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A study on magnetic curves in trans-Sasakian manifolds

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Abstract

In this paper, we focused on biharmonic, f-harmonic and f-biharmonic magnetic curves in trans-Sasakian manifolds. Moreover, we obtain necessary and sufficient conditions for magnetic curves as well as Legendre magnetic curves to be biharmonic, f-harmonic and f-biharmonic. We investigate the states of these conditions in α-Sasakian, β-Kenmotsu and cosymplectic manifolds. Besides, we obtain some nonexistence theorems.

1 Introduction

Recently, interdisciplinary studies have gained importance in terms of keeping up with the changing age and exchanging information. One of the best examples of these interdisciplinary studies is the studies between physics and mathematics. Such that, modeling and formulating physical phenomenon and adding mathematical interpretations are valuable academic studies. One of the interdisciplinary fields of study between physics and differential geometry is magnetic fields on various manifolds and their associated magnetic curves.

There are two main point of view on magnetic curves.

From a physical point of view, when a charged particle enters a magnetic field F_m, its Serret-Frenet vectors are affected by this field. Via this affect, the

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Lorentz force is released and the charged particle begins to follow a trajectory in this field. This trajectory is called a magnetic curve, in [20].

From mathematical point of view, the relation between the magnetic field F_m and the Lorentz force φ given with the following formula

$$F_m(\tilde{X},\tilde{Y}) = g(\varphi\tilde{X},\tilde{Y}), \tag{1.1}$$

for any vector fields $\tilde{X},\tilde{Y} \in K$. Here, $(K,\tilde{k})$ is a $(n \geq 2)-$ dimensional Riemannian manifold, F_m magnetic field is a closed 2-form on $(K,\tilde{k})$ and φ endomorphism field is a Lorentz force related to F_m. The magnetic trajectories of F_m on K are the ζ curves that provide the Lorentz equation given below,

$$\nabla_{\zeta'}\zeta^{'} = \varphi(\zeta^{'}). \tag{1.2}$$

Here "$'$" is the derivative with respect to the arc length parameter, [20].

Equation (1.2) generalizes the geodesic equation. Such that, $\nabla_{\zeta'}\zeta^{'} = 0$ is a natural generalization of geodesics of K where ∇ is the Levi-Civita connection related to the metric $\tilde{k}$. For the trivial magnetic field, namely $\nabla F_m = 0$, from equation (1.2), easily seen that normal magnetic curves are the geodesics. Besides, important to know that magnetic curves are never reduced to geodesics. As mentioned in [3, 5], geodesics are the trajectories of charged particles in free fall which are moving under the influence of solely gravity. Besides, it is worth emphasizing that the homogeneity results known for geodesics are not correct for magnetic curves.

Throughout this study, we will consider that ζ is an arclength parametrized non-geodesic and non-null magnetic curve.

The trend towards magnetic curve studies has increased recently, such that some of the important articles published recently can be summarized as follows. In [15, 14, 25, 8], Munteanu et al. handled Killing magnetic curves in $\mathbb{E}^3_1$, $\mathbb{E}^3$, $S^2 \times \mathbb{R}$ and 3D almost paracontact manifolds, respectively. In [7], Calin et al. handled magnetic curves in 3D quasi-para-Sasakian geometry. In [6, 16, 1], the authors handled magnetic fields in complex space, 3D Sasakian manifold and Sasakian manifold. For more studies see ([16, 29, 4, 11, 32]).

Unlike previous articles, Perktaş et al. ([29]) and Bozdağ et al. ([4]) obtained biharmonicity, biminimality and f-harmonicity, f-biharmonicity, bi-f-harmonicity, f-biminimality conditions, respectively, of a non-null magnetic curve in 3D normal almost paracontact metric manifold.

In this article, different from the studies done so far, we examine non-null magnetic curves on biharmonic, f-harmonic and f-biharmonic maps, which are summarized as below.

In [17, 18], Sampson and Eells gave the definition of harmonic maps between Riemannian manifolds for the first time. Such maps are widely studied as they have wide applications in mathematics, physics and engineering.

Then in [17] they also introduced biharmonic maps between the Riemannian manifolds. Biharmonic maps have an important place in the literature at least as much as harmonic maps. Such that, for some basic studies, see [29, 22, 26, 30].

The definition of f-harmonic maps was introduced by Lemaire and Eells, in [18]. f-Harmonic maps have physical meaning which makes them interesting, see [2].

Finally, f-biharmonic maps were handled by Lu, in [22, 23]. Then, Ou studied f-biharmonic curves in $\mathbb{E}^3$ and their characterization in n-dimensional space forms, [27].

Trans-Sasakian manifolds were revealed by Chinea and Gonzales through the categorization of almost contact metric structures, [10]. Later this type of manifolds appeared as generalization of both Kenmotsu and Sasakian manifolds. Also, via classification of almost Hermite manifolds, trans-Sasakian manifolds introduced as a class W_4 of Hermitian manifolds by Hervella and Gray, [19]. On the other hand, an almost contact metric structure on K called as a trans-Sasakian structure when $K \times \mathbb{R}$ product manifold belongs to the class W_4, [28]. In [24, 13], it is showed that a trans-Sasakian manifold of dimension $n \geq 5$ is either a β-Kenmotsu or a α-Sasakian or a cosymplectic manifold. In addition, studies on 3D trans-Sasakian manifolds are carried out using some restrictions on α, β smooth functions which are given in the definition of trans-Sasakian manifolds.

In this study, we focus on biharmonicity, f-harmonicity and f-biharmonicity of a non-null magnetic curve in three dimensional trans-Sasakian manifolds, unlike the studies done so far. In second section, we remind the basic concepts that will be used in the other sections. In third section, we get the biharmonicity, f-harmonicity and f-biharmonicity conditions of magnetic curves and magnetic Legendre curves. We also determine these conditions in α-Sasakian, β-Kenmotsu and cosymplectic manifolds. Besides, we obtain some important nonexistence theorems.

2 Preliminaries

In this section, we remind fundamental notions about trans-Sasakian manifolds, we briefly remind fundamental instruments about magnetic curves, Legendre curves; biharmonic, f-harmonic and f-biharmonic maps.

Definition 1. Harmonic maps are defined as critical points of energy functional

$$E(\omega) = \frac{1}{2}\int_K |d\omega|^2 v_{\tilde{k}},$$

for maps $\omega : (K,\tilde{k}) \to (L,\tilde{l})$ between Riemannian manifolds where $v_{\tilde{k}}$ is the volume element of K. And also, a map is harmonic if

$$\tau(\omega) := tr\nabla d\omega = 0. \tag{2.1}$$

Here, $\tau(\omega)$ is the tension field of ω, [17].

Definition 2. Biharmonic maps are defined as critical points of bienergy functional

$$E_2(\omega) = \frac{1}{2}\int_K |\tau(\omega)|^2 v_{\tilde{k}},$$

for maps $\omega : (K,\tilde{k}) \to (L,\tilde{l})$ between Riemannian manifolds. Besides, a map is biharmonic if the bitension field $\tau_2(\omega)$ equals to

$$\tau_2(\omega) = trace(\nabla^\omega\nabla^\omega - \nabla^\omega_\nabla)\tau(\omega) - trace(R^L(d\omega,\tau(\omega))d\omega) = 0. \tag{2.2}$$

Here R^L, the curvature tensor field of L, is defined as

$$R^L(\tilde{X},\tilde{Y})\tilde{Z} = \nabla^L_{\tilde{X}}\nabla^L_{\tilde{Y}}\tilde{Z} - \nabla^L_{\tilde{Y}}\nabla^L_{\tilde{X}}\tilde{Z} - \nabla^L_{[\tilde{X},\tilde{Y}]}\tilde{Z},$$

for any $\tilde{X},\tilde{Y},\tilde{Z} \in \Gamma(TL)$, [21].

After these two important definitions, attention should be paid that harmonic maps are always biharmonic, but not vice versa. Specifically, non-harmonic biharmonic maps are called proper biharmonic maps, [29].

Definition 3. f-Harmonic maps are defined as critical points of f-energy functional

$$E_f(\omega) = \frac{1}{2}\int_K f|d\omega|^2 v_{\tilde{k}},$$

for maps $\omega : (K,\tilde{k}) \to (L,\tilde{l})$ between Riemannian manifolds where $f \in C^\infty(K,\mathbb{R})$. Also, a map is f-harmonic if

$$\tau_f(\omega) = f\tau(\omega) + d\omega(gradf) = 0, \tag{2.3}$$

where $\tau_f(\omega)$ is the f-tension field of ω, [12].

Definition 4. f-Biharmonic maps are critical points of f-bienergy functional

$$E_{2,f}(\omega) = \frac{1}{2}\int_K f|\tau(\omega)|^2 v_{\tilde{k}},$$

for maps $\omega : (K,\tilde{k}) \to (L,\tilde{l})$ between Riemannian manifolds. On the other hand, ω is a f-biharmonic map if

$$\tau_{2,f}(\omega) = f\tau_2(\omega) + \Delta f\tau(\omega) + 2\nabla^\omega_{gradf}\tau(\omega) = 0, \tag{2.4}$$

where $\tau_{2,f}(\omega)$ is the f-bitension field of ω. Note that if f is a constant function, an f-biharmonic map becomes a biharmonic map, [9, 31].

Now let give details about the trans-Sasakian manifold.

$K^{(2n+1)}$ is called as an almost contact metric manifold with the almost contact metric structure $(\varphi, \xi, \eta, \tilde{k})$ if

$$\varphi^2 = -I + \eta \otimes \xi,$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$\tilde{k}(\varphi\tilde{X}, \varphi Y) = \tilde{k}(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y}), \quad \forall\, \tilde{X},\, \tilde{Y} \in \Gamma(TK), \tag{2.5}$$

where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form and $\tilde{k}$ is a Riemannian metric. From (2.5), it is obtained that

$$\tilde{k}(\tilde{X}, \varphi\tilde{Y}) = -\tilde{k}(\varphi\tilde{X}, \tilde{Y}), \tag{2.6}$$

$$\tilde{k}(\tilde{X}, \xi) = \eta(\tilde{X}), \tag{2.7}$$

for any $\tilde{X},\, \tilde{Y} \in TK$, [28].

Definition 5. An almost contact metric structure $(\varphi, \xi, \eta, \tilde{k})$ on K is called trans-Sasakian structure if there exist two smooth functions α and β on K satisfying

$$(\nabla_{\tilde{X}}\,\varphi)\,\tilde{Y} = \alpha\left(\tilde{k}(\tilde{X}, \tilde{Y})\xi - \eta(\tilde{Y})\tilde{X}\right) + \beta\left(\tilde{k}(\varphi\tilde{X}, \tilde{Y})\xi - \eta(\tilde{Y})\varphi\tilde{X}\right), \tag{2.8}$$

for any $\tilde{X},\, \tilde{Y} \in \Gamma(TK)$. Then K is called as a trans-Sasakian manifold, [28].

The three types of trans-Sasakian manifolds, varing according to the values of α and β can be given as follows.

- If β equals to zero then the manifold is called as a α-Sasakian manifold.
- If α equals to zero then the manifold is called a β-Kenmotsu manifold.
- If α and β equal to zero then the manifold is a cosymplectic manifold.

For a three dimensional trans-Sasakian manifold K satisfying $\alpha, \beta = cons.$, the curvature tensor field equation given as below,

$$\begin{aligned} R(\tilde{X}, \tilde{Y})\tilde{Z} \;&=\; (\frac{r}{2} - 2(\alpha^2 - \beta^2))(\tilde{k}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{k}(\tilde{X}, \tilde{Z})\tilde{Y}) \\ &+\; (\frac{r}{2} - 3(\alpha^2 - \beta^2))(\tilde{k}(\tilde{X}, \tilde{Z})\eta(\tilde{Y}) - \tilde{k}(\tilde{Y}, \tilde{Z})\eta(\tilde{X}))\xi \\ &+\; (\frac{r}{2} - 3(\alpha^2 - \beta^2))(\eta(\tilde{X})\eta(\tilde{Z})\tilde{Y} - \eta(\tilde{Y})\eta(\tilde{Z})\tilde{X}), \end{aligned} \tag{2.9}$$

where r is the scalar curvature of K and $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(TK)$.

Definition 6. Let $(K, \varphi, \xi, \eta, \tilde{k})$ be a three dimensional almost contact metric manifold and $\zeta(s)$ is an arclength parametrized smooth curve in K. Then the contact angle $\theta(s) \in [0, \pi]$ between ζ and ξ defined as

$$cos(\theta(s)) = g(U(s), \xi) = \eta(U(s)).$$

Then if θ is a constant, $\zeta(s)$ defined as a slant curve and if θ equals to $\frac{\pi}{2}$, $\zeta(s)$ defined as a Legendre curve where $U = \zeta'$, [7].

From Definiton 6, it is easy to see that the following equation is provided for a Legendre curve,

$$\eta(U(s)) = cos(\frac{\pi}{2}) = 0. \tag{2.10}$$

Note that throughout this paper, we will use TSM instead of trans-Sasakian manifold, $3d$ instead of three dimensional for the sake of simplicity.

3 Magnetic Legendre Curves

In this section, we determine biharmonicity, f-harmonicity and f-biharmonicity conditions for not only magnetic curves but also magnetic Legendre curves in $3d$ trans-Sasakian manifolds satisfying $\alpha, \beta = cons$.

A smooth curve $\zeta : I \to K$ on a $3d$ TSM is called as magnetic curve if it satisfies

$$\nabla_U U = \varphi U, \tag{3.1}$$

where $U = \zeta'$.

Then, by using (2.8) and (2.9) in the differentiation of (3.1), we get the following equations;

$$\nabla_U^2 U = -(\alpha\eta(U) + 1)U - \beta\eta(U)\varphi U + (\alpha + \eta(U))\xi, \tag{3.2}$$

$$\begin{aligned} \nabla_U^3 U = & (\alpha\beta\eta(U)^2 + 2\beta\eta(U) + \alpha\beta)U \\ & +(\beta^2\eta(U)^2 - 2\alpha\eta(U) - \alpha^2 - 1)\varphi U \\ & -2\beta(\alpha\eta(U) + \eta(U)^2)\xi, \end{aligned} \tag{3.3}$$

$$\begin{aligned} R(\nabla_U U, U)U &= R(\varphi U, U)U \\ &= \left[\frac{r}{2} - 2(\alpha^2 - \beta^2) - \eta(U)^2(\frac{r}{2} - 3(\alpha^2 - \beta^2))\right]\varphi U. \end{aligned} \tag{3.4}$$

With the help of equations (3.1), (3.2),(3.3) and (3.4), we get bitension, f-tension and f-bitension fields of a magnetic curve in a $3d$ TSM as in following subsections.

3.1 Biharmonic Magnetic Curves

In this subsection, we obtain the conditions for a magnetic curve in a $3d$ TSM K satisfying $\alpha, \beta = cons.$ to be biharmonic.

Let $\zeta : I \to K$ be a magnetic curve. Then, by substituting equations (3.3) and (3.4) into the bitension field formula given by (2.2), we get the biharmonicity condition as below:

$$\begin{aligned}
\tau_2(\zeta) &= \nabla_U^3 U + R(\nabla_U U, U)U \\
&= \left[\alpha\beta\eta(U)^2 + 2\beta\eta(U) + \alpha\beta\right]U \\
&+ \left[\beta^2\eta(U)^2 - 2\alpha\eta(U) - \alpha^2 - 1 + \frac{r}{2} - 2(\alpha^2 - \beta^2) - \eta(U)^2(\frac{r}{2} - 3(\alpha^2 - \beta^2))\right]\varphi U \\
&- 2\beta[\alpha\eta(U) + \eta(U)^2]\xi \\
&= 0. \qquad (3.5)
\end{aligned}$$

From (3.5), we attain the theorem given as below.

Theorem 1. *Let $\zeta : I \to K$ be a magnetic curve in a 3d TSM satisfying $\alpha, \beta = cons.$ Then, ζ is a biharmonic magnetic curve iff the following equations satisfy:*

$$\begin{cases}
\alpha\beta\eta(U)^2 + 2\beta\eta(U) + \alpha\beta = 0, \\
\beta^2\eta(U)^2 - 2\alpha\eta(U) - \alpha^2 - 1 + \frac{r}{2} - 2(\alpha^2 - \beta^2) - \eta(U)^2(\frac{r}{2} - 3(\alpha^2 - \beta^2)) = 0, \\
\beta(\alpha\eta(U) + \eta(U)^2) = 0.
\end{cases} \qquad (3.6)$$

Corollary 1. *Let $\zeta : I \to K$ be a magnetic curve in a 3d α-Sasakian manifold. Then ζ is a biharmonic magnetic curve iff*

$$r = \frac{-6\alpha^2\eta(U)^2 + 6\alpha^2 + 4\alpha\eta(U) + 2}{1 - \eta(U)^2},$$

where $\eta(U)^2 \neq 1$.

Proof. For a α-Sasakian manifold we know that $\beta = 0$. Then, when we substitute zero for β into the equation (3.6), equations 1 and 3 vanish and equation 2 turns into the form given below.

$$-2\alpha\eta(U) - 3\alpha^2 - 1 + \frac{r}{2} - \eta(U)^2(\frac{r}{2} - 3\alpha^2) = 0$$

Here, the result is obtained when r is left alone. □

Corollary 2. *There does not exist non-Legendre biharmonic magnetic curve in 3d β-Kenmotsu manifold.*

Proof. For a β-Kenmotsu manifold we know that $\alpha = 0$. Then, when we substitute zero for α into the equation (3.6), equation 1 reduces to

$$\beta\eta(U) = 0.$$

From this equality easy to see that if $\eta(U) \neq 0$, namely non-Legendre curve, β must be equal to zero but this a contradiction wtih the definition of β-Kenmotsu manifold. □

Corollary 3. *Let $\zeta : I \to K$ be a magnetic curve in a 3d cosymplectic manifold. Then, ζ is a biharmonic magnetic curve iff*

$$r = \frac{2}{1 - \eta(U)^2},$$

where $\eta(U)^2 \neq 1$.

Proof. For a cosymplectic manifold we know that $\alpha, \beta = 0$. Then, when we substitute zero for α and β in equation (3.6), equations 1 and 3 vanish and equation 2 turns into the form given below.

$$\frac{r}{2}(1 - \eta(U)^2) - 1 = 0$$

Here, the result is obtained when r is left alone. □

Now, let ζ be a magnetic Legendre curve in $3d$ TSM satisfying $\alpha, \beta = cons$. Then by substituting (2.10) into the (3.6), biharmonicity conditions reduces to

$$\begin{cases} \alpha\beta = 0, \\ \dfrac{r}{2} + 2\beta^2 - 3\alpha^2 - 1 = 0. \end{cases} \tag{3.7}$$

Then we get the following corollaries.

Corollary 4. *Let $\zeta : I \to K$ be a magnetic curve in a 3d α-Sasakian manifold. Then, ζ is a biharmonic magnetic Legendre curve iff*

$$r = 6\alpha^2 + 2.$$

Corollary 5. *Let $\zeta : I \to K$ be a magnetic curve in a 3d β-Kenmotsu manifold. Then, ζ is a biharmonic magnetic Legendre curve iff*

$$r = -4\beta^2 + 2.$$

Corollary 6. *Let $\zeta : I \to K$ be a magnetic curve in a $3d$ cosymplectic manifold. Then, ζ is a biharmonic magnetic Legendre curve iff*

$$r = 2.$$

3.2 f-Harmonic Magnetic Curves

In this subsection, we investigate the conditions for a magnetic curve in a $3d$ TSM K to be f-harmonic.

Let $\zeta : I \to K$ be a magnetic curve then via f-tension field formula given with equation (2.3), the f-harmonicity condition obtained as below:

$$\tau_f(\zeta) = f^{'}U + f\nabla_U U = f^{'}U + f\varphi U = 0. \tag{3.8}$$

From (3.8), we obtain the nonexistence theorem given as below.

Theorem 2. *There does not exist a proper f-harmonic magnetic curve in a $3d$ TSM.*

Proof. For (3.8) equality to be achieved, the coefficients f and $f^{'}$ must be equal to zero. However, this is not possible according to the f-harmonic map definition. □

3.3 f-Biharmonic Magnetic Curves

Finally in this subsection, we obtain the conditions for a magnetic curve in a $3d$ TSM satisfying $\alpha, \beta = cons.$ to be f-biharmonic. By substituting (3.1), (3.2), (3.3) and (3.4) into f-bitension field formula given by (2.4), we get f-biharmonicity conditions as below:

$$\begin{aligned}
\tau_{2,f}(\zeta) &= f(\nabla_U^3 U + R(\nabla_U U, U)U) + 2f^{'}\nabla_U^2 U + f^{''}\nabla_U U \\
&= \big[f(\alpha\beta\eta(U)^2 + 2\beta\eta(U) + \alpha\beta) - 2f^{'}(\alpha\eta(U) + 1)\big]U \\
&+ \Bigg[f\Bigg(\beta^2\eta(U)^2 - 2\alpha\eta(U) - \alpha^2 - 1 + \frac{r}{2} - 2(\alpha^2 - \beta^2) \\
&\quad -\eta(U)^2(\frac{r}{2} - 3(\alpha^2 - \beta^2))\Bigg) - 2f^{'}\beta\eta(U) + f^{''}\Bigg]\varphi U \\
&+ \big[f(-2\beta(\alpha\eta(U) + \eta(U)^2)) + 2f^{'}(\alpha + \eta(U))\big]\xi \\
&= 0.
\end{aligned} \tag{3.9}$$

From (3.9), we get the theorems and corollaries given as below.

Theorem 3. *Let $\zeta : I \to K$ be a magnetic curve in a 3d TSM satisfying $\alpha, \beta = cons$. Then, ζ is a f-biharmonic magnetic curve iff*

$$\begin{cases} f(\alpha\beta\eta(U)^2 + 2\beta\eta(U) + \alpha\beta) - 2f^{'}(\alpha\eta(U) + 1) = 0, \\ f(2\beta^2 - 3\alpha^2 - 1 + \frac{r}{2}(1 - \eta(U)^2) + \eta(U)^2(3\alpha^2 - 2\beta^2) - 2\alpha\eta(U)) - 2f^{'}\beta\eta(U) + f^{''} = 0, \\ f(-2\beta(\alpha\eta(U) + \eta(U)^2)) + 2f^{'}(\alpha + \eta(U)) = 0. \end{cases} \tag{3.10}$$

Corollary 7. *Let $\zeta : I \to K$ be a magnetic curve in a 3d α-Sasakian manifold. Then, ζ is a proper f-biharmonic slant magnetic curve iff*

$$r = \frac{6\alpha^6 - 14\alpha^4 + 8\alpha^2 - \alpha}{\alpha^4 - 2\alpha^2 + 1},$$

and

$$f(s) = e^{\frac{s}{2\alpha^3 - 2\alpha}},$$

where $\eta(U) = -\dfrac{1}{\alpha}$.

Proof. For a α-Sasakian manifold we know that $\beta = 0$. Then, when we substitute zero for β into the equation (3.10), it is reduced to

$$\begin{cases} f^{'}(\alpha\eta(U) + 1) = 0, \\ f\big(-3\alpha^2 - 1 + \frac{r}{2}(1 - \eta(U)^2) + 3\eta(U)^2\alpha^2 - 2\alpha\eta(U)\big) + f^{''} = 0, \\ f\eta(U)^2 + 2f^{'}(\alpha + \eta(U)) = 0. \end{cases} \tag{3.11}$$

Then from first equation of (3.11) for a proper curve, namely $f^{'} \neq 0$, we obtain that $\eta(U) = -\dfrac{1}{\alpha}$. For a α-Sasakian manifold we know that α is a non-zero constant so this means that $\eta(U)$ is a constant. Then from Definition 6, we get that this a slant curve.
On the other hand from third equation of (3.11), we determined $f(s)$ and by substituting $f(s)$ into the second equation of (3.11) we get the scalar curvature r. $\square$

Corollary 8. *Let $\zeta : I \to K$ be a magnetic curve in a 3d β-Kenmotsu manifold. Then, ζ is a proper f-biharmonic magnetic curve iff*

$$r = \frac{\eta(U)^{'} + 3\eta(U)^2 + 1}{1 - \eta(U)^2},$$

and

$$f(s) = e^{-\frac{1}{2}\int \eta(U)ds}.$$

Proof. For a β-Kenmotsu manifold we know that $\alpha = 0$. Then, when we substitute zero for α into the equation (3.10), it is reduced to

$$\begin{cases} f\beta\eta(U) - f^{'} = 0, \\ f(2\beta^2 - 1 + \frac{r}{2}(1 - \eta(U)^2) - \eta(U)^2\beta^2) - 2f^{'}\beta\eta(U) + f^{''} = 0, \\ f\eta(U)^2 + 2f^{'}\eta(U) = 0. \end{cases} \tag{3.12}$$

Then by solving first and third equations of (3.12) together, we obtain $\beta = \frac{-1}{2}$ and $f(s) = e^{-\frac{1}{2}\int \eta(U)ds}$. Finally, via substituting β, $f(s)$ and derivatives of $f(s)$ we get the scalar curvature r. □

Corollary 9. *There does not exist f-biharmonic magnetic curve in $3d$ cosymplectic manifold.*

Proof. For a cosymplectic manifold we know that $\alpha, \beta = 0$. Then, when we substitute zero for α and β into the first equation of (3.10), we get $f^{'} = 0$. However, this is not possible according to the f-biharmonic map definition. □

Finally, let ζ be a magnetic Legendre curve in $3d$ TSM satisfying $\alpha, \beta = cons.$, so the f-biharmonicity conditions reduces to

$$\begin{cases} \alpha\beta f - 2f^{'} = 0, \\ f(2\beta^2 - 3\alpha^2 - 1 + \frac{r}{2}) + f^{''} = 0, \\ \alpha f^{'} = 0. \end{cases} \tag{3.13}$$

Then, we get the following theorem.

Theorem 4. *There does not exist proper f-biharmonic magnetic Legendre curve in $3d$ TSM satisfying $\alpha, \beta = cons.$*

Proof. It is easy to see that from the third equation of (3.13), for a proper curve α must be equal to zero so this means that there is no f-biharmonic magnetic Legendre curve in α-Sasakian manifold. On the other hand from the first equation of (3.13), for a proper curve there is no f-biharmonic magnetic Legendre curve in not only β-Kenmotsu manifold but also cosymplectic manifold. □

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