa\right)h}{\left(f'\right)^2\kappa^\star\left(1-h^2\right)} \end{aligned}$$

Consequently, we get $h\tau-\kappa \neq 0$. Conversely, let φ be a spacelike curve given by its arc-length s with non-zero curvatures κ, τ. Firstly assume that φ satisfies the equations (3) for differentiable functions μ_1, μ_2 and μ_3. So, a curve $\varphi^\star$ can be stated as

$$\varphi^\star\left(s^\star\right) = \varphi^\star\left(f\left(s\right)\right) = \varphi\left(s\right) + \mu_1\left(s\right)T\left(s\right) + \mu_2\left(s\right)N\left(s\right) + \mu_3\left(s\right)B\left(s\right). \tag{23}$$

If the derivative of equation (23) is taken according to the parameter s, we arrive

$$\frac{d\varphi^\star}{ds} = \left(1+\mu_1^{'}-\mu_2\kappa\right)T. \tag{24}$$

By using (21), we obtain

$$f^{'} = \left\|\frac{d\varphi^\star}{ds}\right\| = m_1\left(1+\mu_1^{'}-\mu_2\kappa\right) > 0$$

where $m_1 = sgn\left(1+\mu_1^{'}-\mu_2\kappa\right)$. Then we obtain

$$\begin{aligned} T^\star &= m_1 T \\ N^\star &= m_1 m_2 N \\ B^\star &= m_1 m_2 m_3 B \end{aligned}$$

and

$$\begin{aligned} \kappa^\star &= \frac{m_2\kappa}{f'} \\ \tau^\star &= \frac{m_3\tau}{f'} \end{aligned}$$

where $m_2, m_3 = \pm 1$. Therefore the curve φ is a Bertrand curve and the curve $\varphi^\star$ is a timelike Bertrand partner curve of the curve φ.

Now, assume that φ satisfies the equation (4) for differentiable functions μ_1, μ_2, μ_3 and $h \in \mathbb{R}$. Then, a curve $\varphi^\star$ given by

$$\varphi^\star(s^\star) = \varphi^\star(f(s)) = \varphi(s) + \mu_1(s)\,T(s) + \mu_2(s)\,N(s) + \mu_3(s)\,B(s) \tag{25}$$

if the derivative of equation (25) is taken according to the parameter s, we have

$$\frac{d\varphi^\star}{ds} = \left(1 + \mu_1^{'} - \mu_2\kappa\right)T + \left(\mu_3^{'} - \mu_2\tau\right)B. \tag{26}$$

Thus, the following equation is obtained

$$f^{'} = \left\|\frac{d\varphi^\star}{ds}\right\| = n_1\left(\mu_3^{'} - \mu_2\tau\right)\sqrt{1-h^2} \tag{27}$$

where $n_1 = sgn\left(\mu_3^{'} - \mu_2\tau\right)$. Rewriting (26) , we get

$$T^\star = \frac{n_1}{\sqrt{1-h^2}}\left(hT + B\right) \quad , g\left(T^\star, T^\star\right) = -1. \tag{28}$$

If the derivative of equation (28) is taken according to the parameter s, we get

$$\frac{dT^\star}{ds^\star} = \frac{n_1\left(h\kappa - \tau\right)}{f'\sqrt{1-h^2}}N \tag{29}$$

so we obtain

$$\kappa^\star = \left\|\frac{dT^\star}{ds^\star}\right\| = \frac{n_2\left(h\kappa - \tau\right)}{f'\sqrt{1-h^2}} \tag{30}$$

where $n_2 = sgn\left(h\kappa - \tau\right)$. Thus, we obtain $N^\star$ as

$$N^\star = n_1 n_2 N, \quad g\left(N^\star, N^\star\right) = 1. \tag{31}$$

If the derivative of equation (31) is taken according to the parameter s and considering the equations (28), (29) we obtain

$$\frac{dN^\star}{ds^\star} - \kappa^\star T^\star = \frac{n_1 n_2\left(h\tau - \kappa\right)}{f'\left(1-h^2\right)}\left(T + hB\right) \tag{32}$$

which bring about that

$$\tau^\star = \frac{n_3\left(h\tau - \kappa\right)}{f'\sqrt{1-h^2}} \tag{33}$$

where $n_3 = sgn\,(h\tau - \kappa)$. Finally, we define $B^\star$ as

$$\begin{aligned} B^\star &= \frac{n_1 n_2 n_3}{\sqrt{1-h^2}}\,(T + hB) \\ g\,(B^\star, B^\star) &= 1. \end{aligned}$$

Then $\varphi^\star$ is a timelike curve and Bertrand partner curve of φ. Thus φ is a Bertrand curve.

(ii) Let $\varphi^\star$ be a spacelike curve with spacelike principal normal. In this case, the proof of the theorem can be done in the same way as the proof the case when $\varphi^\star$ is timelike.

(iii) Let $\varphi^\star$ be a Cartan null curve. Then, if the derivative of equation (8) is taken according to the parameter s and using the (2), we get

$$f^{'} T^\star = \left(1 + \mu_1^{'} - \mu_2\kappa\right) T + \left(\mu_2^{'} + \mu_1\kappa - \mu_3\tau\right) N + \left(\mu_3^{'} - \mu_2\tau\right) B. \quad (34)$$

If the equation (34) is multiplied by N, we have

$$\mu_3\tau = \mu_2^{'} + \mu_1\kappa \quad (35)$$

By using (35) in (34), we find

$$f^{'} T^\star = \left(1 + \mu_1^{'} - \mu_2\kappa\right) T + \left(\mu_3^{'} - \mu_2\tau\right) B. \quad (36)$$

If the equation (36) is multiplied with itself, we get

$$\left(1 + \mu_1^{'} - \mu_2\kappa\right)^2 = \left(\mu_3^{'} - \mu_2\tau\right)^2 \quad (37)$$

and

$$1 + \mu_1^{'} - \mu_2\kappa = h\left(\mu_3^{'} - \mu_2\tau\right)$$

where $h = \pm 1$. If we take

$$\delta = \frac{\mu_3^{'} - \mu_2\tau}{f^{'}}, \quad (38)$$

and writing (38) in (36), we get

$$T^\star = \delta\,(hT + B)\,. \quad (39)$$

If the derivative of equation (39) is taken according to the parameter s and using (2), we obtain

$$f^{'} N^\star = \delta^{'}\,(hT + B) + \delta\,(h\kappa - \tau)\,N. \quad (40)$$

From (40), we get

$$\delta^{'} = 0 \quad \text{and} \quad h\kappa - \tau \neq 0. \tag{41}$$

Substituting (41) in (40), we get

$$f^{'} N^{\star} = \delta \left(h\kappa - \tau \right) N \tag{42}$$

If the equation (42) is multiplied with itself, using (36) and (37), we have

$$\left| \mu_3^{'} - \mu_2 \tau \right| = \delta^2 \left| h\kappa - \tau \right|. \tag{43}$$

Also, since $N^{\star} = \pm N$, we have

$$-\tau^{\star} T^{\star} - B^{\star} = \pm \left(-\kappa T - \tau B \right)$$

and

$$2\tau^{\star} = \kappa^2 - \tau^2$$

which implies that $h\kappa - \tau \neq 0$.

Conversely, let φ be a spacelike curve parametrized by arc-length s with non-zero curvatures κ and τ. Assume that φ provides the conditions of (7) for differentiable functions μ_1, μ_2, μ_3 and real number $h = \pm 1$. Now, we can describe a curve $\varphi^{\star}$ as

$$\varphi^{\star} \left(s^{\star} \right) = \varphi^{\star} \left(f \left(s \right) \right) = \varphi \left(s \right) + \mu_1 \left(s \right) T \left(s \right) + \mu_2 \left(s \right) N \left(s \right) + \mu_3 \left(s \right) B \left(s \right). \tag{44}$$

If the derivative of equation (44) is taken according to the parameter s, we find

$$\frac{d\varphi^{\star}}{ds} = \left(\mu_3^{'} - \mu_2 \tau \right) \left(hT + B \right) \tag{45}$$

and

$$\frac{d^2 \varphi^{*}}{ds^2} = \left(\mu_3^{'} - \mu_2 \tau \right)^{'} \left(hT + B \right) + \left(\mu_3^{'} - \mu_2 \tau \right) \left(h\kappa - \tau \right) N$$

which leads to that

$$f^{'} = \sqrt{m_2 \left(\mu_3^{'} - \mu_2 \tau \right)} \sqrt{m_3 \left(h\kappa - \tau \right)} \tag{46}$$

where $m_2 = sgn \left(\mu_3^{'} - \mu_2 \tau \right)$ and $m_3 = sgn \left(h\kappa - \tau \right)$. Rewriting (45) , we get

$$T^{\star} = m_4 \delta \left(hT + B \right), \quad g \left(T^{\star}, T^{\star} \right) = 0. \tag{47}$$

where $m_4 = sgn \left(\delta \right)$. if the derivative of equation (47) is taken according to the parameter s, we get

$$\frac{dT^{\star}}{ds^{\star}} = \frac{m_4 \delta \left(h\kappa - \tau \right)}{f^{'}} N = m_3 m_4 N \quad , \quad \kappa^{\star} = 1 \tag{48}$$

Thus, we obtain $N^\star$ as

$$N^\star = m_3 m_4 N, \quad g\left(N^\star, N^\star\right) = 1 \tag{49}$$

So we obtain

$$\begin{aligned} B^\star &= \frac{m_4}{-2\delta}\left(-hT + B\right) \\ g\left(B^\star, B^\star\right) &= 0 \\ g\left(T^\star, B^\star\right) &= 1. \end{aligned}$$

Lastly, we get

$$\tau^\star = g\left(\frac{dN^\star}{ds^\star}, B^\star\right) = \frac{m_3\left(h\kappa + \tau\right)}{-2f'\delta} \neq 0$$

Then $\varphi^\star$ is a Cartan null curve and a Bertrand partner curve of φ. Thus φ is a Bertrand curve. □

Theorem 2. *Let φ be a unit speed spacelike curve with timelike principal normal vector and the non-zero curvatures κ, τ in $\mathbb{E}_1^3$. Then the curve φ is a Bertrand curve with Bertrand partner curve φ^* if and only if one of the following condition holds: there exist differentiable functions μ_1, μ_2 and μ_3 providing*

$$\begin{aligned} \mu_1\kappa &= \mu_2^{'} + \mu_3\tau \\ \mu_3^{'} + \mu_2\tau &= 0 \end{aligned} \tag{50}$$

or there are differentiable functions μ_1, μ_2, μ_3 and real number h providing

$$\begin{aligned} \mu_3^{'} + \mu_2\tau &\neq 0 \\ \mu_1\kappa &= \mu_2^{'} + \mu_3\tau \\ 1 + \mu_1^{'} - \mu_2\kappa &= h\left(\mu_3^{'} + \mu_2\tau\right) \\ h\tau - \kappa &\neq 0 \\ h\kappa - \tau &\neq 0 \\ h &\neq 0 \end{aligned} \tag{51}$$

In this situation, the Bertrand partner curve $\varphi^\star$ is a spacelike curve with timelike $N^\star$.

Proof. We omit the proof since it is similar to the proof of theorem 1. □

Corollary 1. *Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Bertrand curve with the Frenet frame $\{T, N, B\}$ and the curve φ^* be a Bertrand partner curve of φ with the Frenet frame $\{T^*, N^*, B^*\}$. If φ is a slant helix, then φ^* is a slant helix if and only if*

$$\mu_3^{'}(s) - \mu_2(s)\tau(s) = constant\ ,$$

where, $\mu_3(s) = g(\varphi^*, B)$ and $\mu_2(s) = g(\varphi^*, N)$.

4 Examples

In this section, examples for Bertrand curves and Bertrand partner curves are constructed according to the new approach described above. Spacelike rectifying curve examples are taken from [1].

Example 1. Let $\varphi : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a spacelike Bertrand curve with the curvatures $\kappa(s), \tau(s)$. In this case, the conditions of theorem 1. are satisfied. By taking $\mu_2 = \mu_0 \in \mathbb{R}$. So, we get,

$$\begin{aligned} \mu_1 \kappa &= \mu_3 \tau \\ 1 + \mu_1' - \mu_0 \kappa &= h \left(\mu_3' - \mu_0 \tau \right) \end{aligned}$$

which implies that

$$\begin{aligned} \mu_3(s) &= \frac{\kappa \left(s - \mu_0 \int (\kappa - h\tau)\, ds \right)}{h\kappa - \tau} \\ \mu_1(s) &= \frac{\tau \left(s - \mu_0 \int (\kappa - h\tau)\, ds \right)}{h\kappa - \tau}. \end{aligned}$$

Thus we get the Bertrand partner curve $\varphi^\star$ as follows

$$\varphi^\star(s) = \varphi(s) + \frac{\tau \left(s - \mu_0 \int (\kappa - h\tau)\, ds \right)}{h\kappa - \tau} T(s) + \mu_0 N(s) + \frac{\kappa \left(s - \mu_0 \int (\kappa - h\tau)\, ds \right)}{h\kappa - \tau} B(s).$$

Here the Bertrand partner curve $\varphi^\star$ is spacelike, timelike or null, respectively if $h^2 > 1$, $h^2 < 1$ or $h^2 = 1$.

Example 2. Let us assess a spacelike curve in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \frac{1}{\sqrt{6}} \left(\sinh \sqrt{3}s, \cosh \sqrt{3}s, 3s \right)$$

with the curvatures $\kappa = \frac{\sqrt{3}}{\sqrt{2}}$ and $\tau = -\frac{3}{\sqrt{2}}$ and the Frenet frame as

$$T(s) = \frac{1}{\sqrt{2}} \left(\cosh \sqrt{3}s, \sinh \sqrt{3}s, \sqrt{3} \right),$$

$$N(s) = \left(\sinh \sqrt{3}s, \cosh \sqrt{3}s, 0 \right),$$

$$B(s) = \frac{1}{\sqrt{2}} \left(\sqrt{3} \cosh \sqrt{3}s, \sqrt{3} \sinh \sqrt{3}s, 1 \right).$$

(i) If we take $\mu_1 = -1, \mu_2 = \sqrt{6}, \mu_3 = \frac{1}{\sqrt{3}}$ and $h = \frac{-2}{3\sqrt{3}}$ in (i) of theorem 1, then we get the curve $\varphi^{\star}$ as follows;

$$\varphi^{\star}(s) = \frac{1}{\sqrt{23}\sqrt{6}}\left(7\sinh\sqrt{3}s, 7\cosh\sqrt{3}s, 3s-2\right).$$

After careful and exhausting calculations, we acquire

$$T^{\star}(s) = \left(\frac{7}{\sqrt{46}}\cosh\sqrt{3}s, \frac{7}{\sqrt{46}}\sinh\sqrt{3}s, \frac{\sqrt{3}}{\sqrt{46}}\right),$$

$$N^{\star}(s) = \left(\sinh\sqrt{3}s, \cosh\sqrt{3}s, 0\right),$$

$$B^{\star}(s) = \left(\frac{\sqrt{3}}{\sqrt{46}}\cosh\sqrt{3}s, \frac{-\sqrt{3}}{\sqrt{46}}\sinh\sqrt{3}s, \frac{7}{\sqrt{46}}\right)$$

and $\kappa^{\star} = 7\sqrt{3}$, $\tau^{\star} = -\frac{3}{\sqrt{46}}$. It can be easily checked that the curve $\varphi^{\star}$ is a timelike Bertrand partner curve of the curve φ.

(ii) If we take $\mu_1 = \sqrt{3}, \mu_2 = \frac{\sqrt{2}}{3\sqrt{3}}, \mu_3 = -1$ and $h = \frac{2}{\sqrt{3}}$ in (ii) of the theorem 1, then we get the curve $\varphi^{\star}$ as follows;

$$\varphi^{\star}(s) = \left(\frac{5}{\sqrt{6}}\sinh\sqrt{3}s, \frac{5}{\sqrt{6}}\cosh\sqrt{3}s, \frac{3(3s+2\sqrt{3})}{\sqrt{6}}\right).$$

After careful calculations, we get

$$T^{\star}(s) = \left(\frac{5}{\sqrt{2}}\cosh\sqrt{3}s, \frac{5}{\sqrt{2}}\sinh\sqrt{3}s, \frac{3\sqrt{3}}{\sqrt{2}}\right),$$

$$N^{\star}(s) = \left(\sinh\sqrt{3}s, \cosh\sqrt{3}s, 0\right),$$

$$B^{\star}(s) = \left(\frac{-3\sqrt{3}}{\sqrt{2}}\cosh\sqrt{3}s, \frac{3\sqrt{3}}{\sqrt{2}}\sinh\sqrt{3}s, \frac{5}{\sqrt{2}}\right)$$

and $\kappa^{\star} = \frac{5\sqrt{3}}{\sqrt{2}}$, $\tau^{\star} = -\frac{9}{\sqrt{2}}$. Easily checked that the curve $\varphi^{\star}$ is a spacelike Bertrand partner curve of the curve φ.

(iii) If we take $\mu_1(s) = \left(\frac{3}{\sqrt{2}} - \frac{3}{3+\sqrt{3}}\right) s, \mu_2 = 1, \mu_3(s) = \left(-\frac{3}{\sqrt{2}} + \frac{3}{3+\sqrt{3}}\right) s$ and $h = 1$ in theorem 1, then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s) = \left(\frac{1}{3}\sinh\sqrt{3}s, \frac{1}{3}\cosh\sqrt{3}s, \frac{\sqrt{3}}{3}s\right).$$

After careful calculations, we acquire

$$T^\star(s) = \left(\frac{\sqrt{3}}{3}\cosh\sqrt{3}s, \frac{\sqrt{3}}{3}\sinh\sqrt{3}s, \frac{\sqrt{3}}{3}\right),$$

$$N^\star(s) = \left(\sinh\sqrt{3}s, \cosh\sqrt{3}s, 0\right),$$

$$B^\star(s) = \left(-\frac{\sqrt{3}}{2}\cosh\sqrt{3}s, \frac{-\sqrt{3}}{2}\sinh\sqrt{3}s, \frac{\sqrt{3}}{2}\right)$$

and $\kappa^\star = 1$, $\tau^\star = -\frac{3}{2}$. Easily checked that the curve $\varphi^\star$ is a Cartan null Bertrand partner curve of the curve φ.

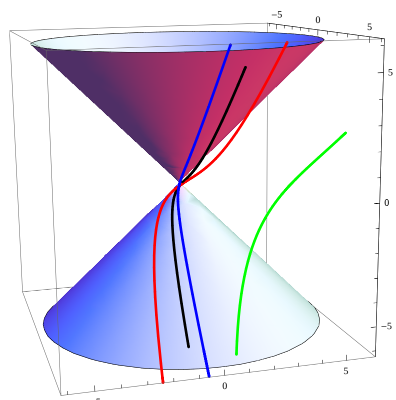

Figure 1: The red graphic is φ, the blue graphic is the timelike Bertand partner curve, the green graphic is the spacelike Bertand partner curve and the black graphic is the null Bertand partner curve in Example 2.

Example 3. Let us consider a spacelike general helix in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \left(-\frac{s^5}{40}, -\frac{s^5}{40} + s, \frac{s^3}{6}\right)$$

with the curvatures $\kappa(s) = s$ and $\tau(s) = -s$ and the Frenet vectors as

$$T(s) = \left(-\frac{s^4}{8}, -\frac{s^4}{8} + 1, \frac{s^2}{2}\right),$$

$$N(s) = \left(-\frac{s^2}{2}, -\frac{s^2}{2}, 1\right),$$

$$B(s) = \left(-\frac{s^4}{8} - 1, -\frac{s^4}{8}, \frac{s^2}{2}\right).$$

(i) If we take $\mu_1(s) = \frac{s^2}{2} - \frac{2s}{3}, \mu_2 = 1, \mu_3(s) = -\frac{s^2}{2} + \frac{2s}{3}$ and $h = \frac{1}{2}$ in (i) of theorem 1, then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s) = \left(-\frac{s^5}{40} - \frac{2s}{3}, -\frac{s^5}{40} + \frac{s}{3}, \frac{s^3}{6} + 1\right).$$

After correct calculations, we obtain

$$T^\star(s) = \left(-\frac{\sqrt{3}s^4}{8} - \frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}s^4}{8} + \frac{\sqrt{3}}{3}, \frac{\sqrt{3}s^2}{8}\right),$$

$$N^\star(s) = \left(-\frac{s^2}{2}, -\frac{s^2}{2}, 1\right),$$

$$B^\star(s) = \left(-\frac{\sqrt{3}s^4}{8} - \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}s^4}{8} + \frac{2\sqrt{3}}{3}, \frac{\sqrt{3}s^2}{2}\right)$$

and $\kappa^\star(s) = 3s$, $\tau^\star(s) = -3s$. Easily checked that the curve $\varphi^\star$ is a timelike Bertrand partner curve of the curve φ.

(ii) If we take $\mu_1(s) = \frac{s^2}{2} + 2s, \mu_2 = 1, \mu_3(s) = -\frac{s^2}{2} - 2s$ and $h = -\frac{3}{2}$ in (ii) of theorem 1, then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s) = \left(-\frac{s^5}{40} + 2s, -\frac{s^5}{40} + 3s, \frac{s^3}{6} + 1\right).$$

After correct calculations, we find

$$T^\star(s) = \left(-\frac{s^4}{8\sqrt{5}} + \frac{2}{\sqrt{5}}, -\frac{s^4}{8\sqrt{5}} + \frac{3}{\sqrt{5}}, \frac{s^2}{2\sqrt{5}}\right),$$

$$N^\star(s) = \left(-\frac{s^2}{2}, -\frac{s^2}{2}, 1\right),$$

$$B^{\star}(s)=\left(-\frac{s^4}{8\sqrt{5}}-\frac{3}{\sqrt{5}},-\frac{s^4}{8\sqrt{5}}-\frac{2}{\sqrt{5}},\frac{s^2}{2\sqrt{5}}\right)$$

and $\kappa^{\star}(s)=\frac{s}{5}$, $\tau^{\star}(s)=-\frac{s}{\sqrt{5}}$Easily checked that the curve $\varphi^{\star}$ is a spacelike Bertrand partner curve of the curve φ.
(iii) If we take $\mu_1(s)=-\frac{s^6}{12}+\frac{s^5}{15}+s^2-s$, $\mu_2(s)=-\frac{s^4}{2}+\frac{s^3}{3}$, $\mu_3(s)=\frac{s^6}{12}-\frac{s^5}{15}+s^2$, $\gamma=1$ and $h=1$ in *(iii)* of theorem 1, then we get the curve $\varphi^{\star}$as follows;

$$\varphi^{\star}(s)=\left(-\frac{s^6}{12}-s^2,-\frac{s^6}{12}+s^2,\frac{s^4}{2}\right)$$

With correct calculations, we have,

$$T^{\star}(s)=\left(-\frac{s^4}{4}-1,-\frac{s^4}{4}+1,s^2\right),$$

$$N^{\star}(s)=\left(-\frac{s^2}{2},-\frac{s^2}{2},1\right),$$

$$B^{\star}(s)=\left(\frac{1}{2},\frac{1}{2},0\right)$$

and $\kappa^{\star}=1$, $\tau^{\star}=0$. Easily checked that the curve $\varphi^{\star}$ is a Cartan null Bertrand partner curve of the curve φ.

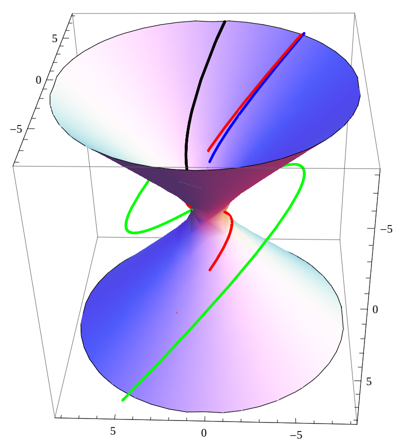

Figure 2: The red graphic is φ, the blue graphic is the timelike Bertand partner curve, the green graphic is the spacelike Bertand partner curve and the black graphic is the null Bertand partner curve in Example 3.

Example 4. Let us consider a spacelike general helix in $\mathbb{E}_1^3$ with the equation

$$\varphi\left(s\right)=\left(\frac{1}{12}\left(16s+5s^2-10\ln s\right),\frac{\sqrt{5}}{6}\left(4s+s^2-2\ln s\right),\frac{\sqrt{5}}{12}\left(s^2+2\ln s\right)\right)$$

with the curvatures $\kappa(s)=\frac{\sqrt{5}}{3s}$ and $\tau(s)=\frac{2}{3s}$ and the Frenet frame, with timelike principal normal vector as

$$\begin{aligned}
T(s)=&\quad\left(\tfrac{5s^2+8s-5}{6s},\tfrac{\sqrt{5}\left(s^2+2s-1\right)}{3s},\tfrac{\sqrt{5}(s^2+1)}{6s}\right)\\
N(s)=&\quad\left(-\tfrac{\sqrt{5}(s^2+1)}{2s},-\tfrac{s^2+1}{s},\tfrac{1-s^2}{2s}\right)\\
B(s)=&\quad\left(\tfrac{\sqrt{5}\left(1-s^2+2s\right)}{3s},\tfrac{2-2s^2+5s}{3s},-\tfrac{1+s^2}{3s}\right)
\end{aligned}$$

If we take $\mu_1(s)=\frac{14}{3\sqrt{5}}\ln s-2s,\mu_2=1,\mu_3(s)=\frac{7}{3}\ln s-\sqrt{5}s$ and $h=\frac{1}{\sqrt{5}}$ in theorem 2, then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star\left(s\right)=\left(\begin{array}{c}-\tfrac{(6\sqrt{5}+(56+6\sqrt{5})s^2-5s^3}{12s}+\tfrac{(-25+84\sqrt{5})\ln s}{30},\tfrac{-6-2(3+7\sqrt{5})s^2+\sqrt{5}s^3}{6s}+\tfrac{(-21+\sqrt{5})\ln s}{3},\\ \tfrac{6-6s^2+\sqrt{5}s^3}{12s}+\tfrac{\sqrt{5}\ln s}{6}\end{array}\right)$$

With correct calculations, we acquire

$$\begin{aligned}
T^\star(s)=&\quad\left(-\tfrac{\left(3\sqrt{5}-5s\right)(-5+s(-28+5s))}{6\sqrt{30}s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}},\tfrac{\sqrt{\frac{5}{6}}\left(-3+\sqrt{5}s\right)\left(-1-7s+s^2\right)}{3s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}},\tfrac{\sqrt{\frac{5}{6}}\left(-3+\sqrt{5}s\right)\left(1+s^2\right)}{6s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}}\right)\\
N^\star(s)=&\quad\left(-\tfrac{\sqrt{5}(s^2+1)}{2s},-\tfrac{s^2+1}{s},\tfrac{1-s^2}{2s}\right)\\
B^\star(s)=&\quad\left(\tfrac{\left(-3\sqrt{5}+5s\right)(-59+s(-28+59s))}{18\sqrt{46}s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}},\tfrac{\left(-3\sqrt{5}+5s\right)(-59+s(-35+59s))}{9\sqrt{46}s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}},\tfrac{59\left(-3+\sqrt{5}s\right)\left(1+s^2\right)}{18\sqrt{46}s^2\sqrt{5+\frac{9-6\sqrt{5}s}{s^2}}}\right)
\end{aligned}$$

and $\kappa^\star(s)=\frac{5\sqrt{5}}{18\left(3\sqrt{5}-5s\right)}$, $\tau^\star(s)=\frac{5\sqrt{\frac{23}{3}}}{6\left(3\sqrt{5}-5s\right)}$. Easily checked that the curve $\varphi^\star$ is a spacelike Bertrand partner curve with timelike principal normal vector of the curve φ.

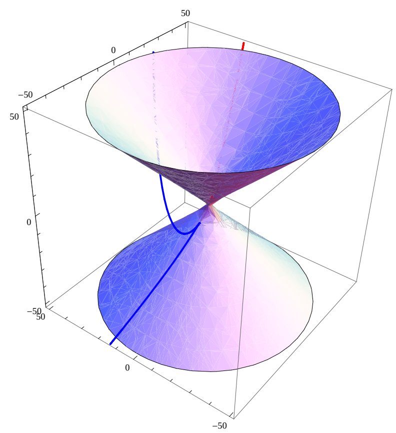

Figure 3: The red graphic is φ, the blue graphic is the spacelike Bertand partner curve with timelike principal normal vector in Example 4.

Example 5. Let us consider a spacelike rectifying slant helix in $\mathbb{E}_1^3$ with the equation

$$\varphi(s) = \left(-\sinh(1)\sqrt{1+s^2}, \cosh(1)\sqrt{1+s^2}\cos[\mathrm{sech}(1)\arctan(s)], \cosh(1)\sqrt{1+s^2}\sin[\mathrm{sech}(1)\arctan(s)]\right)$$

with the curvatures $\kappa(s) = \frac{\tanh(1)}{(1+s^2)^{\frac{3}{2}}}$ and $\tau(s) = \frac{s\tanh(1)}{(1+s^2)^{\frac{3}{2}}}$ and the Frenet frame, with timelike principal normal vector as

$$\begin{aligned}
T(s) &= \begin{pmatrix} -\frac{\sinh(1)s}{1+s^2}, \frac{1}{\sqrt{1+s^2}}\left(\cosh(1)s\cos[\mathrm{sech}(1)\arctan(s)] - \sin[\mathrm{sech}(1)\arctan(s)]\right), \\ \frac{1}{\sqrt{1+s^2}}\left(\cos[\mathrm{sech}(1)\arctan(s)] + \cosh(1)s\sin[\mathrm{sech}(1)\arctan(s)]\right) \end{pmatrix} \\
N(s) &= \left(\cosh(1), -\sinh(1)\cos[\mathrm{sech}(1)\arctan(s)], -\sinh(1)\sin[\mathrm{sech}(1)\arctan(s)]\right) \\
B(s) &= \begin{pmatrix} -\frac{\sinh(1)}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}\left(\cosh(1)\cos[\mathrm{sech}(1)\arctan(s)] + s\sin[\mathrm{sech}(1)\arctan(s)]\right), \\ \frac{1}{\sqrt{1+s^2}}\left(s\cos[\mathrm{sech}(1)\arctan(s)] + \cosh(1)\sin[\mathrm{sech}(1)\arctan(s)]\right) \end{pmatrix}
\end{aligned}$$

If we take $\mu_1(s) = \frac{s\tanh(1)}{\sqrt{1+s^2}} - \frac{s^2}{s-1}, \mu_2 = 1, \mu_3(s) = \frac{\tanh(1)}{\sqrt{1+s^2}} - \frac{s}{s-1}$ and $h = 1$ in of theorem 2, then we get the curve $\varphi^\star$ as follows;

$$\varphi^\star(s) = \begin{pmatrix} \mathrm{sech}(1) + \frac{\sinh(1)\sqrt{1+s^2}}{-1+s}, -\frac{\cosh(1)\sqrt{1+s^2}\cos[\mathrm{sech}(1)\arctan(s)]}{-1+s}, \\ -\frac{\cosh(1)\sqrt{1+s^2}\sin[\mathrm{sech}(1)\arctan(s)]}{-1+s} \end{pmatrix}$$

After correct calculations, we acquire

$$T^{\star}(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}} \begin{pmatrix} -\sinh(1)\,(1+s)\,, \\ \cosh(1)\,(1+s)\cos[\operatorname{sech}(1)\arctan(s)] + (-1+s)\sin[\operatorname{sech}(1)\arctan(s)], \\ (-1+s)\cos[\operatorname{sech}(1)\arctan(s)] + \cosh(1)\,(1+s)\sin[\operatorname{sech}(1)\arctan(s)] \end{pmatrix},$$

$$N^{\star}(s) = (-\cosh(1), \sinh(1)\cos[\operatorname{sech}(1)\arctan(s)], \sinh(1)\sin[\operatorname{sech}(1)\arctan(s)])\,,$$

$$B^{\star}(s) = \frac{1}{\sqrt{2}\sqrt{1+s^2}} \begin{pmatrix} -\sinh(1)\,(-1+s)\,, \\ \cosh(1)\,(-1+s)\cos[\operatorname{sech}(1)\arctan(s)] - (1+s)\sin[\operatorname{sech}(1)\arctan(s)], \\ (1+s)\cos[\operatorname{sech}(1)\arctan(s)] + \cosh(1)\,(-1+s)\sin[\operatorname{sech}(1)\arctan(s)] \end{pmatrix}$$

and $\kappa^{\star}(s) = \frac{\tanh(1)(-1+s)^3}{2(1+s^2)^{\frac{3}{2}}}$, $\tau^{\star}(s) = \frac{\tanh(1)(-1+s)^2(1+s)}{2(1+s^2)^{\frac{3}{2}}}$. Easily seen that the curve $\varphi^{\star}$ is a spacelike Bertrand partner curve with timelike principal normal vector of the curve φ.

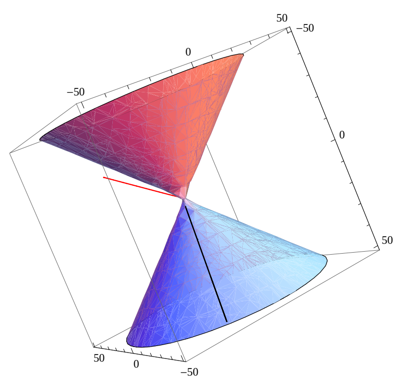

Figure 4: The red graphic is φ, the black graphic is the spacelike Bertand partner curve with timelike principal normal vector in Example 5.

Acknowledgments: The authors thank the anonymous referees and the Editor for valuable comments and suggestions that improved the paper.

References

[1] Altunkaya, B., Kula, L., On spacelike rectifying slant helices in Minkowski 3-space. *Turkish J. Math.*, **42**(3) (2018), 10981110.

[2] Ali, TA., Lopez, R., Slant helices in Minkowski 3-space. *J. Korean Math Soc.*, **48** (2011), 159-167.

[3] Balgetir, H., Bektaş, M., Ergüt, M., Bertrand curves for non-null curves in 3-dimensional Lorentzian space. *Hadronic J.*, **27**(2) (2004), 229-236.

[4] Balgetir,H., Bektaş, M., Inoguchi, J.,Null Bertrand curves in Minkowski 3-space and their characterizations. *Note Mat.*, **23** (1)(2004/05), 7-13.

[5] Bertrand, J. M., Mémoire sur la théorie des courbes á double courbure. *Comptes Rendus.*, **36** (1850).

[6] Camcı, Ç., Uçum, A., İlarslan, K., A New Approach to Bertrand Curves in Euclidean 3-Space. *J. Geom.*, **111**:49,(2020).

[7] Chen,B.Y., When does the position vector of a space curve always lie in its rectifying plane? *Am Math Mon.*, **110** (2003), 147-152.

[8] Chen, B.Y., Dillen, F., Rectifying curves as centrodes and extremal curves. *Bull Ins Math Aca Sinica*, **33** (2005),77-90.

[9] Cheng,Y. M., Lin, C. C., On the generalized bertrand curves in Euclidean -spaces. *Note di Matematica*, **29** (2) (2009), 33-39.

[10] Duggal, K. L., Jin, D. H., Null curves and Hypersurfaces of Semi-Riemannian Manifolds. *World Scientific Publishing*, (2007).

[11] Ekmekci, N., İlarslan, K., On Bertrand curves and their characterization. *Differ. Geom. Dyn. Syst.*, **3**(2) (2001), 17-24.

[12] Gökcek, F., Altın Erdem, H., On Cartan null Bertrand curves in Minkowski 3-space. *Facta Univ. Ser. Math. Inform.*, **36**(5) (2021), 10791088.

[13] Görgülü, A., Özdamar, E., A generalization of the Bertrand curves as general inclined curves in $\mathbb{E}^n$. *Communications of the Faculty of Sciences of the University of Ankara, Series A1: Mathematics and Statistics*, **35** (12) (1986), pp. 53-60

[14] Huang, J., Chen, L., Izumiya, S., Pei, D., Geometry of special curves and surfaces in 3-space form. *J. Geom. Phys.*, **136** (2019), 3138.

[15] Inoguchi, J., Lee, S., Null curves in Minkowski 3-space. *Int. Electron. J. Geom.*, **1**(2) (2008), 40-83.

[16] İlarslan, K., Klç Aslan, N., *On generalized null Mannheim curves in* $\mathbb{E}^4_2$. Math. Methods Appl. Sci. 44 (9) (2021), 75887600.

[17] Izumiya, S., Takeuchi, N., *New special curves and developable surfaces.* Turk J Math. 28(2004), 153-163.

[18] Jin,D. H., *Null Bertrand curves in a Lorentz manifold.* J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 15(3) (2008), 209-215.

[19] Kuhnel,W., *Differential geometry: curves-surfaces-manifolds.* Braunschweig, Wiesbaden, (1999).

[20] Liu, H., Curves in Three Dimensional Riemannian Space Forms. *Results. Math.*, **66** (2014), 469480.

[21] Lopez, R., Differential geometry of curves and surfaces in Lorentz-Minkowski space. *Int. Electron. J. Geom.*, **7**(1) 2014, 44-107.

[22] Lucas, P., Ortega-Yagues, J. A., Bertrand curves in the three-dimensional sphere. *Journal of Geometry and Physics* , **62**(9) 2012, 1903-1914.

[23] Lucas, P., Ortega-Yagues, J. A., Bertrand curves in non-flat three-dimensional(Riemannian or Lorentzian) space forms. *Bull. Korean Math. Soc.* **50**(2013), 11091126.

[24] Matsuda, H., Yorozu, S., Notes on Bertrand curves. *Yokohama Math. J.*, **50** (1-2) (2003), 41-58.

[25] O'Neill, B., Semi-Riemannian geometry with applications to relativity. *Academic Press, New York*, (1983).

[26] Öztekin, H.B., Weakened Bertrand curves in the Galilean space G3. *Journal of Advanced Mathematical Studies*, **2** (2) (2009), 6976.

[27] Öğrenmi ş, A. O., Öztekin, H., Ergüt, M., Bertrand curves in Galilean space and their characterizations. *Kragujevac J. Math.*, **32**(2009), 139-147.

[28] Pears, L. R., Bertrand curves in Riemannian space. *J. London Math. Soc.*, **1-10**(2) (1935), 180-183.

[29] Ravani,B., Ku, T.S., Bertrand offsets of ruled and developable surfaces. *Computer-Aided Design*, **23**(2) (1991), 145-152.

[30] Uçum, A., Keçilioğlu, O., İlarslan, K., Generalized Pseudo Null Bertrand curves in Semi-Euclidean 4-Space with index 2. *Rend. Circ. Mat. Palermo, II. Ser.*, **65**(2016), 459472.

[31] Uçum, A., Keçilioğlu, O., İlarslan, K., Generalized Bertrand curves with timelike (1,3)-normal plane in Minkowski space-time. *Kuwait J. Sci.*, **42** (3)(2015), 10-27.

[32] Uçum, A., İlarslan, K., On timelike Bertrand curves in Minkowski 3-space. *Honam Mathematical J.*, **38**(3) (2016), 467-477.

[33] Uçum, A., İlarslan, K., Sakaki, M., Generalized Cartan Null Bertrand curves in semi-Euclidean 4-Space with index 2. *J. Geom.* **107** (2016), 579591.

[34] Saint Venant, B., Mémoire sur les lignes courbes non planes. *Journal de l'Ecole Polytechnique*, **18**(1845),1-76.

[35] Schief, W. K., On the integrability of Bertrand curves and Razzaboni surfaces. *J. Geom. Phys.*, **45**(1-2) (2003), 130150.

[36] Xu, C., Cao, X., Zhu, P., Bertrand curves and Razzaboni surfaces in Minkowski 3-space. *Bull. Korean Math. Soc.* **52**(2) (2015), 377394.

Hatice Altın Erdem
Department of Mathematics,
Kırıkkale University,
Kırıkkale-Turkey
Email: hatice_altin@yahoo.com

Kazım İlarslan,
Department of Mathematics,
Kırıkkale University,
Kırıkkale-Turkey
Email: kilarslan@kku.edu.tr