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Some Fixed Point Results for Sehgal-Proinov Type Contractions in Modular $b-$Metric Spaces

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Abstract

In this paper, inspired by Proinov type contractions, we intend to acquire novel definitions and results that expand Sehgals [3] metric fixed point theory in the sense of modular $b-$metric space. To demonstrate the theorems, we employ a general form of (α, β) $-$admissible and multi-valued mappings and obtain some general results for single-valued mapping in the context of modular $b-$metric space.

1 Introduction

In the course of this study, the notations $\mathbb{N}$, $\mathbb{Z}^+$, and $\mathbb{R}_+$ will symbolize the set of natural numbers, the set of positive integers, and the set of all non-negative real numbers, respectively.

Let $\mathcal{Q}$ be a nonvoid set and $\mathcal{F}, \mathcal{S} : \mathcal{Q} \to \mathcal{Q}$ be self-mappings. Thereby, the following ones represent the set of fixed points of $\mathcal{F}$ and the set of common fixed points of $\mathcal{F}$ and $\mathcal{S}$, respectively:

- $Fix\,(\mathcal{F}) = \{\mathfrak{j} \in \mathcal{Q} : \ \mathcal{F}\mathfrak{j} = \mathfrak{j}\}$;
- $C_{Fix}\,(\mathcal{F}, \mathcal{S}) = \{\mathfrak{j} \in \mathcal{Q} : \ \mathcal{F}\mathfrak{j} = \mathcal{S}\mathfrak{j} = \mathfrak{j}\}$.

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Also, let $\mathcal{P}(\mathcal{Q})$ be the family of all nonempty subsets of $\mathcal{Q}$ and let $\mathcal{F}, \mathcal{S} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ be a multi-valued mapping. So, the following one is the set of fixed points of multi-valued mapping $\mathcal{F}$ and the set of common fixed points of multi-valued mappings of $\mathcal{F}$ and $\mathcal{S}$, respectively:

- $M_{Fix}(\mathcal{F}) = \{\mathfrak{j} \in \mathcal{Q} : \mathfrak{j} \in \mathcal{F}(\mathfrak{j})\}$;
- $M^{C}_{Fix}(\mathcal{F}, \mathcal{S}) = \{\mathfrak{j} \in \mathcal{Q} : \mathfrak{j} \in \mathcal{F}(\mathfrak{j}) \text{ and } \mathfrak{j} \in \mathcal{S}(\mathfrak{j})\}$.

The Banach fixed point theorem [1] has been one of the remarkable and most productive consequences of metric fixed point theory, which put forward that every mapping $\mathcal{F}$ on a complete metric space $(\mathcal{Q}, \mathsf{d})$ satisfying for all $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$

$$\mathsf{d}(\mathcal{F}\mathfrak{j}, \mathcal{F}\mathfrak{s}) \le \varsigma \mathsf{d}(\mathfrak{j}, \mathfrak{s}), \quad \textit{where} \quad \varsigma \in (0,1) \tag{1}$$

owns a unique fixed point, and for every $\mathfrak{j}_0 \in \mathcal{Q}$, the sequence $\{\mathcal{F}^n \mathfrak{j}_0\}$ convergence to this fixed point.

A natural generalization of Banach's fixed point theorem is Bryants fixed point theorem [2], proved by Bryant in 1968, as noted below.

Theorem 1.1. [2] *Let $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ be a self-mapping on complete metric space $(\mathcal{Q}, \mathsf{d})$. If so, the set $Fix(\mathcal{F})$ owns exactly one element provided that $\mathcal{F}^N$ is a contraction mapping for some $N \in \mathbb{Z}^+$.*

One can clearly say that $\mathcal{F}^N$ is continuous. However, the fact that $\mathcal{F}^N$ is continuous does not necessarily mean that $\mathcal{F}$ is continuous. Bryant gave an example illustrating this observation in [2].

In 1969, Sehgal [3] asserted a novel result, an extension of Theorem 1.1, with respect to "the contractive iteration of each point" in the sense of complete metric space, as follows.

Theorem 1.2. [3] *$\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ be a continuous self-mapping on a complete metric space $(\mathcal{Q}, \mathsf{d})$ and $q \in [0,1)$. If there exists $n = n(\mathfrak{j}) \in \mathbb{Z}^+$ for each $\mathfrak{j} \in \mathcal{Q}$ such that*

$$\mathsf{d}\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{s}\right) \le q \mathsf{d}(\mathfrak{j}, \mathfrak{s}), \tag{2}$$

for all $\mathfrak{s} \in \mathcal{Q}$, the set $Fix(\mathcal{F})$ possesses exactly one element.

Moreover, Sehgal [3] came up with an example, which not satisfy the inequality (1), that is, not a contraction, yet it admits (2) and owns a fixed point. Subsequently, Guseman [4] removed the continuity condition on the mapping and resubmitted the results. In 2018, Alqahtani et al. [5] verified the subsequent common fixed point theorem regarding Sehgal's consequences in complete $b-$metric space (see definition 1.8 for $b-$metric spaces), as indicated below.

Theorem 1.3. [5] *Let* $\mathcal{F}, \mathcal{S}$ *be two self-mappings on a complete* $b-$*metric space* $(\mathcal{Q}, \flat, \hbar)$. *For each* $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$, *if* $\chi \in \left(0, \frac{1}{2\hbar-1}\right)$ *consists such that* , $n(\mathfrak{j}), m(\mathfrak{s}) \in \mathbb{Z}^+$ *exists such that*

$$\mathsf{d}\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) \leq \chi\left[\mathsf{d}(\mathfrak{j}, \mathfrak{s}) + \left|\mathsf{d}\left(\mathfrak{j}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right) - \mathsf{d}\left(\mathfrak{s}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right)\right|\right].$$

Thereby, $\mathcal{F}$ *and* $\mathcal{S}$ *possess exactly one common fixed point.*

Also, for the latest study involving Sehgal's fixed point result, refer to [6]-[11].

Proinov [12] recently constituted a novel and interesting contraction condition in a metric space, as many authors have tried to put forward in metric fixed point theory. Proinov indicated a fixed point theorem, considering proper auxiliary functions, which develops and has various consequences in the substantial literature.

Definition 1.4. [12] Let $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ be a mapping defined on a metric space $(\mathcal{Q}, \mathsf{d})$. Presume that $\Sigma, \Omega : (0, \infty) \to \mathbb{R}$ are two functions such that the features

(p_1) Σ is a non-decreasing function,

(p_2) $\Omega(\mathfrak{a}) < \Sigma(\mathfrak{a})$ for all $\mathfrak{a} > 0$,

(p_3) $\limsup_{\mathfrak{a} \to \mathfrak{a}_0+} \Omega(\mathfrak{a}) < \Sigma(\mathfrak{a}_0+)$ for any $\mathfrak{a}_0 > 0$

are provided. Thereby, for all $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$, if the inequality

$$\Sigma(\mathsf{d}(\mathcal{F}\mathfrak{j}, \mathcal{F}\mathfrak{s})) \leq \Omega(\mathsf{d}(\mathfrak{j}, \mathfrak{s})),$$

is satisfied, the mapping $\mathcal{F}$ is termed a Proinov type contraction.

Theorem 1.5. [12] *Presume that* $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ *is a Proinov type contraction defined on a complete metric space* $(\mathcal{Q}, \mathsf{d})$. *The set* $Fix(\mathcal{F})$ *includes just one element.*

Diverse fixed point consequences appear in the literature, including Proinov type contraction. Some instances of these studies are [13], [14], and [15].

In 2014, Alizadeh et al. [16] gained the construction of cyclic $(\alpha, \beta)-$admissible mappings to the literature.

Definition 1.6. [16] Let $\mathcal{F}$ be a self-mapping defined on a nonempty set $\mathcal{Q}$ and $\alpha, \beta : \mathcal{Q} \to \mathbb{R}_+$ be two functions. $\mathcal{F}$ is a cyclic $(\alpha, \beta)-$admissible mapping if the given two circumstances

(i) $\alpha(\mathfrak{j}) \geq 1 \Rightarrow \alpha(\mathcal{F}\mathfrak{j}) \geq 1,$

(ii) $\alpha(\mathfrak{j}) \geq 1 \Rightarrow \alpha(\mathcal{F}\mathfrak{j}) \geq 1$

are provided for some $\mathfrak{j} \in \mathcal{Q}$.

Subsequently, Latif et al. [17] have generalized Definition 1.6 as indicated below by taking notice of two self-mappings.

Definition 1.7. [17] Presume that $\mathcal{Q}$ is a nonvoid set, the self-mappings $\mathcal{F}$, $\mathcal{S}$ be defined on this set, α and β are two functions from $\mathcal{Q}$ to $\mathbb{R}_+$. $(\mathcal{F}, \mathcal{S})$ is a cyclic (α, β) $-$admissible pair provided that the subsequent two situations are ensured:

(i) $\alpha(\mathfrak{j}) \geq 1 \Rightarrow \alpha(\mathcal{F}\mathfrak{j}) \geq 1,$

(ii) $\alpha(\mathfrak{j}) \geq 1 \Rightarrow \alpha(\mathcal{S}\mathfrak{j}) \geq 1,$

for some $\mathfrak{j} \in \mathcal{Q}$.

Remark 1. If $\mathcal{S} = \mathcal{F}$ in the definition afore, Definition 1.6 is achieved.

In addition to introducing new contractions to the fixed point theory, the extension of the concept of a metric was of great interest to the authors, and many studies have been done in this direction. One of these is the naturally-formed b-metric function, which appears first in Bakhtin's [18] study and then in Czerwik's [19, 20].

Definition 1.8. [19] Assume that $\mathcal{Q}$ is a nonempty set and $\hbar \geq 1$ is a real-valued constant. For all $\mathfrak{j}, \mathfrak{s}, \mathfrak{r} \in \mathcal{Q}$, if the circumstances

$(\flat_1)$ $\flat(\mathfrak{j}, \mathfrak{s}) = 0 \Leftrightarrow \mathfrak{j} = \mathfrak{s},$

$(\flat_2)$ $\flat(\mathfrak{j}, \mathfrak{s}) = \flat(\mathfrak{s}, \mathfrak{j}),$

$(\flat_3)$ $\flat(\mathfrak{j}, \mathfrak{s}) \leq \hbar[\flat(\mathfrak{j}, \mathfrak{r}) + \flat(\mathfrak{r}, \mathfrak{s})]$

are satisfied, the mapping $\flat : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}_+$ is termed as $b-$metric. The pair $(\mathcal{Q}, \flat)$ is entitled $b-$metric space.

If $\hbar = 1$, the $b-$metric is treated as a metric function.

Furthermore, outside of the continuity, nearly all of the topological features of $b-$metric space are counterparts to the metric ones. The following crucial lemma is fundamental for employing the continuity of $b-$metric.

Lemma 1.9. [21] *Let the triple* $(\mathcal{Q}, \flat, \hbar \geq 1)$ *be a* $b-$*metric space. Presume that the sequences belong to the space* $\{\mathfrak{j}_\mathfrak{z}\}$, $\{\mathfrak{s}_\mathfrak{z}\}$ *convergence to* $\mathfrak{j}$, $\mathfrak{s} \in \mathcal{Q}$, *respectively. Then*

$$\frac{1}{\hbar^2}\flat(\mathfrak{j}, \mathfrak{s}) \leq \liminf_{\mathfrak{z}\to\infty} \flat(\mathfrak{j}_\mathfrak{z}, \mathfrak{s}_\mathfrak{z}) \leq \limsup_{\mathfrak{z}\to\infty} \flat(\mathfrak{j}_\mathfrak{z}, \mathfrak{s}_\mathfrak{z}) \leq \hbar^2 \flat(\mathfrak{j}, \mathfrak{s}).$$

Especially, if $\mathfrak{j} = \mathfrak{s}$, *then* $\lim_{\mathfrak{z}\to\infty} \flat(\mathfrak{j}_\mathfrak{z}, \mathfrak{s}_\mathfrak{z}) = 0$. *Also, for* $\mathfrak{r} \in \mathcal{Q}$, *we attain*

$$\frac{1}{\hbar}\flat(\mathfrak{j}, \mathfrak{r}) \leq \liminf_{\mathfrak{z}\to\infty} \flat(\mathfrak{j}_\mathfrak{z}, \mathfrak{r}) \leq \limsup_{\mathfrak{z}\to\infty} \flat(\mathfrak{j}_\mathfrak{z}, \mathfrak{r}) \leq \hbar \flat(\mathfrak{j}, \mathfrak{r}).$$

The studies [22]-[25] by Chistyakov constitute the basis of the studies on modular metrics, a very recent and intriguing concept.

Primarily, let $\mathscr{M} : (0, \infty) \times \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ be a function provided to be the $\mathcal{Q}$ is a nonempty set. If so, for clarity, we will prefer the notions of $\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{s})$ rather than $\mathscr{M}(\kappa, \mathfrak{j}, \mathfrak{s})$ for all $\kappa > 0$ and $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$.

Definition 1.10. [23, 24] Presume that $\mathcal{Q}$ is a nonempty set. The mapping $\mathscr{M} : (0, \infty) \times \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ is entitled to modular metric provided that the circumstances are provided for all $\mathfrak{j}, \mathfrak{s}, \mathfrak{r} \in \mathcal{Q}$, and $\kappa, \varsigma > 0$

$(\mathscr{M}_1)$ $\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{s}) = 0$ if and only if $\mathfrak{j} = \mathfrak{s}$,

$(\mathscr{M}_2)$ $\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{s}) = \mathscr{M}_\kappa(\mathfrak{s}, \mathfrak{j})$,

$(\mathscr{M}_3)$ $\mathscr{M}_{\kappa+\varsigma}(\mathfrak{j}, \mathfrak{s}) \leq \mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{r}) + \mathscr{M}_\varsigma(\mathfrak{r}, \mathfrak{s})$.

Thereupon, $(\mathcal{Q}, \mathscr{M})$ is a modular metric space abbreviated as $\mathfrak{mms}$.

Instead of $(\mathscr{M}_1)$, if we consider the following statement, then $\mathscr{M}$ is a (metric) pseudo-modular on $\mathcal{Q}$ for all $\kappa > 0$

$(\mathscr{M}_1')$ $\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{j}) = 0$.

Moreover, $\mathscr{M}$ defined on $\mathcal{Q}$ has the property of regularity if the new statement, which is a weaker version of $(\mathscr{M}_1)$, for some $\kappa > 0$,

$(\mathscr{M}_1'')$ $\mathfrak{j} = \mathfrak{s}$ if and only if $\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{j}) = 0$

is provided. The function $\mathscr{M}$, owning the following feature if for $\kappa, \varsigma > 0$ and $\mathfrak{j}, \mathfrak{s}, \mathfrak{r} \in \mathcal{Q}$, is termed a convex modular on $\mathcal{Q}$

$$\mathscr{M}_{\kappa+\varsigma}(\mathfrak{j}, \mathfrak{s}) \leq \frac{\kappa}{\kappa+\varsigma}\mathscr{M}_\kappa(\mathfrak{j}, \mathfrak{r}) + \frac{\varsigma}{\kappa+\varsigma}\mathscr{M}_\varsigma(\mathfrak{r}, \mathfrak{s}).$$

On the other hand, whenever $\mathscr{M}$ is a metric pseudo-modular on a set $\mathcal{Q}$, the function $\kappa \to \mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{s})$ is non-increasing on $(0,\infty)$ for any $\mathfrak{j},\mathfrak{s} \in \mathcal{Q}$. For $0<\varsigma<\kappa$, it is verified that

$$\mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{s}) \leq \mathscr{M}_{\kappa-\varsigma}(\mathfrak{j},\mathfrak{j}) + \mathscr{M}_{\varsigma}(\mathfrak{j},\mathfrak{s}) = \mathscr{M}_{\varsigma}(\mathfrak{j},\mathfrak{s}).$$

Definition 1.11. [23, 24] Consider $\mathscr{M}$ is a pseudo-modular on $\mathcal{Q}$ and $\mathfrak{j}_0$ be a fixed element belonging to $\mathcal{Q}$. Thereby, the following sets are mentioned as modular spaces (around $\mathfrak{j}_0$):

- $\mathcal{Q}_{\mathscr{M}} = \mathcal{Q}_{\mathscr{M}}(\mathfrak{j}_0) = \{\mathfrak{j} \in \mathcal{Q} : \mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{j}_0) \to 0\}$ as $\kappa \to \infty$, and
- $\mathcal{Q}^*_{\mathscr{M}} = \mathcal{Q}^*_{\mathscr{M}}(\mathfrak{j}_0) = \{\mathfrak{j} \in \mathcal{Q} : \exists \kappa = \kappa(\mathfrak{j}) > 0 \text{ such that } \mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{j}_0) < \infty\}$.

Note that $\mathcal{Q}_{\mathscr{M}} \subset \mathcal{Q}^*_{\mathscr{M}}$, but it is not in general. Accordingly, from [23, 24], a (nontrivial) metric $\mathsf{d}_{\mathscr{M}}$, which is presented in follows and generated by the modular $\mathscr{M}$, for any $\mathfrak{j},\mathfrak{s} \in \mathcal{Q}_{\mathscr{M}}$

$$\mathsf{d}_{\mathscr{M}}(\mathfrak{j},\mathfrak{s}) = \inf\{\kappa > 0 : \mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{s}) \leq \kappa\}$$

is identified on $\mathcal{Q}_{\mathscr{M}}$. Furthermore, if we consider a convex modular $\mathscr{M}$ on $\mathcal{Q}$, then $\mathcal{Q}_{\mathscr{M}} = \mathcal{Q}^*_{\mathscr{M}}$ thereupon, these sets are endowed with the metric

$$\mathsf{d}^*_{\mathscr{M}}(\mathfrak{j},\mathfrak{s}) = \inf\{\kappa > 0 : \mathscr{M}_{\kappa}(\mathfrak{j},\mathfrak{s}) \leq 1\},$$

withal proverbial as the Luxembourg distance, for any $\mathfrak{j},\mathfrak{s} \in \mathcal{Q}_{\mathscr{M}}$.

Definition 1.12. [23, 24] Let $\mathcal{Q}^*_{\mathscr{M}}$ be an $\mathfrak{mms}$, $\{\mathfrak{j}_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}} \in \mathcal{Q}^*_{\mathscr{M}}$ be a sequence, and $\mathcal{Y}$ be a subset of $\mathcal{Q}^*_{\mathscr{M}}$.

1. $\{\mathfrak{j}_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ is an $\mathscr{M}-$convergent sequence to $\mathfrak{j} \in \mathcal{Q}^*_{\mathscr{M}}$ if and only if for all $\kappa > 0$, $\mathscr{M}_{\kappa}(\mathfrak{j}_{\mathfrak{z}},\mathfrak{j}) \to 0$, as n tends to infinity, and the point $\mathfrak{j}$ is named the $\mathscr{M}-$limit of $\{\mathfrak{j}_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$.
2. If $\lim\limits_{\mathfrak{z},m\to\infty} \mathscr{M}_{\kappa}(\mathfrak{j}_{\mathfrak{z}},\mathfrak{j}_m) = 0$, for all $\kappa > 0$, $\{\mathfrak{j}_{\mathfrak{z}}\}_{\mathfrak{z}\in\mathbb{N}}$ in $\mathcal{Q}^*_{\mathscr{M}}$ is named as an $\mathscr{M}-$Cauchy sequence.
3. If any $\mathscr{M}-$Cauchy sequence $\mathscr{M}-$convergences to the element of $\mathcal{Q}^*_{\mathscr{M}}$, $\mathcal{Q}^*_{\mathscr{M}}$ is termed an $\mathscr{M}-$complete space.
4. The set $\mathcal{Y}$ is $\mathscr{M}-$closed, provided that the $\mathscr{M}-$limit of an $\mathscr{M}-$convergent sequence of $\mathcal{Y}$ always belongs to $\mathcal{Y}$.
5. $\mathcal{F} : \mathcal{Q}^*_{\mathscr{M}} \to \mathcal{Q}^*_{\mathscr{M}}$ is an $\mathscr{M}-$continuous mapping if $\mathscr{M}_{\kappa}(\mathfrak{j}_{\mathfrak{z}},\mathfrak{j}) \to 0$, provided to $\mathscr{M}_{\kappa}(\mathcal{F}\mathfrak{j}_{\mathfrak{z}},\mathcal{F}\mathfrak{j}) \to 0$ as $k \to \infty$.

6. $\mathcal{Y}$ is an $\mathscr{M}-$bounded set, provided that

$$\delta_{\mathscr{M}}(\mathcal{Y})=\sup\{\mathscr{M}_1(\mathfrak{j},\mathfrak{s}):\mathfrak{j},\mathfrak{s}\in\mathcal{Y}\}<\infty.$$

7. $\mathcal{Y}$ is an $\mathscr{M}-$compact set if, for any $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ in $\mathcal{Y}$, there exists a subsequence $\left\{\mathfrak{j}_{\mathfrak{z}_\mathfrak{z}}\right\}$ and a point $\mathfrak{j}$ in $\mathcal{Y}$ such that $\mathscr{M}_1\left(\mathfrak{j}_{\mathfrak{z}_\mathfrak{z}},\mathfrak{j}\right)\to 0$.

8. $\mathscr{M}$ holds the Fatou property $\Leftrightarrow$ for any sequence $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ in $\mathcal{Q}^*_{\mathscr{M}}$ $\mathscr{M}-$convergences to $\mathscr{M}$, then

$$\mathscr{M}_1(\mathfrak{j},\mathfrak{s})\le\liminf_{\mathfrak{z}\to\infty}\mathscr{M}_1(\mathfrak{j}_\mathfrak{z},\mathfrak{s})$$

for any $\mathfrak{s}\in\mathcal{Q}^*_{\mathscr{M}}$.

Definition 1.13. [26] The modular $\mathscr{M}$ fulfills the Δ_2-condition if the condition

($\mathcal{D}$) $\lim_{\mathfrak{z}\to\infty}\mathscr{M}_\kappa(\mathfrak{j}_\mathfrak{z},\mathfrak{j})=0$ for some $\kappa>0$ implies $\lim_{\mathfrak{z}\to\infty}\mathscr{M}_\kappa(\mathfrak{j}_\mathfrak{z},\mathfrak{j})=0$, for all $\kappa>0$

is realized.

However, the converse of condition ($\mathcal{D}$) is not always valid.

Now, we will recall the following sets.

- $\mathscr{CB}(\mathcal{Y})=\{\mathscr{X}:\ \mathscr{X}$ is nonvoid, $\mathscr{M}-$ closed, and $\mathscr{M}-$ bounded subset of $\mathcal{Y}\}$.
- $\mathscr{K}(\mathcal{Y})=\{\mathscr{X}:\ \mathscr{X}$ is nonvoid, $\mathscr{M}-$ compact subset of $\mathcal{Y}\}$.
- On $\mathscr{CB}(\mathcal{Y})$, the Hausdorff-Pompei modular metric is identified by

$$\mathscr{H}_{\mathscr{M}}(\mathcal{Z},\mathcal{L})=\max\left\{\sup_{\mathfrak{j}\in\mathcal{Z}}\mathscr{M}_1(\mathfrak{j},\mathcal{L}),\sup_{\mathfrak{s}\in\mathcal{L}}\mathscr{M}_1(\mathcal{Z},\mathfrak{s})\right\}$$

for $\mathscr{M}_1(\mathfrak{j},\mathcal{L})=\inf_{\mathfrak{s}\in\mathcal{L}}\mathscr{M}_1(\mathfrak{j},\mathfrak{s})$.

The Banach fixed point theorem for multi-valued mappings in the metric space setting by handling the notion of the Hausdorff-Pompei metric was demonstrated by Nadler [27]. Moreover, this concept is also discussed in modular metric spaces. As noted in [26], Abdou and Khamsi characterized the multi-valued Lipschitzian mapping in this space.

Definition 1.14. [26] Let $(\mathcal{Q},\mathscr{M})$ be an $\mathfrak{mms}$, $\mathcal{F}:\mathcal{Y}\to\mathscr{CB}(\mathcal{Y})$ be a mapping, and $\mathcal{Y}$ be a nonvoid subset of $\mathcal{Q}_{\mathscr{M}}$. For any $\mathfrak{j},\mathfrak{s}\in\mathcal{Y}$ and $\gamma\ge 0$, if the inequality

$$\mathscr{H}_{\mathscr{M}}(\mathcal{F}(\mathfrak{j}),\mathcal{F}(\mathfrak{s}))\le\gamma\mathscr{M}_1(\mathfrak{j},\mathfrak{s})$$

is provided, then the mapping $\mathcal{F}$ is entitled to a multi-valued Lipschitzian.

The following lemmas are essential for multi-valued mappings in $\mathfrak{mms}$.

Lemma 1.15. [26] *Let $(\mathcal{Q},\mathscr{M})$ be an $\mathfrak{mms}$ and $\mathcal{Y}$ be a nonvoid subset of $\mathcal{Q}_{\mathscr{M}}$. Assume that $\mathcal{R},\mathcal{S}\in\mathcal{C}\mathfrak{B}(\mathcal{Y})$. For each $\varepsilon>0$ and $\mathfrak{j}\in\mathcal{R}$, an element $\mathfrak{s}$ exists in $\mathcal{S}$ such that*

$$\mathscr{M}_1(\mathfrak{j},\mathfrak{s})\leq\mathscr{H}_{\mathscr{M}}(\mathcal{R},\mathcal{S})+\varepsilon.$$

Furthermore, provided that $\mathcal{S}$ is $\mathscr{M}-$compact and $\mathscr{M}$ fulfills the Fatou property, then for any $\mathfrak{j}$ in $\mathcal{R}$, $\mathfrak{s}\in\mathcal{S}$ comes into existence such that

$$\mathscr{M}_1(\mathfrak{j},\mathfrak{s})\leq\mathscr{H}_{\mathscr{M}}(\mathcal{R},\mathcal{S}).$$

Lemma 1.16. [26] *Let $(\mathcal{Q},\mathscr{M})$ be an $\mathfrak{mms}$ and $\mathcal{Y}$ be a nonvoid subset of $\mathcal{Q}_{\mathscr{M}}$. Presume that $\mathscr{M}$ admits the condition $(\mathcal{D})$, and $\mathcal{R}_{\mathfrak{z}}$ is a sequence of sets $\mathscr{C}\mathscr{B}(\mathcal{Y})$ provided that $\lim_{\mathfrak{z}\to\infty}\mathscr{H}_{\mathscr{M}}(\mathcal{R}_{\mathfrak{z}},\mathcal{R}_0)=0$, where $\mathcal{R}_0\in\mathscr{C}\mathscr{B}(\mathcal{Y})$. If $\mathfrak{j}_{\mathfrak{z}}\in\mathcal{R}_{\mathfrak{z}}$ and $\lim_{\mathfrak{z}\to\infty}\mathfrak{j}_{\mathfrak{z}}=\mathfrak{j}_0$, it follows that $\mathfrak{j}_0\in\mathcal{R}_0$.*

Also, to have more knowledge of $\mathfrak{mms}$, see [28]-[31].

In 2018, Ege and Alaca [32], contemplating modular metric and $b-$metric, identified a novel concept named the modular $b-$metric space, as pointed out below.

Definition 1.17. [32] Let $\zeta:(0,\infty)\times\mathcal{Q}\times\mathcal{Q}\to[0,\infty]$ be a mapping, where $\mathcal{Q}$ is a nonvoid set. The function ζ is mentioned as modular $b-$metric, if there exists $\hbar\in\mathbb{R}$ with $\hbar\geq1$, and also, for all $\kappa,\varsigma>0$ and $\mathfrak{j},\mathfrak{s},\mathfrak{r}\in\mathcal{Q}$, the axioms

(ζ_1) $\zeta_\kappa(\mathfrak{j},\mathfrak{s})=0\Leftrightarrow\mathfrak{j}=\mathfrak{s}$,

(ζ_2) $\zeta_\kappa(\mathfrak{j},\mathfrak{s})=\zeta_\kappa(\mathfrak{s},\mathfrak{j})$,

(ζ_3) $\zeta_{\kappa+\varsigma}(\mathfrak{j},\mathfrak{s})\leq\hbar\left[\zeta_\kappa(\mathfrak{j},\mathfrak{r})+\zeta_\varsigma(\mathfrak{r},\mathfrak{s})\right]$

are satisfied. In addition, the pair $(\mathcal{Q},\zeta)$ is a *modular $b-$metric space*, abbreviated as $\mathfrak{m}_{\flat}\mathfrak{ms}$.

Note that we can achieve the concept of $\mathfrak{mms}$ if we accept $\hbar=1$ in the above definition.

Whenever ζ is a modular $b-$metric, the set

$$\mathcal{Q}_\zeta=\left\{\mathfrak{s}\in\mathcal{Q}:\mathfrak{s}\overset{\zeta}{\sim}\mathfrak{j}\right\}$$

is entitled as a modular set on $\mathcal{Q}$ such that $\overset{\zeta}{\sim}$ is a binary relation described with $\mathfrak{j}\sim\mathfrak{s}\Leftrightarrow\lim_{\kappa\to\infty}\zeta_\kappa(\mathfrak{j},\mathfrak{s})=0$, for $\mathfrak{j},\mathfrak{s}\in\mathcal{Q}$.

Furthermore, the set

$$\mathcal{Q}_{\zeta}^{*}=\{\mathsf{j}\in\mathcal{Q}:\exists\kappa=\kappa(\mathsf{j})>0\text{ such that }\zeta_{\kappa}(\mathsf{j},\mathsf{j}_{0})<\infty\}\ \ (\mathsf{j}_{0}\in\mathcal{Q})$$

is a $\mathfrak{m}_{\flat}\mathfrak{ms}$ (around j_0).

The subsequent examples can be given to comprehend the concept of $\mathfrak{m}_{\flat}\mathfrak{ms}$.

Example 1.18. [32] Let us regard the space

$$\ell_{\mathsf{p}}=\left\{(\mathsf{j}_{j})\subset\mathbb{R}:\sum_{\mathfrak{z}=1}^{\infty}|\mathsf{j}_{\mathfrak{z}}|^{\mathsf{p}}<\infty\right\},\quad 0<\mathsf{p}<1.$$

For $\kappa\in(0,\infty)$, if we specify $\zeta_{\kappa}(\mathsf{j},\mathfrak{s})=\frac{\mathsf{d}(\mathsf{j},\mathfrak{s})}{\kappa}$ that

$$\mathsf{d}(\mathsf{j},\mathfrak{s})=\left(\sum_{\mathfrak{z}=1}^{\infty}|\mathsf{j}_{\mathfrak{z}}-\mathfrak{s}_{\mathfrak{z}}|^{\mathsf{p}}\right)^{\frac{1}{\mathsf{p}}},\quad \mathsf{j}=\mathsf{j}_{\mathfrak{z}},\ \mathfrak{s}=\mathfrak{s}_{\mathfrak{z}}\in\ell_{\mathsf{p}}$$

then the pair $(\mathcal{Q},\zeta)$ is an $\mathfrak{m}_{\flat}\mathfrak{ms}$.

Example 1.19. [33] Let $(\mathcal{Q},\mathscr{M})$ be an $\mathfrak{mms}$ and $\tau\geq 1$ with $\tau\in\mathbb{R}$. Let $\zeta_{\kappa}(\mathsf{j},\mathfrak{s})=(\mathscr{M}_{\kappa}(\mathsf{j},\mathfrak{s}))^{\tau}$. Using the convexity of the function $\mathcal{F}(\iota)=\iota^{\tau}$ for $\iota\geq 0$, and Jensen inequality, we get

$$(\omega+\upsilon)^{\tau}\leq 2^{\tau-1}(\omega^{\tau}+\upsilon^{\tau})$$

for $\omega,\upsilon\geq 0$. Thus, $(\mathcal{Q},\zeta)$ is an $\mathfrak{m}_{\flat}\mathfrak{ms}$ with $\hbar=2^{\tau-1}$.

Some fundamental topological properties in $\mathfrak{m}_{\flat}\mathfrak{ms}$ can be defined as in $\mathfrak{mms}$. Also, all of the properties of $\mathfrak{mms}$ are valid in $\mathfrak{m}_{\flat}\mathfrak{ms}$.

2 Some Fixed Point Results for Multi-Valued Mappings

This section proposes a novel idea, extending the $(\alpha,\beta)-$admissible mappings. Then, by using this construction, a common fixed point theorem has been verified in the sense of $\mathfrak{m}_{\flat}\mathfrak{ms}$.

Initially, the following notion is essential for the outcomes of this part.

Choudhury et al. [34] have extended the concept of cyclic $(\alpha,\beta)-$admissible mapping to a multi-valued version, as noted below.

Definition 2.1. [34] Let $\mathcal{Q}$ be a nonvoid set, $\mathcal{F}:\mathcal{Q}\to\mathcal{P}(\mathcal{Q})$ be a multi-valued mapping, and $\alpha,\beta:\mathcal{Q}\to[0,\infty)$ be two functions. Then, $\mathcal{F}$ is a multi-valued cyclic $(\alpha,\beta)-$admissible mapping if for $\mathsf{j},\mathfrak{s}\in\mathcal{Q}$, (*i*) $\alpha(\mathfrak{j}) \geq 1 \;\Rightarrow\; \beta(u) \geq 1$ for all $u \in \mathcal{F}\mathfrak{j}$,

(*ii*) $\beta(\mathfrak{s}) \geq 1 \;\Rightarrow\; \alpha(v) \geq 1$ for all $v \in \mathcal{F}\mathfrak{s}$.

The following concept can be easily defined.

Definition 2.2. Let $\mathcal{F}, \mathcal{S} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ be multi-valued mappings, where $\mathcal{Q}$ is a nonvoid set, and $\alpha, \beta : \mathcal{Q} \to [0,\infty)$ be two functions. Then, $(\mathcal{F}, \mathcal{S})$ is a multi-valued cyclic $(\alpha,\beta)-$admissible pair if for $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$,

(*i*) $\alpha(\mathfrak{j}) \geq 1 \;\Rightarrow\; \beta(u) \geq 1$ for all $u \in \mathcal{F}\mathfrak{j}$,

(*ii*) $\beta(\mathfrak{s}) \geq 1 \;\Rightarrow\; \alpha(v) \geq 1$ for all $v \in \mathcal{S}\mathfrak{s}$.

We are ready to present an extension of Definition 2.1 and 2.2.

Definition 2.3. Let $\mathcal{F}, \mathcal{S} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ be multi-valued mappings, where $\mathcal{Q}$ is a nonvoid set, and $\alpha, \beta : \mathcal{Q} \to [0,\infty)$ be two functions. Also, for $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}$, positive integers $n = n(\mathfrak{j})$ and $m = m(\mathfrak{s})$ exits. We contemplate the following circumstances.

$(\alpha\beta_1)$ $\alpha(\mathfrak{j}) \geq 1$ for some $\mathfrak{j} \in \mathcal{Q}$ implies $\beta(u) \geq 1$ for all $u \in \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}$.

$(\alpha\beta_2)$ $\beta(\mathfrak{j}) \geq 1$ for some $\mathfrak{j} \in \mathcal{Q}$ implies $\alpha(v) \geq 1$ for all $v \in \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}$.

$(\alpha\beta_3)$ $\beta(\mathfrak{s}) \geq 1$ for some $\mathfrak{s} \in \mathcal{Q}$ implies $\alpha(v) \geq 1$ for all $v \in \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}$.

Taking into account the function $(\alpha\beta_i)$, we assert that

- $i = 1, 2$, $\mathcal{F}$ is a multi-valued cyclic $(\alpha,\beta) - n-$admissible mapping.
- $i = 1, 3$, $(\mathcal{F}, \mathcal{S})$ is a multi-valued cyclic $(\alpha,\beta) - (n,m)-$admissible pair.

Remark 2. Let us consider $n = n(\mathfrak{j}) = 1$ in the above definition; then we obtain the definition of multi-valued cyclic $(\alpha,\beta)-$admissible mapping defined by [34] and, in case of $n = m = m(\mathfrak{j}) = 1$, multi-valued cyclic $(\alpha,\beta)-$admissible pairs.

Definition 2.4. Let $(\mathcal{Q}, \zeta)$ be an $\mathfrak{m}_\mathfrak{b}\mathfrak{ms}$ with $\hbar \geq 1$, $\mathcal{Y}$ be a nonempty bounded subset of $\mathcal{Q}_\zeta$, and $\alpha, \beta : \mathcal{Q}_\zeta \to \mathbb{R}_+$ be two functions. Two multi-valued mappings $\mathcal{F}, \mathcal{S} : \mathcal{Y} \to \mathscr{CB}(\mathcal{Y})$ are called multi-valued Sehgal-Proinov-type $(\alpha,\beta) - (n,m)-$contraction if there exist $\Sigma, \Omega : (0,\infty) \to \mathbb{R}$ such that for each $\mathfrak{j}, \mathfrak{s} \in \mathcal{Y}$, there exist $n(\mathfrak{j}), m(\mathfrak{s}) \in \mathbb{Z}^+$ such that

$$\alpha(\mathfrak{j}).\beta(\mathfrak{s}) \geq 1 \;\Rightarrow\; \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right)\right) \leq \Omega\left(\mathcal{C}(\mathfrak{j},\mathfrak{s})\right), \tag{3}$$

where

$$\mathcal{C}(\mathfrak{j},\mathfrak{s}) = \max\left\{ \begin{array}{c} \zeta_1(\mathfrak{j},\mathfrak{s}), \delta_1\left(\mathfrak{j}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right), \delta_1\left(\mathfrak{s}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right), \\ \frac{\delta_2\left(\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) + \delta_2\left(\mathfrak{s}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right)}{2\hbar} \end{array} \right\},$$

for all $\mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) > 0$.

Theorem 2.5. *Let* $(\mathcal{Q},\zeta)$ *be a* $\zeta-$*complete* $\mathfrak{m}_{\mathfrak{b}}\mathfrak{ms}$ *with* $\hbar \geq 1$ *and* ζ *be a convex regular modular which fulfills the Fatou property and* Δ_2-*condition. Let* $\mathcal{Y}$ *be a nonempty* $\zeta-$*complete subset of* $\mathcal{Q}_\zeta$*, and* $\mathcal{F},\mathcal{S}:\mathcal{Y}\to\mathscr{K}(\mathcal{Y})$ *be multi-valued Sehgal-Proinov-type* $(\alpha,\beta)-(n,m)-$*contraction mappings. If the circumstances*

(i) *there exist* $\mathfrak{j}_0\in\mathcal{Y}$ *such that* $\alpha(\mathfrak{j}_0)\geq 1$,

(ii) $(\mathcal{F},\mathcal{S})$ *is a multi-valued cyclic* $(\alpha,\beta)-(n,m)-$*admissible pair,*

(iii_a) $\mathcal{F}$ *or* $\mathcal{S}$ *is* $\zeta-$*continuous, or*

(iii_b) *if* $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ *is a sequence in* $\mathcal{Y}$ *such that* $\mathfrak{j}_\mathfrak{z}\to\mathfrak{j}$ *and* $\alpha(\mathfrak{j}_{2\mathfrak{z}})\geq 1$, $\beta(\mathfrak{j}_{2\mathfrak{z}-1})\geq 1$ *for all* $\mathfrak{z}\in\mathbb{N}$*, then* $\alpha(\mathfrak{j})\geq 1$ *and* $\beta(\mathfrak{j})\geq 1$,

(iv) Σ *is non-decreasing and* $\Omega(\mathfrak{a})<\Sigma(\mathfrak{a})$ *for all* $\mathfrak{a}>0$,

(v) $\limsup_{\mathfrak{a}\to\mathfrak{a}_0+}\Omega(\mathfrak{a})<\Sigma(\mathfrak{a}_0+)$ *for any* $\mathfrak{a}_0>0$

are provided, $\mathcal{F}$ *and* $\mathcal{S}$ *own exactly one common fixed point* x^* *in* $\mathcal{Y}\subseteq\mathcal{Q}_\zeta$*, where* $\zeta_1(\mathfrak{j}_0,\mathfrak{j}_1)<\infty$ *for some* $\mathfrak{j}_0,\mathfrak{j}_1\in\mathcal{Q}_\zeta$*. Additionally, if* $\alpha(\mathfrak{j})\beta(\mathfrak{s})\geq 1$ *for all* $\mathfrak{j},\mathfrak{s}\in M^C_{Fix}(\mathcal{F}^{n(\mathfrak{j}^*)},\mathcal{S}^{m(\mathfrak{j}^*)})$*, then the set* $M^C_{Fix}(\mathcal{F}^{n(\mathfrak{j}^*)},\mathcal{S}^{m(\mathfrak{j}^*)})$ *has a exactly one element. Moreover, if* $\alpha(\mathcal{F}\mathfrak{j}^*)\beta(\mathfrak{j}^*)\geq 1$ *and* $\alpha(\mathfrak{j}^*)\beta(\mathcal{S}\mathfrak{j}^*)\geq 1$*, then* $M^C_{Fix}(\mathcal{F},\mathcal{S})=\{\mathfrak{j}^*\}$.

Proof. Let $\mathfrak{j}_0\in\mathcal{Y}$ be a point mentioned in condition (i) such that $\alpha(\mathfrak{j}_0)\geq 1$. From the fact that $(\mathcal{F},\mathcal{S})$ is a multi-valued cyclic $(\alpha,\beta)-(n,m)-$admissible pair and by choosing $\mathfrak{j}_1\in\mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0$, we get

$$\alpha(\mathfrak{j}_0)\geq 1 \;\Rightarrow\; \beta\left(\mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0\right)=\beta(\mathfrak{j}_1)\geq 1,$$

and so, there exists $\mathfrak{j}_2\in\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1$ such that

$$\beta(\mathfrak{j}_1)\geq 1 \;\Rightarrow\; \alpha\left(\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1\right)=\alpha(\mathfrak{j}_2)\geq 1.$$

Thereby, we gain that $\alpha(\mathfrak{j}_0)\beta(\mathfrak{j}_1)\geq 1$ such that

$$\zeta_1(\mathfrak{j}_1,\mathfrak{j}_2)\leq\mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0,\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1\right)\leq\hbar^3\mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0,\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1\right),$$

and in this way, by using the assumption of (iv) and the inequality (3), we attain

$$\begin{aligned}\Sigma(\zeta_1(\mathfrak{j}_1,\mathfrak{j}_2))\leq\Sigma\left(\hbar^3\mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0,\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1\right)\right)&\leq\Omega(\mathcal{C}(\mathfrak{j}_0,\mathfrak{j}_1))\\&<\Sigma(\mathcal{C}(\mathfrak{j}_0,\mathfrak{j}_1)).\end{aligned}$$

On the other hand, we have the point $\mathrm{j}_3 \in \mathcal{F}^{n(\mathrm{j}_2)}\mathrm{j}_2$ such that

$$\alpha(\mathrm{j}_2) \geq 1 \Rightarrow \beta\left(\mathcal{F}^{n(\mathrm{j}_2)}\mathrm{j}_2\right) = \beta(\mathrm{j}_3) \geq 1,$$

that is, we acquire $\alpha(\mathrm{j}_2)\beta(\mathrm{j}_1) \geq 1$ which implies

$$\Sigma(\zeta_1(\mathrm{j}_3, \mathrm{j}_2)) \leq \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathrm{j}_2)}\mathrm{j}_2, \mathcal{S}^{m(\mathrm{j}_1)}\mathrm{j}_1\right)\right) \leq \Omega(\mathcal{C}(\mathrm{j}_2, \mathrm{j}_1))$$
$$\leq \Sigma(\mathcal{C}(\mathrm{j}_2, \mathrm{j}_1)).$$

Likewise, it follows that $\alpha(\mathrm{j}_2)\beta(\mathrm{j}_3) \geq 1$ and

$$\Sigma(\zeta_1(\mathrm{j}_3, \mathrm{j}_4)) \leq \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathrm{j}_2)}\mathrm{j}_2, \mathcal{S}^{m(\mathrm{j}_3)}\mathrm{j}_3\right)\right) \leq \Omega(\mathcal{C}(\mathrm{j}_2, \mathrm{j}_3))$$
$$\leq \Sigma(\mathcal{C}(\mathrm{j}_2, \mathrm{j}_3)).$$

Consequently, repeating this procedure, we set up a sequence $\{\mathrm{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ with the initial point j_0 such that

$$\mathrm{j}_{2\mathfrak{z}+1} \in \mathcal{F}^{n(\mathrm{j}_{2\mathfrak{z}})}\mathrm{j}_{2\mathfrak{z}} \text{ and } \mathrm{j}_{2\mathfrak{z}+2} \in \mathcal{S}^{m(\mathrm{j}_{2\mathfrak{z}+1})}\mathrm{j}_{2\mathfrak{z}+1},$$

or, if we use the notation $n_\mathfrak{z} = n(\mathrm{j}_{2\mathfrak{z}})$ and $m_\mathfrak{z} = m(\mathrm{j}_{2\mathfrak{z}+1})$, we can again write

$$\mathrm{j}_{2\mathfrak{z}+1} \in \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}} \text{ and } \mathrm{j}_{2\mathfrak{z}+2} \in \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}.$$

If we presume $\mathrm{j}_{\mathfrak{z}_0} = \mathrm{j}_{\mathfrak{z}_0+1}$, for some $\mathfrak{z}_0 \in \mathbb{N}$, then $\mathrm{j}_{\mathfrak{z}_0} \in M^C_{Fix}(\mathcal{F}, \mathcal{S})$. Hence, we consider $\mathrm{j}_\mathfrak{z} \neq \mathrm{j}_{\mathfrak{z}+1}$ for each $\mathfrak{z} \in \mathbb{N}$. Moreover, we procure that $\alpha(\mathrm{j}_{2\mathfrak{z}}) \geq 1$ and $\beta(\mathrm{j}_{2\mathfrak{z}+1}) \geq 1$ for all $\mathfrak{z} \in \mathbb{N}$. Thereupon, we achieve $\alpha(\mathrm{j}_{2\mathfrak{z}})\beta(\mathrm{j}_{2\mathfrak{z}+1}) \geq 1$ and, by using the (3), we write

$$\Sigma(\zeta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2})) \leq \Sigma\left(\hbar^3 \mathscr{H}_\zeta(\mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1})\right) \leq \Omega(\mathcal{C}(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1})), \quad (4)$$

where

$$\mathcal{C}(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}) = \max\left\{\begin{array}{c} \zeta_1(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}), \delta_1(\mathrm{j}_{2\mathfrak{z}}, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}), \delta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}), \\ \frac{\delta_2(\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}) + \delta_2(\mathrm{j}_{2\mathfrak{z}+1}, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}})}{2\hbar} \end{array}\right\},$$

and also,

- $\delta_1(\mathrm{j}_{2\mathfrak{z}}, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}) = \inf_{\mathrm{j}_{2\mathfrak{z}+1} \in \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}} \{\zeta_1(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1})\} \leq \zeta_1(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1})$,
- $\delta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}) = \inf_{\mathrm{j}_{2\mathfrak{z}+2} \in \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}} \{\zeta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2})\} \leq \zeta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2})$,

- $\delta_2(\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}) = \inf_{\mathrm{j}_{2\mathfrak{z}+2}\in\mathcal{S}^{m_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}+1}} \{\zeta_2(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+2})\} \leq \zeta_2(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+2})$,
- $\delta_2(\mathrm{j}_{2\mathfrak{z}+1}, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}) = \inf_{\mathrm{j}_{2\mathfrak{z}+1}\in\mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}} \{\zeta_2(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+1})\} = 0$.

Accordingly, we gain

$$\mathcal{C}(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}) = \max\left\{\zeta_1(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}), \zeta_1(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2}), \frac{\zeta_2(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+2})}{2\hbar}\right\}.$$

We substitute $\zeta_1(\mathrm{j}_\mathfrak{z}, \mathrm{j}_{\mathfrak{z}+1})$ with $\vartheta_\mathfrak{z}$. Then, we get

$$\begin{aligned}\mathcal{C}(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}) &= \max\left\{\vartheta_{2\mathfrak{z}}, \vartheta_{2\mathfrak{z}+1}, \tfrac{\zeta_2(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+2})}{2\hbar}\right\}\\ &\leq \max\left\{\vartheta_{2\mathfrak{z}}, \vartheta_{2\mathfrak{z}+1}, \tfrac{\vartheta_{2\mathfrak{z}}+\vartheta_{2\mathfrak{z}+1}}{2}\right\}\\ &= \max\{\vartheta_{2\mathfrak{z}}, \vartheta_{2\mathfrak{z}+1}\}.\end{aligned}$$

Let us presume that $\max\{\vartheta_{2\mathfrak{z}}, \vartheta_{2\mathfrak{z}+1}\} = \vartheta_{2\mathfrak{z}+1}$. Thus, by hypothesis (iv), taking into account (4), we attain

$$\Sigma(\vartheta_{2\mathfrak{z}+1}) \leq \Sigma\left(\hbar^3\mathscr{H}_\zeta(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2})\right) \leq \Omega(\vartheta_{2\mathfrak{z}+1}) < \Sigma(\vartheta_{2\mathfrak{z}+1}).$$

However, a contradiction arises as Σ is a non-decreasing map. Then, $\max\{\vartheta_{2\mathfrak{z}}, \vartheta_{2\mathfrak{z}+1}\} = \vartheta_{2\mathfrak{z}}$ and in this case, we achieve

$$\Sigma(\vartheta_{2\mathfrak{z}+1}) \leq \Sigma\left(\hbar^3\mathscr{H}_\zeta(\mathrm{j}_{2\mathfrak{z}+1}, \mathrm{j}_{2\mathfrak{z}+2})\right) \leq \Omega(\vartheta_{2\mathfrak{z}}) < \Sigma(\vartheta_{2\mathfrak{z}}). \tag{5}$$

Eventually, we conclude that $\vartheta_{2\mathfrak{z}+1} < \vartheta_{2\mathfrak{z}}$. By a similar step, one can deduce that $\vartheta_{2\mathfrak{z}} < \vartheta_{2\mathfrak{z}-1}$. Thereupon, it is ensured that the sequence $\{\vartheta_\mathfrak{z}\} = \{\zeta_1(\mathrm{j}_\mathfrak{z}, \mathrm{j}_{\mathfrak{z}+1})\}$ is positively decreasing. Thus, the equality $\lim_{\mathfrak{z}\to\infty} \vartheta_\mathfrak{z} = h+$ is provided for $h \geq 0$. Now, we aim to achieve that $h = 0$. Opposite this one, we presume $h > 0$. Then, by (5), we get

$$\Sigma(h+) = \lim_{\mathfrak{z}\to\infty} \Sigma(\vartheta_{2\mathfrak{z}+1}) \leq \limsup_{\mathfrak{z}\to\infty} \Omega(\vartheta_{2\mathfrak{z}}) \leq \limsup_{u\to h+} \Omega(u)$$

such that this contradicts to supposition (v). So, we gain

$$\lim_{\mathfrak{z}\to\infty} \zeta_1(\mathrm{j}_\mathfrak{z}, \mathrm{j}_{\mathfrak{z}+1}) = 0. \tag{6}$$

Now, we need to demonstrate that $\{\mathrm{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a $\zeta-$Cauchy sequence. It is sufficient to indicate $\{\mathrm{j}_{2\mathfrak{z}}\}$ is a $\zeta-$Cauchy sequence. Unlike our assertion, considering $\{\mathfrak{j}_{2\mathfrak{z}}\}$ is not a $\zeta-$Cauchy sequence, then for $\varepsilon > 0$, we constitute two subsequences $\left\{\mathfrak{j}_{2\mathfrak{b}_q}\right\}$ and $\left\{\mathfrak{j}_{2\mathfrak{z}_q}\right\}$ of positive integers fulfilling $\mathfrak{z}_q > \mathfrak{b}_q > q$ such that $\mathfrak{z}_q$ is the smallest index for which

$$\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q}\right) \geq \varepsilon \quad \text{and} \quad \zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q-2}\right) < \varepsilon. \tag{7}$$

As $(\mathcal{F}, \mathcal{S})$ is a multi-valued cyclic $(\alpha, \beta) - (n, m)-$admissible pair and $\hbar \geq 1$, then, $\alpha\left(\mathfrak{j}_{2\mathfrak{b}_q}\right)\beta\left(\mathfrak{j}_{2\mathfrak{z}_q+1}\right) \geq 1$ which implies that

$$\begin{aligned}\Sigma\left(\hbar^3\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q+1}, \mathfrak{j}_{2\mathfrak{z}_q+2}\right)\right) &\leq \Sigma\left(\hbar^3\mathscr{H}_\zeta\left(\mathcal{F}^{n\left(\mathfrak{j}_{2\mathfrak{b}_q}\right)}\mathfrak{j}_{2\mathfrak{b}_q}, \mathcal{S}^{m\left(\mathfrak{j}_{2\mathfrak{z}_q+1}\right)}\mathfrak{j}_{2\mathfrak{z}_q+1}\right)\right) \\ &\leq \Omega\left(\mathcal{C}\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right)\right),\end{aligned} \tag{8}$$

where

$$\begin{aligned}&\mathcal{C}\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right) \\ &= \max\left\{\begin{array}{l}\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right), \delta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathcal{F}^{n\left(\mathfrak{j}_{2\mathfrak{b}_q}\right)}\mathfrak{j}_{2\mathfrak{b}_q}\right), \delta_1\left(\mathfrak{j}_{2\mathfrak{z}_q+1}, \mathcal{S}^{m\left(\mathfrak{j}_{2\mathfrak{z}_q+1}\right)}\mathfrak{j}_{2\mathfrak{z}_q+1}\right), \\ \frac{\delta_2\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathcal{S}^{m\left(\mathfrak{j}_{2\mathfrak{z}_q+1}\right)}\mathfrak{j}_{2\mathfrak{z}+1}\right) + \delta_2\left(\mathfrak{j}_{2\mathfrak{z}+1}, \mathcal{F}^{n\left(\mathfrak{j}_{2\mathfrak{b}_q}\right)}\mathfrak{j}_{2\mathfrak{b}_q}\right)}{2\hbar}\end{array}\right\} \\ &= \max\left\{\begin{array}{l}\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right), \zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{b}_q+1}\right), \zeta_1\left(\mathfrak{j}_{2\mathfrak{z}_q+1}, \mathfrak{j}_{2\mathfrak{z}_q+2}\right), \\ \frac{\zeta_2\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+2}\right) + \zeta_2\left(\mathfrak{j}_{2\mathfrak{z}_q+1}, \mathfrak{j}_{2\mathfrak{b}_q+1}\right)}{2\hbar}\end{array}\right\}.\end{aligned} \tag{9}$$

Considering (6),(7) and the modular inequality, we have

$$\begin{aligned}\varepsilon \leq \zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q}\right) &\leq \hbar\zeta_{\frac{1}{2}}\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{b}_q+1}\right) + \hbar^2\zeta_{\frac{1}{4}}\left(\mathfrak{j}_{2\mathfrak{b}_q+1}, \mathfrak{j}_{2\mathfrak{z}_q+2}\right) \\ &+ \hbar^3\zeta_{\frac{1}{8}}\left(\mathfrak{j}_{2\mathfrak{z}_q+2}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right) + \hbar^3\zeta_{\frac{1}{8}}\left(\mathfrak{j}_{2\mathfrak{z}_q+1}, \mathfrak{j}_{2\mathfrak{z}_q}\right)\end{aligned}$$

such that

$$\limsup_{q\to\infty}\zeta_{\frac{1}{4}}\left(\mathfrak{j}_{2\mathfrak{b}_q+1}, \mathfrak{j}_{2\mathfrak{z}_q+2}\right) \geq \frac{\varepsilon}{\hbar^2}. \tag{10}$$

Also likewise, we get

$$\begin{aligned}\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right) &\leq \hbar\zeta_{\frac{1}{2}}\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q-2}\right) + \hbar^2\zeta_{\frac{1}{4}}\left(\mathfrak{j}_{2\mathfrak{z}_q-2}, \mathfrak{j}_{2\mathfrak{z}_q-1}\right) \\ &+ \hbar^3\zeta_{\frac{1}{8}}\left(\mathfrak{j}_{2\mathfrak{z}_q-1}, \mathfrak{j}_{2\mathfrak{z}_q}\right) + \hbar^3\zeta_{\frac{1}{8}}\left(\mathfrak{j}_{2\mathfrak{z}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right)\end{aligned}$$

and by (6), we obtain

$$\limsup_{q\to\infty}\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q}, \mathfrak{j}_{2\mathfrak{z}_q+1}\right) \leq \hbar\varepsilon. \tag{11}$$

Moreover, note that

$$\zeta_2\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+2}\right)\le\hbar\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+1}\right)+\hbar\zeta_1\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathfrak{j}_{2\mathfrak{z}_q+2}\right),$$

$$\zeta_2\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathfrak{j}_{2\mathfrak{b}_q+1}\right)\le\hbar\zeta_1\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathfrak{j}_{2\mathfrak{b}_q}\right)+\hbar\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{b}_q+1}\right),$$

and by using (6) and (11), we can easily achieve

$$\limsup_{q\to\infty}\zeta_2\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+2}\right)=\limsup_{q\to\infty}\zeta_2\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathfrak{j}_{2\mathfrak{b}_q+1}\right)\le\hbar^2\varepsilon. \tag{12}$$

Taking into (11) and (12) account, the expression (9) turns into

$$\limsup_{q\to\infty}\mathcal{C}\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+1}\right)\le\max\left\{\hbar\varepsilon,0,0,\frac{\hbar^2\varepsilon+\hbar^2\varepsilon}{2\hbar}\right\}=\hbar\varepsilon. \tag{13}$$

Thereupon, by using (10) and (13), taking the limit superior in the inequality (8), we get

$$\begin{aligned}\Sigma\left(\hbar\varepsilon\right)\le\limsup_{q\to\infty}\Sigma\left(\hbar^3\zeta_1\left(\mathfrak{j}_{2\mathfrak{b}_q+1},\mathfrak{j}_{2\mathfrak{z}_q+2}\right)\right)&\le\limsup_{q\to\infty}\Omega\left(\mathcal{C}\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+1}\right)\right)\\&<\Sigma\left(\limsup_{q\to\infty}\mathcal{C}\left(\mathfrak{j}_{2\mathfrak{b}_q},\mathfrak{j}_{2\mathfrak{z}_q+1}\right)\right)\\&\le\Sigma\left(\hbar\varepsilon\right).\end{aligned}$$

Nevertheless, it is a contradiction. Thereby, we say $\{\mathfrak{j}_{2\mathfrak{z}}\}$ is a $\zeta-$Cauchy sequence, also $\{\mathfrak{j}_{\mathfrak{z}}\}$ is a $\zeta-$Cauchy sequence on $\zeta-$complete $\mathfrak{m}_\flat\mathfrak{ms}$. Then, a point $\mathfrak{j}^*\in\mathcal{Y}$ exits such that

$$\lim_{\mathfrak{z}\to\infty}\zeta_1\left(\mathfrak{j}_{\mathfrak{z}},\mathfrak{j}^*\right)=0. \tag{14}$$

Let $\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}_{2\mathfrak{z}}$ be a sequence in $\mathscr{CB}(\mathcal{Y})$. Owing to the fact that the mapping S is $\zeta-$continuous, we have $\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}_{2\mathfrak{z}}\to\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*$, and so $\lim_{\mathfrak{z}\to\infty}\mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}_{2\mathfrak{z}},\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\right)=0$, where $\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\in\mathscr{CB}(\mathcal{Y})$. If $\mathfrak{j}_{2\mathfrak{z}+1}\in\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}_{2\mathfrak{z}}$ and $\lim_{\mathfrak{z}\to\infty}\mathfrak{j}_{2\mathfrak{z}+1}=\mathfrak{j}^*$, then, considering Lemma 1.16, we conclude that $\mathfrak{j}^*\in\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*$, that is, $\mathfrak{j}^*$ is a fixed point of $\mathcal{F}^{n(\mathfrak{j}^*)}$. Similarly, one can achieve that $\mathfrak{j}^*\in\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*$.

On the other hand, if we assume the condition (iii_b) is satisfied, then we have $\beta\left(\mathfrak{j}^*\right)\ge 1$, and so, it follows that $\alpha\left(\mathfrak{j}_{2\mathfrak{z}}\right)\beta\left(\mathfrak{j}^*\right)\ge 1$. Moreover, to show the existence of a fixed point, we presume that $\mathfrak{j}^*\notin M_{Fix}\left(\mathcal{S}^{m(\mathfrak{j}^*)}\right)$. Because $\mathscr{K}\left(\mathcal{Y}\right)$ is compact, there exists a $\mathfrak{j}^*\in\mathscr{K}\left(\mathcal{Y}\right)\subseteq\mathcal{Q}_\zeta$ such that $\mathfrak{j}_{\mathfrak{z}}\to\mathfrak{j}^*$. Then, from the Fatou property, we get

$$\delta_1\left(\mathrm{j}^*, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right) \leq \liminf_{\mathfrak{z}\to\infty} \zeta_1\left(\mathrm{j}_{2\mathfrak{z}+1}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right) = \liminf_{\mathfrak{z}\to\infty} \zeta_1\left(\mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)$$
$$\leq \hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right),$$

and because Σ is a non-decreasing map, we have

$$\Sigma\left(\delta_1\left(\mathrm{j}^*, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)\right) \leq \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)\right) \leq \Omega\left(\mathcal{C}\left(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}^*\right)\right), \quad (15)$$

where

$$\mathcal{C}\left(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}^*\right) = \max\left\{\begin{array}{l} \zeta_1\left(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}^*\right), \delta_1\left(\mathrm{j}_{2\mathfrak{z}}, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}\right), \delta_1\left(\mathrm{j}_{2\mathfrak{z}_q+1}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right), \\ \frac{\delta_2\left(\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)+\delta_2\left(\mathrm{j}^*, \mathcal{F}^{n_\mathfrak{z}}\mathrm{j}_{2\mathfrak{z}}\right)}{2\hbar} \end{array}\right\}$$
$$= \max\left\{\begin{array}{l} \zeta_1\left(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}^*\right), \delta_1\left(\mathrm{j}_{2\mathfrak{z}}, \mathrm{j}_{2\mathfrak{z}+1}\right), \delta_1\left(\mathrm{j}_{2\mathfrak{z}_q+1}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right), \\ \frac{\delta_2\left(\mathrm{j}_{2\mathfrak{z}}, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)+\delta_2\left(\mathrm{j}^*, \mathrm{j}_{2\mathfrak{z}+1}\right)}{2\hbar} \end{array}\right\}. \quad (16)$$

Now, taking the limit in (15) and (16) and employing (14) and (iv), we obtain

$$\Sigma\left(\delta_1\left(\mathrm{j}^*, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)\right) \leq \Omega\left(\delta_1\left(\mathrm{j}^*, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)\right) < \Sigma\left(\delta_1\left(\mathrm{j}^*, \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*\right)\right),$$

which causes a contradiction. Thereby, we achieve $\mathrm{j}^* \in \mathcal{S}^{m(\mathrm{j}^*)}\mathrm{j}^*$, that is, $\mathrm{j}^* \in M_{Fix}\left(\mathcal{S}^{m(\mathrm{j}^*)}\right)$. In a similar way, one can show $\mathrm{j}^* \in M_{Fix}\left(\mathcal{F}^{n(\mathrm{j}^*)}\right)$.

For the uniqueness of the fixed point, we presume that there exists $z^* \in M^C_{Fix}\left(\mathcal{F}^{n(z^*)}, \mathcal{S}^{m(z^*)}\right)$ such that $\mathrm{j}^* \neq z^*$. From the hypothesis, we gain $\alpha\left(\mathrm{j}^*\right)\beta\left(z^*\right) \geq 1$. Thereupon, we acquire

$$\Sigma\left(\zeta_1\left(\mathrm{j}^*, z^*\right)\right) \leq \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathrm{j}^*)}\mathrm{j}^*, \mathcal{S}^{m(z^*)}z^*\right)\right) \leq \Omega\left(\mathcal{C}\left(\mathrm{j}^*, z^*\right)\right),$$

where

$$\mathcal{C}\left(\mathrm{j}^*, z^*\right) = \max\left\{\begin{array}{l} \zeta_1\left(\mathrm{j}^*, z^*\right), \delta_1\left(\mathrm{j}^*, \mathcal{F}^{n(\mathrm{j}^*)}\mathrm{j}^*\right), \delta_1\left(z^*, \mathcal{S}^{m(z^*)}z^*\right), \\ \frac{\delta_2\left(\mathrm{j}^*, \mathcal{S}^{m(z^*)}z^*\right)+\delta_2\left(z^*, \mathcal{F}^{n(\mathrm{j}^*)}\mathrm{j}^*\right)}{2\hbar} \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{l} \zeta_1\left(\mathrm{j}^*, z^*\right), \delta_1\left(\mathrm{j}^*, \mathrm{j}^*\right), \delta_1\left(z^*, z^*\right), \\ \frac{\delta_2\left(\mathrm{j}^*, z^*\right)+\delta_2\left(z^*, \mathrm{j}^*\right)}{2\hbar} \end{array}\right\} = \zeta_1\left(\mathrm{j}^*, z^*\right).$$

Using (iv), we conclude

$$\Sigma\left(\zeta_1\left(\mathfrak{j}^*, z^*\right)\right) \leq \Omega\left(\zeta_1\left(\mathfrak{j}^*, z^*\right)\right) < \Sigma\left(\zeta_1\left(\mathfrak{j}^*, z^*\right)\right),$$

which means $\mathfrak{j}^* = z^*$. Lastly, we demonstrate that $\mathfrak{j}^* \in S\mathfrak{j}^*$ and $\mathfrak{j}^* \in T\mathfrak{j}^*$. Conversely, we presume that $\mathfrak{j}^* \notin S\mathfrak{j}^*$. Hence, considering the uniqueness of the set $M_{Fix}^{C}\left(\mathcal{F}^{n(\mathfrak{j}^*)}, \mathcal{S}^{m(\mathfrak{j}^*)}\right)$ and $\alpha\left(\mathcal{F}\mathfrak{j}^*\right)\beta\left(\mathfrak{j}^*\right) \geq 1$, we have

$$\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right) \leq \hbar^3 \mathscr{H}_{\mathscr{M}}\left(S\left(\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\right), \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right) \leq \hbar^3 \mathscr{H}_{\mathscr{M}}\left(\mathcal{F}^{n(\mathfrak{j}^*)}\left(\mathcal{F}\mathfrak{j}^*\right), \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right),$$

for $\mathfrak{j}^* \in \mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*$and $\mathfrak{j}^* \in \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*$. Thus, from the properties of the functions Σ and (3), it follows that

$$\Sigma\left(\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)\right) \leq \Sigma\left(\hbar^3 \mathscr{H}_{\mathscr{M}}\left(\mathcal{F}^{n(\mathfrak{j}^*)}\left(\mathcal{F}\mathfrak{j}^*\right), \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right) \leq \Omega\left(\mathcal{C}\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)\right), \quad (17)$$

where

$$\mathcal{C}\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right) = \max\left\{\begin{array}{c} \delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right), \delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathcal{F}^{n(\mathfrak{j}^*)}\left(\mathcal{F}\mathfrak{j}^*\right)\right), \delta_1\left(\mathfrak{j}^*, \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right), \\ \frac{\delta_2\left(\mathcal{F}\mathfrak{j}^*, \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right) + \delta_2\left(\mathfrak{j}^*, \mathcal{F}^{n(\mathfrak{j}^*)}\left(\mathcal{F}\mathfrak{j}^*\right)\right)}{2\hbar} \end{array}\right\}$$

$$\leq \max\left\{\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right), \frac{\delta_2\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)}{s}\right\} = \delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right).$$

Therefore, by using assumption (iv), the inequality (17) turns into

$$\Sigma\left(\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)\right) \leq \Omega\left(\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)\right) < \Sigma\left(\delta_1\left(\mathcal{F}\mathfrak{j}^*, \mathfrak{j}^*\right)\right),$$

which is a contradiction. Then, we achieve $\mathfrak{j}^* \in S\mathfrak{j}^*$. Likewise, if we presume that $\alpha\left(\mathfrak{j}^*\right)\beta\left(\mathcal{S}\mathfrak{j}^*\right) \geq 1$, then $\mathfrak{j}^* \in \mathcal{S}\mathfrak{j}^*$. We get $M_{Fix}^{C}\left(\mathcal{F}, \mathcal{S}\right) = \{\mathfrak{j}^*\}$. So, the proof is accomplished. □

We have the next outcomes by applying $\mathcal{F}$ equals to $\mathcal{S}$ and $m\left(\mathfrak{s}\right) = n\left(\mathfrak{s}\right)$ in the above.

Definition 2.6. Let $(\mathcal{Q}, \zeta)$ be a $\mathfrak{m}_{\flat}\mathfrak{ms}$ with $\hbar \geq 1$, $\mathcal{Y}$ be a nonempty bounded subset of $\mathcal{Q}_\zeta$, and $\alpha, \beta : \mathcal{Q}_\zeta \to \mathbb{R}_+$ be two functions. A multi-valued mapping $\mathcal{F} : \mathcal{Y} \to \mathscr{CB}(\mathcal{Y})$ is called multi-valued Sehgal-Proinov-type $(\alpha, \beta)-n-$contraction if $\Sigma, \Omega : (0, \infty) \to \mathbb{R}$ exist such that for each $\mathfrak{j}, \mathfrak{s} \in \mathcal{Y}$, there exists $n\left(\mathfrak{j}\right) \in \mathbb{Z}^+$ such that

$$\alpha\left(\mathfrak{j}\right).\beta\left(\mathfrak{s}\right) \geq 1 \ \Rightarrow \ \Sigma\left(\hbar^3 \mathscr{H}_\zeta\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{F}^{n(\mathfrak{s})}\mathfrak{s}\right)\right) \leq \Omega\left(\mathcal{C}\left(\mathfrak{j}, \mathfrak{s}\right)\right), \quad (18)$$

where

$$\mathcal{C}(\mathfrak{j},\mathfrak{s}) = \max\left\{ \begin{array}{c} \zeta_1(\mathfrak{j},\mathfrak{s}), \delta_1\left(\mathfrak{j},\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right), \delta_1\left(\mathfrak{s},\mathcal{F}^{n(\mathfrak{s})}\mathfrak{s}\right), \\ \frac{\delta_2\left(\mathfrak{j},\mathcal{F}^{n(\mathfrak{j})}\mathfrak{s}\right)+\delta_2\left(\mathfrak{s},\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right)}{2\hbar} \end{array} \right\},$$

for all $\mathscr{H}_\zeta\left(\mathcal{S}^{n(\mathfrak{j})}\mathfrak{j},\mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) > 0$.

Corollary 2.7. *Let $(\mathcal{Q},\zeta)$ be a $\zeta-$complete $\mathfrak{m}_\flat\mathfrak{ms}$ with $\hbar \geq 1$. Assume that ζ is a convex regular modular with the Fatou property and Δ_2-condition. Let $\mathcal{Y}$ be a nonempty $\zeta-$complete subset of $\mathcal{Q}_\zeta$, and $\mathcal{F}: \mathcal{Y} \to \mathscr{K}(\mathcal{Y})$ be a multi-valued Sehgal-Proinov-type $(\alpha,\beta)-n-$contraction mapping. If the conditions*

- (i) *there exist $\mathfrak{j}_0 \in \mathcal{Y}$ such that $\alpha(\mathfrak{j}_0) \geq 1$,*
- (ii) *$\mathcal{F}$ is a multi-valued cyclic $(\alpha,\beta)-n-$admissible mapping,*
- (iii_a) *$\mathcal{F}$ is $\zeta-$continuous, or*
- (iii_b) *if $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a sequence in $\mathcal{Y}$ such that $\mathfrak{j}_\mathfrak{z} \to \mathfrak{j}$ and $\beta(\mathfrak{j}_\mathfrak{z}) \geq 1$ for all $\mathfrak{z} \in \mathbb{N}$, then $\beta(\mathfrak{j}) \geq 1$,*
- (iv) *Σ is non-decreasing and $\Omega(\mathfrak{a}) < \Sigma(\mathfrak{a})$ for all $\mathfrak{a} > 0$,*
- (v) $\limsup_{\mathfrak{a}\to\mathfrak{a}_0+} \Omega(\mathfrak{a}) < \Sigma(\mathfrak{a}_0+)$ *for any* $\mathfrak{a}_0 > 0$

are provided, $\mathcal{F}$ owns exactly one fixed point x^* in $\mathcal{Y} \subseteq \mathcal{Q}_\zeta$, where $\zeta_1(\mathfrak{j}_0,\mathfrak{j}_1) < \infty$ for some $\mathfrak{j}_0,\mathfrak{j}_1 \in \mathcal{Q}_\zeta$. Additionally, if $\alpha(\mathfrak{j})\beta(\mathfrak{s}) \geq 1$ for all $\mathfrak{j},\mathfrak{s} \in M_{Fix}(\mathcal{F}^{n(\mathfrak{j}^*)})$, then the set $M_{Fix}(\mathcal{F}^{n(\mathfrak{j}^*)})$ has exactly one element. Moreover, if $\alpha(\mathcal{F}\mathfrak{j}^*)\beta(\mathfrak{j}^*) \geq 1$ or $\alpha(\mathfrak{j}^*)\beta(\mathcal{F}\mathfrak{j}^*) \geq 1$, then $M_{Fix}(\mathcal{F},\mathcal{S}) = \{\mathfrak{j}^*\}$.

3 Some Fixed Point Results for Single-Valued Mappings

This section indicates some concepts that generalize conclusions commonly used in metric fixed point theory for single-valued mappings. Next, we set up a new common fixed point theorem by employing the newly defined construction in the setting of $\mathfrak{m}_\flat\mathfrak{ms}$. Also, this theorem can be considered a consequence of Theorem 2.5.

Primarily, we acquaint a novel extension of the notation of $(\alpha,\beta)-$admissible, as noted below.

Definition 3.1. Let $\mathcal{F},\mathcal{S}$ be two self-mappings on a nonempty set $\mathcal{Q}$, and $\alpha,\beta : \mathcal{Q} \to [0,\infty)$ be two functions. Also, for $\mathfrak{j},\mathfrak{s} \in \mathcal{Q}$, positive integers $n = n(\mathfrak{j})$ and $m = m(\mathfrak{s})$ exist. We contemplate the following circumstances.

$(\alpha\beta_1)$ $\alpha(\mathfrak{j}) \geq 1$ for some $\mathfrak{j} \in \mathcal{Q}$ implies $\beta\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right) \geq 1$.

$(\alpha\beta_2)$ $\beta(\mathfrak{j}) \geq 1$ for some $\mathfrak{j} \in \mathcal{Q}$ implies $\alpha\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right) \geq 1$.

$(\alpha\beta_3)$ $\beta(\mathfrak{j}) \geq 1$ for some $\mathfrak{j} \in \mathcal{Q}$ implies $\alpha\left(\mathcal{S}^{m(\mathfrak{j})}\mathfrak{j}\right) \geq 1$.

Considering the function $(\alpha\beta_i)$, we assert that

- $i = 1, 2$, $\mathcal{S}$ is a cyclic $(\alpha, \beta) - n-$admissible mapping.
- $i = 1, 3$, $(\mathcal{F}, \mathcal{S})$ is a cyclic $(\alpha, \beta) - (n, m)-$admissible pair.

Remark 3. Taking into account $n = n(\mathfrak{j}) = 1$ and $n = m = m(\mathfrak{j}) = 1$ in the above definitions, then we obtain the definitions of cyclic $(\alpha, \beta)-$admissible and cyclic $(\alpha, \beta)-$admissible pairs defined by Alizadeh et al. [16] and Latif et al. [17], respectively.

Definition 3.2. Let $\mathcal{Q}^*_\zeta$ be an $\mathfrak{m}_\flat\mathfrak{ms}$ with $\hbar \geq 1$, $\mathcal{F}, \mathcal{S} : \mathcal{Q}^*_\zeta \to \mathcal{Q}^*_\zeta$ be two self-mappings and $\alpha, \beta : \mathcal{Q}^*_\zeta \to \mathbb{R}_+$ be two functions. The pair $(\mathcal{F}, \mathcal{S})$ is called Sehgal-Proinov-type $(\alpha, \beta)-(n, m)-$contraction if there exist $\Sigma, \Omega : (0, \infty) \to \mathbb{R}$ such that for each $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}^*_\zeta$, there exist $n(\mathfrak{j}), m(\mathfrak{s}) \in \mathbb{Z}^+$ such that

$$\alpha(\mathfrak{j}) . \beta(\mathfrak{s}) \geq 1 \Rightarrow \Sigma\left(\hbar^3 \zeta_\kappa\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right)\right) \leq \Omega\left(\mathcal{C}^*(\mathfrak{j}, \mathfrak{s})\right), \tag{19}$$

where

$$\mathcal{C}^*(\mathfrak{j}, \mathfrak{s}) = \max\left\{\begin{array}{c} \zeta_\kappa(\mathfrak{j}, \mathfrak{s}), \zeta_\kappa\left(\mathfrak{j}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right), \zeta_\kappa\left(\mathfrak{s}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right), \\ \frac{\zeta_{2\kappa}\left(\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) + \zeta_{2\kappa}\left(\mathfrak{s}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right)}{2\hbar} \end{array}\right\}$$

for all $\zeta_\kappa\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{S}^{m(\mathfrak{s})}\mathfrak{s}\right) > 0$ and all $\kappa > 0$.

Theorem 3.3. *Let $\mathcal{Q}^*_\zeta$ be a $\zeta-$complete $\mathfrak{m}_\flat\mathfrak{ms}$ and the pair $(\mathcal{F}, \mathcal{S})$ be a Sehgal-Proinov-type $(\alpha, \beta) - (n, m)-$contraction. Presume the statements*

(*i*) *$(\mathcal{F}, \mathcal{S})$ is a cyclic $(\alpha, \beta) - (n, m)-$admissible pair,*

(*ii*) *there exist $\mathfrak{j}_0 \in \mathcal{Q}^*_\zeta$ such that $\alpha(\mathfrak{j}_0) \geq 1$,*

(*iii$_a$*) *$\mathcal{F}$ or $\mathcal{S}$ is $\zeta-$continuous, or*

(*iii$_b$*) *if $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a sequence in $\mathcal{Q}^*_\zeta$ such that $\mathfrak{j}_\mathfrak{z} \to \mathfrak{j}$ and $\alpha(\mathfrak{j}_{2\mathfrak{z}}) \geq 1$, $\beta(\mathfrak{j}_{2\mathfrak{z}-1}) \geq 1$, for all $\mathfrak{z} \in \mathbb{N}$, then $\alpha(\mathfrak{j}) \geq 1$ and $\beta(\mathfrak{j}) \geq 1$,*

(*iv*) *Σ is non-decreasing and $\Omega(\mathfrak{a}) < \Sigma(\mathfrak{a})$ for all $\mathfrak{a} > 0$,*

(v) $\limsup_{\mathfrak{a}\to\mathfrak{a}_0+} \Omega(\mathfrak{a}) < \Sigma(\mathfrak{a}_0+)$ *for any* $\mathfrak{a}_0 > 0$

are provided and there exists $\mathfrak{j}_0, \mathfrak{j}_1 \in \mathcal{Q}^*_\zeta$ *such that* $\zeta_\kappa(\mathfrak{j}_0, \mathfrak{j}_1) < \infty$ *for all* $\kappa > 0$. *If* $\alpha(\mathfrak{j})\beta(\mathfrak{s}) \geq 1$ *for all* $x, y \in C_{Fix}\left(\mathcal{F}^{n(\mathfrak{j}^*)}, \mathcal{S}^{m(\mathfrak{j}^*)}\right)$, *then the set* $C_{Fix}\left(\mathcal{F}^{n(\mathfrak{j}^*)}, \mathcal{S}^{m(\mathfrak{j}^*)}\right)$ *has exactly one element, which means that* $\mathcal{F}$ *and* $\mathcal{S}$ *own a unique common fixed point.*

Proof. Let $\mathfrak{j}_0 \in \mathcal{Q}^*_\zeta$ be an arbitrary point and, beginning from $\mathfrak{j}_0$, we set up a sequence $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ by

$$\mathfrak{j}_1 = \mathcal{F}^{n(\mathfrak{j}_0)}\mathfrak{j}_0, \mathfrak{j}_2 = \mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1, ..., \mathfrak{j}_{2\mathfrak{z}+1} = \mathcal{F}^{n(\mathfrak{j}_{2\mathfrak{z}})}\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+2} = \mathcal{S}^{m(\mathfrak{j}_{2\mathfrak{z}+1})}\mathfrak{j}_{2\mathfrak{z}+1}, ...$$

or if we denote $n_\mathfrak{z} = n(\mathfrak{j}_{2\mathfrak{z}})$ and $m_\mathfrak{z} = m(\mathfrak{j}_{2\mathfrak{z}+1})$, then we can write

$$\mathfrak{j}_{2\mathfrak{z}+1} = \mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}} \text{ and } \mathfrak{j}_{2\mathfrak{z}+2} = \mathcal{S}^{m_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}+1}.$$

Also, because $(\mathcal{F}, \mathcal{S})$ is a cyclic $(\alpha, \beta) - (n, m) -$admissible pair and $\alpha(\mathfrak{j}_0) \geq 1$, we have

$$\beta\left(\mathcal{S}^{n(\mathfrak{j}_0)}\mathfrak{j}_0\right) = \beta(\mathfrak{j}_1) \geq 1$$

which means that

$$\alpha\left(\mathcal{S}^{m(\mathfrak{j}_1)}\mathfrak{j}_1\right) = \alpha(\mathfrak{j}_2) \geq 1.$$

Thereupon, by pursuing this procedure, we achieve $\alpha(\mathfrak{j}_{2\mathfrak{z}}) \geq 1$ and $\beta(\mathfrak{j}_{2\mathfrak{z}+1}) \geq 1$, which entails $\alpha(\mathfrak{j}_{2\mathfrak{z}})\beta(\mathfrak{j}_{2\mathfrak{z}+1}) \geq 1$, for all $\mathfrak{z} \in \mathbb{N}$. Thus, from (19), we get

$$\Sigma\left(\hbar^3\zeta_\kappa\left(\mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}}, \mathcal{S}^{m_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}+1}\right)\right) \leq \Omega\left(\mathcal{C}^*(\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+1})\right),$$

where

$$\begin{aligned} &\mathcal{C}^*(\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+1}) \\ &= \max\left\{\begin{array}{l} \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+1}), \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}}, \mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}}), \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}+1}, \mathcal{S}^{m_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}+1}), \\ \frac{\zeta_{2\kappa}(\mathfrak{j}_{2\mathfrak{z}}, \mathcal{S}^{m_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}+1}) + \zeta_{2\kappa}(\mathfrak{j}_{2\mathfrak{z}+1}, \mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}})}{2\hbar} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{l} \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+1}), \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}+1}, \mathfrak{j}_{2\mathfrak{z}+2}), \\ \frac{\zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}}, \mathfrak{j}_{2\mathfrak{z}+1}) + \zeta_\kappa(\mathfrak{j}_{2\mathfrak{z}+1}, \mathfrak{j}_{2\mathfrak{z}+2})}{2} \end{array}\right\}. \end{aligned}$$

At this stage, if we proceed in the manner of the proof of Theorem 2.5, $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is yielded as a $\zeta-$Cauchy sequence on a $\zeta-$complete $\mathfrak{m}_\flat\mathfrak{ms}$, which expresses that a point $\mathfrak{j}^* \in \mathcal{Q}^*_\zeta$ exists such that

$$\lim_{\mathfrak{z}\to\infty} \zeta_1(\mathfrak{j}_\mathfrak{z}, \mathfrak{j}^*) = 0, \tag{20}$$

for all $\kappa > 0$.

We will explain that $\{\mathfrak{j}^*\} = C_{Fix}(\mathcal{F},\mathcal{S})$. For this, from (iii_a), if $\mathcal{F}$ is $\zeta-$continuous, then it is straightforward to realize that $\mathfrak{j}^* \in C_{Fix}(\mathcal{F},\mathcal{S})$. Therefore, assumption (iii_b) is fulfilled. Hence, we get $\alpha(\mathfrak{j}_{2\mathfrak{z}})\beta(\mathfrak{j}^*) \geq 1$ for all $\mathfrak{z} \in \mathbb{N}$. We presume that $\mathfrak{j}^* \neq \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*$. Then, from (19), we have

$$\Sigma\left(\zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}+1},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right) \leq \Sigma\left(\hbar^3\zeta_\kappa\left(\mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right) \leq \Omega\left(\mathcal{C}^*\left(\mathfrak{j}_{2\mathfrak{z}},\mathfrak{j}^*\right)\right), \tag{21}$$

where

$$\mathcal{C}^*\left(\mathfrak{j}_{2\mathfrak{z}},\mathfrak{j}^*\right) = \max\left\{\begin{array}{l} \zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}},\mathfrak{j}^*\right),\zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}},\mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}}\right),\zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right), \\ \frac{\zeta_{2\kappa}\left(\mathfrak{j}_{2\mathfrak{z}},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)+\zeta_{2\kappa}\left(\mathfrak{j}^*,\mathcal{F}^{n_\mathfrak{z}}\mathfrak{j}_{2\mathfrak{z}}\right)}{2\hbar} \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} \zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}},\mathfrak{j}^*\right),\zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}},\mathfrak{j}_{2\mathfrak{z}+1}\right),\zeta_\kappa\left(\mathfrak{j}_{2\mathfrak{z}_q+1},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right), \\ \frac{\zeta_{2\kappa}\left(\mathfrak{j}_{2\mathfrak{z}},\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)+\zeta_{2\kappa}\left(\mathfrak{j}^*,\mathfrak{j}_{2\mathfrak{z}+1}\right)}{2\hbar} \end{array}\right\}. \tag{22}$$

Thereby, letting $k \to \infty$ in (21) and (22) and using (20) and (iv), we gain

$$\Sigma\left(\zeta_\kappa\left(\mathfrak{j}^*,\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right) \leq \Omega\left(\zeta_\kappa\left(\mathfrak{j}^*,\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right) < \Sigma\left(\zeta_\kappa\left(\mathfrak{j}^*,\mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*\right)\right),$$

such that a contradiction arises. So, we achieve $\mathfrak{j}^* = \mathcal{S}^{m(\mathfrak{j}^*)}\mathfrak{j}^*$. Likewise, we conclude that $\mathfrak{j}^* = \mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*$. Thus, $\mathfrak{j}^* \in C_{Fix}\left(\mathcal{F}^{n(\mathfrak{j}^*)},\mathcal{S}^{m(\mathfrak{j}^*)}\right)$. Now, we prove $\{\mathfrak{j}^*\} = C_{Fix}\left(\mathcal{F}^{n(\mathfrak{j}^*)},\mathcal{S}^{m(\mathfrak{j}^*)}\right)$. On the contrary, a point z^* exits differ from $\mathfrak{j}^*$ such that

$$z^* = \mathcal{F}^{n(z^*)}z^* \text{ and } z^* = \mathcal{S}^{m(z^*)}z^*.$$

Also, from the assumption, we have $\alpha(\mathfrak{j}^*)\beta(z^*) \geq 1$. Thereupon, by (19)

$$\begin{aligned} \Sigma\left(\zeta_\kappa\left(\mathfrak{j}^*,z^*\right)\right) &\leq \Sigma\left(\hbar^3\zeta_\kappa\left(\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*,\mathcal{S}^{m(z^*)}z^*\right)\right) \leq \Omega\left(\mathcal{C}^*\left(\mathfrak{j}^*,z^*\right)\right) \\ &< \Sigma\left(\max\left\{\begin{array}{l} \zeta_\kappa\left(\mathfrak{j}^*,z^*\right),\zeta_\kappa\left(\mathfrak{j}^*,\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\right),\zeta_\kappa\left(z^*,\mathcal{S}^{m(z^*)}z^*\right), \\ \frac{\zeta_{2\kappa}\left(\mathfrak{j}^*,\mathcal{S}^{m(z^*)}z^*\right)+\zeta_{2\kappa}\left(z^*,\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\right)}{2\hbar} \end{array}\right\}\right) \\ &\leq \Sigma\left(\max\left\{\zeta_\kappa\left(\mathfrak{j}^*,z^*\right),\frac{\zeta_\kappa\left(\mathfrak{j}^*,z^*\right)}{2}\right\}\right) = \Sigma\left(\zeta_\kappa\left(\mathfrak{j}^*,z^*\right)\right) \end{aligned}$$

is obtained. This implies that $\{\mathfrak{j}^*\} = C_{Fix}\left(\mathcal{F}^{n(\mathfrak{j}^*)},\mathcal{S}^{m(\mathfrak{j}^*)}\right)$. On the other hand, $\mathcal{F}\mathfrak{j}^* = \mathcal{F}\left(\mathcal{F}^{n(\mathfrak{j}^*)}\mathfrak{j}^*\right) = \mathcal{F}^{n(\mathfrak{j}^*)}\left(S\mathfrak{j}^*\right)$ and from the uniqueness of $\{\mathfrak{j}^*\}$, we reason out $\mathfrak{j}^* = \mathcal{F}\mathfrak{j}^*$. Similarly, we get $\mathfrak{j}^* = \mathcal{S}\mathfrak{j}^*$. In conclusion, $\{\mathfrak{j}^*\} = C_{Fix}(\mathcal{F}, \mathcal{S})$, that is, $\mathcal{F}$ and $\mathcal{S}$ own a unique common fixed point. □

We possess the subsequent consequence, which is gained instantly from Theorem 3.3, on the condition that $\mathcal{F} = \mathcal{S}$ and $m(\mathfrak{s}) = n(\mathfrak{s})$.

Definition 3.4. Let $\mathcal{Q}_\zeta^*$ be an $\mathfrak{m}_\flat\mathfrak{ms}$ with $\hbar \geq 1$, $\mathcal{F} : \mathcal{Q}_\zeta^* \to \mathcal{Q}_\zeta^*$ be a self-mapping and $\alpha, \beta : \mathcal{Q}_\zeta^* \to \mathbb{R}_+$ be two functions. The mapping $\mathcal{F}$ is called a Sehgal-Proinov-type $(\alpha, \beta) - n-$contraction if the functions $\Sigma, \Omega : (0, \infty) \to \mathbb{R}$ exist such that for each $\mathfrak{j}, \mathfrak{s} \in \mathcal{Q}_\zeta^*$, $n(\mathfrak{j}) \in \mathbb{Z}^+$ exists such that

$$\alpha(\mathfrak{j})\beta(\mathfrak{s}) \geq 1 \Rightarrow \Sigma\left(\hbar^3\zeta_\kappa\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{F}^{n(\mathfrak{s})}\mathfrak{s}\right)\right) \leq \Omega\left(\mathcal{C}^*(\mathfrak{j}, \mathfrak{s})\right), \tag{23}$$

where

$$\mathcal{C}^*(\mathfrak{j}, \mathfrak{s}) = \max\left\{\begin{array}{l} \zeta_\kappa(\mathfrak{j}, \mathfrak{s}), \zeta_\kappa\left(\mathfrak{j}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right), \zeta_\kappa\left(\mathfrak{s}, \mathcal{F}^{m(\mathfrak{s})}\mathfrak{s}\right), \\ \frac{\zeta_{2\kappa}\left(\mathfrak{j}, \mathcal{F}^{m(\mathfrak{s})}\mathfrak{s}\right) + \zeta_{2\kappa}\left(\mathfrak{s}, \mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}\right)}{2\hbar} \end{array}\right\}$$

for all $\zeta_\kappa\left(\mathcal{F}^{n(\mathfrak{j})}\mathfrak{j}, \mathcal{F}^{m(\mathfrak{s})}\mathfrak{s}\right) > 0$ and all $\kappa > 0$.

Corollary 3.5. *Let $\mathcal{Q}_\zeta^*$ be a $\zeta-$complete $\mathfrak{m}_\flat\mathfrak{ms}$ with a constant $\hbar \geq 1$ and $\mathcal{F}$ be Sehgal-Proinov-type $(\alpha, \beta) - n-$contraction mapping. Presume that the statements*

- *(i) $\mathcal{F}$ is a cyclic $(\alpha, \beta) - n-$admissible mapping,*
- *(ii) there exist $\mathfrak{j}_0 \in \mathcal{Q}_\zeta^*$ such that $\alpha(\mathfrak{j}_0) \geq 1$,*
- *(iii_a) $\mathcal{F}$ is $\zeta-$continuous, or*
- *(iii_b) if $\{\mathfrak{j}_\mathfrak{z}\}_{\mathfrak{z}\in\mathbb{N}}$ is a sequence in $\mathcal{Q}_\zeta^*$ such that $\mathfrak{j}_\mathfrak{z} \to \mathfrak{j}$ and $\beta(\mathfrak{j}_\mathfrak{z}) \geq 1$ for all $\mathfrak{z} \in \mathbb{N}$, then $\beta(\mathfrak{j}) \geq 1$,*
- *(iv) Σ is non-decreasing and $\Omega(\mathfrak{a}) < \Sigma(\mathfrak{a})$ for all $\mathfrak{a} > 0$,*
- *(v) $\limsup_{\mathfrak{a}\to\mathfrak{a}_0+} \Omega(\mathfrak{a}) < \Sigma(\mathfrak{a}_0+)$ for any $\mathfrak{a}_0 > 0$*

are provided and there exist $\mathfrak{j}_0, \mathfrak{j}_1 \in \mathcal{Q}_\zeta^*$ such that $\zeta_\kappa(\mathfrak{j}_0, \mathfrak{j}_1) < \infty$ for all $\kappa > 0$. If $\alpha(\mathfrak{j})\beta(\mathfrak{s}) \geq 1$ for all $x, y \in Fix\left(\mathcal{F}^{n(\mathfrak{j}^*)}\right)$, then the set $Fix\left(\mathcal{F}^{n(\mathfrak{j}^*)}\right)$ has exactly one element, which means that $\mathcal{F}$ owns a unique fixed point.

4 Conclusions

In consequence, appraising Proinov's outcomes [12], we have extended the result of Sehgal [3], which comprises a more general form of the Ciric contractive condition [35] for multi-valued mappings in the context of $\mathfrak{m}_{\mathfrak{b}}\mathfrak{ms}$. The main theorem has been verified for single-valued mappings. Corresponding outcomes are acquired in the context of modular metric spaces when $n(\mathfrak{j}) = 1$ and $\hbar = 1$ are applied.

References

[1] S. Banach, *Sur les operations dans les emsembles abstraits et leurs applications aux equations integrales*, Fund. Math., 1, (1922), 133181.

[2] V. W. Bryant, *A remark on a fixed point theorem for iterated mappings*, The American Mathematical Monthly, 75, (1968), 399400.

[3] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc., 23, (1969), 631634.

[4] L. F. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*. Proc. Am. Math. Soc., 26, (1970), 615618.

[5] B. Alqahtani, A. Fulga, E. Karapınar, *Sehgal type contractions on b-metric space*, Symmetry, 10(11), (2018), 560.

[6] A. Öztürk, *A fixed point theorem for mappings with an $F-$contractive iterate*, Adv. Theory Nonlinear Anal. Appl., 3(4), (2019), 231236.

[7] A. P. Farajzadeh, M. Delfani, Y. H. Wang, *Existence and uniqueness of fixed points of generalized $F-$contraction mappings*, Journal of Mathematics, 2021, (2021), 19, ID:6687238.

[8] B. Alqahtani, A. Fulga, E. Karapınar, P. S. Kumari, *Sehgal type contractions on dislocated spaces*, Mathematics, 7(2), (2019), 153.

[9] D. Zheng, G. Ye, D. Liu, *Sehgal–Guseman type fixed point theorem in $b-$rectangular metric spaces*, Mathematics, 9, (2021), 3149.

[10] N. A. Secelean, D. Wardowski, M. Zhou, *The Sehgals fixed point result in the framework of $\rho-$Space*, Mathematics, 10, (2022), 459.

[11] J. Zhou, W. Yuan, *Sehgal-Guseman-type fixed point theorem on $b-$metric spaces*, Int. J. Math. Anal., 16(2), (2022), 7380.

[12] P. Proinov, *Fixed point theorems for generalized contractive mappings in metric spaces*, J. Fixed Point Theory Appl., 22, (2020), 21.

[13] E. Karapınar, J. Martinez-Moreno, N. Shahzad, A. F. Roldan Lopez de Hierro, *Extended Proinov $X-$contractions in metric spaces and fuzzy metric spaces satisfying the property NC by avoiding the monotone condition*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., 116(4), (2022), 128.

[14] E. Karapınar, M. De La Sen, A. Fulga, *A note on the Gornicki-Proinov type contraction*, J. Funct. Spaces, 2021, (2021), DOI: org/10.1155/2021/6686644.

[15] A. F. Roldan Lopez de Hierro, A. Fulga, E. Karapınar, N. Shahzad, *Proinov type fixed point results in non-Archimedean fuzzy metric spaces.* Mathematics, 9(14), (2021), 1594.

[16] S. Alizadeh, F. Moradlou, P. Salimi, *Some fixed point results for $(\alpha, \beta)-(\psi, \varphi)-$contractive mappings*, Filomat, 28, (2014), 635647.

[17] A. Latif, A. H. Ansari, *Fixed points and functional equation problems via cyclic admissible generalized contractive type mappings*, J. Nonlinear Sci. Appl., 9, (2016), 11291142.

[18] I. A. Bakhtin, *The contraction mapping principle in quasi metric spaces*, Funct. Anal. Unianowsk Gos. Ped. Inst., 30, (1989), 2637.

[19] S. Czerwik, *Contraction mappings in $b-$metric spaces*, Acta. Math. Inform. Univ. Ostrav., 1(1), (1993), 511.

[20] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ.Modena, 46, (1998), 263276.

[21] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered $b-$metric spaces*, Math. Slovaca, 64(4), (2014), 941960.

[22] V. V. Chistyakov, *Modular metric spaces generated by $F-$modulars*, Folia Math., 15, (2008), 324.

[23] V. V. Chistyakov, *Modular metric spaces, I: Basic concepts*, Nonlinear Anal., 72, (2010), 114.

[24] V. V. Chistyakov, *Modular metric spaces, II: Application to superposition operators*, Nonlinear Anal., 72, (2010), 1530.

[25] V. V. Chistyakov, *Fixed points of modular contractive maps*, Dokl. Math., 86, (2012), 515518.

[26] A. A. N. Abdou, M. A. Khamsi, *Fixed points of multi-valued contraction mappings in modular metric spaces*, Fixed Point Theory Appl., 2014, (2014), 249.

[27] S. B. Nadler, *Multi-valued contraction mappings*, Pac. J. Math., 30, (1969), 475488.

[28] A. Padcharoen, P. Kumam, G. Gopal, P. Chaipunya, *Fixed points and periodic point results for $\alpha-$type $F-$contractions in modular metric spaces*, Fixed Point Theory Appl., 2016, (2016), 39.

[29] C. Mongkolkeha, W. Sintunavarat, P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl., 93, (2011), 19.

[30] U. Aksoy, E. Karapınar, I. M. Erhan, V. Rakocevic, *Meir-Keeler Type Contractions on Modular Metric Spaces*, Filomat, 32(10), (2018), 36973707.

[31] M. Jleli, E. Karapınar, B. Samet, *A best proximity point result in modular spaces with the Fatou property*, Abstr. Appl. Anal., 2013, (2013), DOI: org/10.1155/2013/329451.

[32] M. E. Ege, C. Alaca, *Some results for modular $b-$metric spaces and an application to system of linear equations*, Azerbaijan J. Math., 8(1), (2018), 314.

[33] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki, H. Aydi, *Fixed point results for generalized $F-$contractions in modular $b-$metric spaces with applications*, Mathematics, 7(10), (2019), 116.

[34] B. S. Choudhury, N. Metiya, S. Kundu, D. Khatua, *Fixed points of multi-valued mappings in metric spaces*, Surv. Math. Appl., 14 (2019), 116.

[35] L. Ciric, *On contraction type mappings*, Math. Balcanica, 1, (1971), 5257

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