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Fixed Point Index for Simulation Mappings and Applications

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Abstract

In this paper, we construct the fixed point index for a class of contractive mapping defined by a simulation mapping and a measure of noncompact-ness noted by $Z_\mu-$ contraction maps. Then we establish some fixed point theorem for this mapping of the Krasnoselskii type. An Application to the integral equation is presented to support the results.

1 Introduction

In all this paper we let $(E, \leq)$ be an ordered real Banach space, where the order $\leq$ is induced by the cone K and we write write for all $x, y \in E$, $x \leq y$ if $y - x \in K$, $x < y$ if $y - x \in K$ and $x \neq y$, $x \nleq y$ if $y - x \notin K$, $x \ll y$ if $intK \neq \emptyset$ and $y - x \in intK$. The notations $\geq, >, \ngeq, \gg$ are defined similarly. Let also G be a bounded open subset of K.

The concept of fixed point index for positive compact mappings is described in [11] by the Leary-Shauder degree and is actually one of the most important and useful tools to study the existence of positive fixed point for the equation

$$fu = u,\ u \in E. \tag{1}$$

where $f : \overline{G} \to K$ is a compact mapping. The introduction of the measure of noncompactness (MNC for short) by Kuratowski has motivated Nussbaum

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and Sadovski to extend the definition of the fixed point index to the case of k-set contraction or condensing mappings, see [16, 17]. Recall that a mapping $f : E \to E$ is said to be a k-set contraction if $f(\mu(B)) \leq k\mu(B)$ for any bounded subset B of E with $k \in [0, 1)$, where μ is the Kuratowski MNC and it said to be condensing if $f(\mu(B)) \leq \mu(B)$ for any bounded subset B of E. More generally, Nussbaum proved in [17] that the fixed point index is well defined even μ is any abstract MNC. Furthermore, Nussbaum's construction of the fixed point index theory for k-set contraction or condensing mappings is based on the Leary-shauder degree.

Recently, many researchers used the concept of fixed point index for a class of strictly set contraction, condensing and 1-set contraction or compact mappings to obtain many fixed point theorems, see ([2, 3, 5, 6, 8, 12]).

Motivated by the above cited works, we present in this paper an extension of the definition of the fixed point index to the class of simulation mappings (see Definition 2.15). In particular, we show that if $0 \in G$ and a simulation mapping $f : \overline{G} \to K$ satisfies the Leray-Schauder boundary condition $(\mathcal{LS})$

$$(\mathcal{LS}) : \qquad fx \neq \lambda x \text{ for all } x \in \partial G \text{ and } \lambda \geq 1 \text{ with } .$$

then the fixed point index $i(f, G, K) = 1$. We present also some situations where for such a mapping f the fixed point index $i(f, G, K) = 0$. Therefore a variant of the compression and expansion of a cone principle is proved.

We end the paper with application to a nonlinear integral equation.

2 Preliminaries

Let us recall some notations and tools that will be used in this paper. Let $\mathcal{B}$ and $Q(E)$ be respectively the set of all bounded subsets of E and the set of all compact subsets of E.

Definition 2.1. *([1]) A mapping $\mu : \mathcal{B} \to [0, +\infty[$ is called a measure of non-compactness in the space E, if it satisfies the following conditions,*

(C_1) $Ker(u) = \{B \in \mathcal{B}, \mu(B) = 0\}$ *is nonempty and* $Ker(u) \subset Q(E)$. *(where* $Ker(\mu)$ *denotes the kernel of the measure of non-compactness* μ*)*

(C_2) $\forall B_1, B_2 \in \mathcal{B}, B_1 \subset B_2 \Rightarrow \mu(B_1) \leq \mu(B_2)$.

(C_3) $\mu(\overline{Con}(B)) = \mu(B) = \mu(\overline{B})$, *where Con is the convex hull of B.*

Note that the MNC μ may meet some other conditions, such as;

(C_4) $\forall B \in \mathcal{B}, b \in E, \mu(B \cup \{b\}) = \mu(B)$.

(C_5) *If $\{B_n\}$ is a decreasing sequence of nonempty closed and bounded subset of E and if $\lim_{n\to\infty} \mu(B_n) = 0$ then the intersection $B_\infty = \cap_{n=0}^{\infty} B_n$ is nonempty and compact.*

(C_6) $\forall B \in \mathcal{B}, b \in E, \mu(B + \{b\}) = \mu(B)$.

(C_7) $\forall B_1, B_2 \in \mathcal{B}, \mu(B_1 \cup B_2) = max\{\mu(B_1), \mu(B_2)\}$.

Remark 2.2. *1. If $Ker(u) = Q(E)$, the MNC is called full and in the case that μ fulfills all the conditions $(C_1) - -(C_6)$, μ is called regular.*

2. In case the MNC may be a full or its kernel is reduced of the singleton sets, then the condition (C_4) is straightforwardly from the condition (C_7).

Notice, that the Kuratwoski and Hausdaurf MNCs, (see [1]), are regular and full measures. On the other hand, one can find in the literature other measures of non-compactness like The diameter MNC, denoted by μ_d and defined by $\mu_d(X) = diam(X)$. The spermium MNC, denoted by μ_s and defined by $\mu_s(X) = \sup\{\| x \|, x \in X\}$. μ_d, μ_s do not fulfill the condition (C_5), and finally, the MNC μ_{si} defined by

$$\mu_{si} = \sup\{\| x \|, x \in X\} - \inf\{\| y \|, y \in \overline{conv}(X)\}.$$

This MNC does not satisfy the condition (C_5) but in [1] we can find a results when μ_{si} holds the condition (C_5).

Definition 2.3. *([11]) Let E be a Banach space and consider μ is a MNC on E, $f : E \to E$ be a mapping,*

1. *f is called $(\mu) - k$ set contraction if there exists a constant $k \in [0, 1)$ such that $\mu(f(B)) \leq k\mu(B)$ for all bounded sets $B \subset E$.*
2. *f is called $(\mu)-$ condensing if $\mu(f(B)) < \mu(B)$ for all bounded sets $B \subset E$ with $\mu(B) > 0$.*
3. *f is called non-expansive if $\| f(x) - f(y) \| \leq \| x - y \|$ holds for every $x, y \in E$.*

In the following, we make a few reviews about the cones and fixed point index, (see [11]).

Definition 2.4. *([11]) The cone K is said*

1. *normal if there exists a positive constant N such that for all $x, y \in K$, $x \leq y$ implies $\| x \| \leq N \| y \|$.*

2. *total if* $\overline{K-K} = E$.

Definition 2.5. *([11]) A subset* $X \subset E$ *is called a retract of* E *if there exists a continuous mapping* $r : E \to X$*, such that* $r(x) = x$*, for all* $x \in X$*,* r *is called a retraction.*

Remark 2.6. *([11]) Every closed convex subset of* E *is a retract of* E*, in particular every ordered cone of* E *is a retract of* E*.*

Definition 2.7. *([11]) Let* X *be a retract of* E *and* G *be a bounded open subset of* X *such that* $G \subset B(0,R)$*, where* $B(0,R)$ *is the ball centered at* 0 *of radius* R*. For any completely continuous mapping* $f : \overline{G} \to K$ *with* $f(x) \neq x$ *for all* $x \in \partial U$*, the integer given by:*

$$i(f,G,K) = deg(I - f \circ r, B(0,R) \cap r^{-1}(G), 0). \tag{2}$$

where deg is the Leray-Schauder degree and r *is a retraction, is well defined and is called the fixed point index.*

Proposition 2.8. *([11]) [Properties of fixed point index]*

1. *Normality:* $i(f,G,K) = 1$ *if* $f(x) = x_0 \in \overline{G}$ *for all* $x \in \overline{G}$*.*

2. *Homotopy invariance: Let* $H : [0,1] \times \overline{G} \to K$ *be a completely continuous mapping such that* $H(t,x) \neq x$ *for all* $(t,x) \in [0,1] \times \partial G$*, the integer* $i(H(t,.),G,K)$ *is independent of* t*.*

3. *Additivity :* $i(f,G,K) = i(f,G_1,K) + i(f,G_2,K)$ *whenever* U_1 *and* U_2 *are two disjoint open subsets of* G *such that* f *has no fixed point in* $\overline{G} \setminus (G_1 \cup G_2)$*.*

4. *Permanence : if* K' *is a retract of* K *with* $f(\overline{G}) \subset K'$*, then* $i(f,G,K) = i(f,G,K')$*.*

Notice that the fixed point index has been generalized for k-set contraction and $\mu-$ condensing mapping, where μ is Hausdorff or Kuratowski MNC. For more details we refer to [11]. So, we can find in the literature the following result: if K is a cone of real Banach space E, G is a bounded open set of K and $f : \overline{G} \to K$ is a condensing mapping then, one has,

Lemma 2.9. *(Lemma 2.3.1 in [11]) Let* $0 \in G$ *and suppose that* $fx \neq \lambda x$ *for all* $x \in \partial G$ *and* $\lambda \geq 1$ *then* $i(f,G,K) = 1$*.*

Now, we introduce the concept of simulation mapping.

Definition 2.10. *([14]) A simulation function is a mapping* $\xi : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ *satisfying the following conditions:*

(ξ_1) $\xi(0,0) = 0$,

(ξ_2) $\xi(t,s) < s - t$, *for all* t, $s > 0$.

(ξ_3) *if* (t_n), (s_n) *are sequences in* $(0,+\infty)$ *such that* $\lim_{n\longrightarrow+\infty} t_n = \lim_{n\longrightarrow+\infty} s_n > 0$, *then* $\limsup_{n\longrightarrow+\infty} \xi(t_n, s_n) < 0$.

In [18], the authors slightly modify the simulation function, as follows.

Definition 2.11. *A simulation function is a mapping* $\xi : [0,+\infty)\times[0,+\infty) \longrightarrow \mathbb{R}$ *satisfying the following conditions:*

(ξ_1) $\xi(0,0) = 0$,

(ξ_2) $\xi(t,s) < s - t$,

(ξ_3) *if* (t_n), (s_n) *are sequences in* $(0,+\infty)$ *such that* $\lim_{n\longrightarrow+\infty} t_n = \lim_{n\longrightarrow+\infty} s_n > 0$ *and* $t_n < s_n$, *then* $\limsup_{n\longrightarrow+\infty} \xi(t_n, s_n) < 0$.

Example 2.12. *(see [7] and [18])*
If $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ *is an upper semi-continuous mapping such that* $\varphi(t) < t$ *for all* $t > 0$ *and* $\varphi(0) = 0$, *and we define* $\xi : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ *by* $\xi(t,s) = \varphi(s) - t$ *for all* $s,t \in [0,+\infty)$, *then* ξ *is a simulation function.*

Example 2.13. *(see [7] and [18])*
Let ϕ *and* ψ *be two altering distance functions such that* $\psi(t) < t \leq \phi(t)$ *for all* $t > 0$. *Then the mapping* $\xi(t,s) = \psi(s) - \phi(t)$ *for all* $t,s \in [0,+\infty$ *is a simulation function. If,* $\phi(t) = t$ *and* $\psi(t) = \kappa t$ *for all* $t \geq 0$, *where* $\kappa \in [0,1)$, *then we obtain the following particular case of simulation function:* $\xi(t,s) = \kappa s - t$ *for all* $t \geq 0$, *where* $\kappa \in [0,1)$.

Definition 2.14. *(see [7]) A generalized simulation function is a mapping* $\xi : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ *satisfying the following conditions:*

$$\xi(t,s) \leq s - t \text{ for all } s,\, t > 0.$$

Let Z be the family of all simulation functions $\xi : [0,+\infty)\times[0,+\infty) \longrightarrow \mathbb{R}$ following the Definition 2.10. Let Z' be the family of all simulation functions $\xi : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ following the Definition 2.14.

Definition 2.15. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \longrightarrow \Omega$ be a continuous operator. We say that T is a Z_μ-contraction if there exists $\xi \in Z$ such that*

$$\xi(\mu(T(\Omega)), \mu(\Omega)) \geq 0,$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness.

Remark 2.16.

1. *It is easy to see that if T is a Z_μ-contraction map then we have*

$$0 \leq \xi(\mu(T(\Omega)), \mu(\Omega)) < \mu(\Omega) - \mu(T(\Omega)),$$

so

$$\mu(T(\Omega)) < \mu(\Omega).$$

2. *Also, if $\Theta \subseteq \Omega$ then $\mu(\Theta) \leq \mu(\Omega)$ and $\mu(T(\Theta)) \leq \mu(T(\Omega))$ which implies that*

$$0 \leq \xi(\mu(T(\Omega)), \mu(\Omega)) < \mu(\Omega) - \mu(T(\Omega)) \leq \mu(\Omega) - \mu(T(\Theta)).$$

According to Remark 2.16, we introduce the monotony property of function ξ.

Definition 2.17. *Let $\xi : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ a simulation functions, we say that is decreasing if for all t_1, $t_2 \in [0, +\infty)$ such that $t_1 \leq t_2$ then $\xi(t_1, s) \leq \xi(t_2, s)$ for $s \in [0, +\infty.)$*

Example 2.18. *Let ρ and χ be two altering distance functions such that $\chi(t) < t \leq \rho(t)$ for all $t > 0$. Then the mapping*

$$\zeta(t, s) = \chi(s) - \rho(t)$$

is a nondecreasing simulation mappind when ρ is an decreasing map on $[0, +\infty)$.

Example 2.19. *Let $\rho : [0, +\infty) \longrightarrow\longrightarrow [0, +\infty)$ be an decreasing lower semi-continuous function such that $\rho^{-1}(0) = 0$. Then the mapping*

$$\zeta(t, s) = s - \rho(s) - t$$

is a nondecreasing simulation mapping when ρ is an increasing map on $[0, +\infty)$.

Definition 2.20. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \longrightarrow \Omega$ be a continuous operator. We say that T is a Z'_μ-contraction if there exists $\xi \in Z'$ such that*

$$\xi(\mu(T(\Omega)), \phi(\mu(\Omega))) \geq 0,$$

for any nonempty subset X of Ω, where μ is an arbitrary MNC and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing mapping with $\lim_{n \longrightarrow +\infty} \phi^n(t) = 0$, for all $t > 0$.

Remark 2.21. *Let $\xi : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ a simulation mapping. If we suppose that the sequence (t_n) given in (ξ_3) of Definition 2.10, satisfies $\lim\limits_{n \longrightarrow +\infty} t_n = \delta > 0$ and $t_n \geq \delta$ for all $n \in \mathbb{N}$ we get that the mapping $\xi(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, +\infty$ is a simulation function, when ψ is nondecreasing mapping and ϕ is a lower semi-continuous mapping with $\phi(t) < \psi(t)$ for all $t > 0$, (see [4]).*

Definition 2.22. *([12]) Suppose that $T : \overline{U} \to C$ is $1-$set contraction mapping with $0_E \notin \overline{(I - T)(\partial D)}$, so there exists $\delta > 0$ such that*

$$\inf_{x \in \partial U} ||x - Tx|| \geq 0.$$

Let $M = \sup_{x \in \overline{U}} ||Tx|| + \delta$ and take $k \in]1 - \frac{\delta}{M}, 1[$. Obviously $T_k = kT$ is strict set contraction mapping, so we defined the fixed point index $i(T, U, C) = i(T_k, U, C)$

Definition 2.23. *Let D be a subset of the Banach space X, the mapping $T : D \to X$ is said to be $\phi-$ expansive if there exists a function $\phi : [0, +\infty[\to [0, +\infty[$ such that for all x, y in D*

$$||Tx - Ty|| \geq \phi\big(||x - y||\big)$$

with $\phi(0) = 0$ and $\phi(r) > 0, \forall r > 0$ and ψ is either continuous and nondecreasing.

Lemma 2.24. *([9]) Let G be a closed bounded subset of a Banach space E and let $T : G \longrightarrow G$ be a continuous mapping such that $I - T : G \longrightarrow E$ is ϕ-expansive. If there exists an almost fixed point (a.f.p. in short) sequence (x_n) of T in G, then T has a unique fixed point $x_0 \in G$. Furthermore, $x_n \longrightarrow x_0$ when $n \longrightarrow +\infty$.*

3 Main results

3.1 Fixed point index for Z_μ-contraction mapping

Let X be a nonempty convex closed subset of the space E, G be an open bounded subset of X.

Proposition 3.1. *Let $f : \overline{G} \to X$ be a Z_μ-contraction mapping where μ is a MNC . Let*

$$M_1 = \overline{Con}[f(\overline{G})], \quad M_{n+1} = \overline{Con}[f(\overline{G} \cap M_n)]. \tag{3}$$

Suppose that for any $n \in \mathbb{N}$, $M_n \cap G \neq \emptyset$, then $M_\infty = \bigcap_{n\geq 1} M_n$ is a nonempty convex and compact.

Proof. Obviously, we have $M_\infty \subset X$ and it is easy to see that $\{M_n\}_n$ is a decreasing sequence. Indeed, we have $f(\overline{G} \cap M_1) \subset f(\overline{G})$ and so $M_2 \subset M_1$, also if we suppose that $M_n \subset M_{n-1}$ we find that $f(M_n \cap \overline{G}) \subset f(M_{n-1} \cap \overline{G})$ and so $M_{n+1} \subset M_n$. Next, consider the sequence $\{\mu(M_n)\}_n$, using the conditions (C_2) and (C_3) of the MNC μ, we find that $\{\mu(M_n)\}_n$ is a positive decreasing sequence, then there exists a positive integer l such that

$$\lim_{n\to\infty} \mu(M_n) = l.$$

Now, we show that $l = 0$. At first, using the fact that f is Z_μ-contraction map, check that $\xi(\mu(M_{n+1}), \mu(M_n)) \geq 0$.

$$\begin{aligned} \xi(\mu(M_{n+1}), \mu(M_n)) &= \xi(\mu\big(Con\big[f\big(M_n \cap \overline{G}\big)\big], \mu(M_n))), \\ &= \xi\big(\mu\big(\big[f\big(M_n \cap \overline{G}\big)\big]\big), \mu(M_n)\big), \\ &\geq \xi\big(\mu\big(\big[f\big(M_n\big)\big]\big), \mu(M_n)\big) \geq 0. \end{aligned}$$

Now, on the contrary, suppose that $l \neq 0$ and put $t_n = \mu(M_{n+1})$ and $s_n = \mu(M_n)$ for all $n \in \mathbb{N}$. So we get that

$$0 \leq \limsup_{n\longrightarrow+\infty} \xi\big(\mu\big(\big[f\big(M_n\big)\big]\big), \mu(M_n)\big) = \limsup_{n\longrightarrow+\infty} \xi\big(t_n, s_n\big) < 0,$$

which is a contradiction so $l = 0$. Finally, since μ satisfies (C_5) we get that $M_\infty = \cap_{n=1}^{\infty} M_n$ is a convex and compact subset of X and more over we have $f(M_\infty \cap \overline{G}) \subset M_\infty$. □

Proposition 3.2. *Let $f : \overline{G} \to X$ be a Z'_μ-contraction mapping. Let $(M_n)_n$ the sequence defined in (3), then $M_\infty = \bigcap_{n\geq 1} M_n$ is convex and compact.*

Proof. As in the precedent proof, we show that $l = 0$. At first using the fact that f is a Z'_μ-contraction map, we check that $\xi(\mu(M_{n+1}), \phi(\mu(M_n))) \geq 0$ as follows:

$$\begin{aligned} \xi(\mu(M_{n+1}), \mu(M_n)) &= \xi(\mu\big(Con\big[f\big(M_n \cap \overline{G}\big)\big], \mu(M_n))), \\ &= \xi\big(\mu\big(\big[f\big(M_n \cap \overline{G}\big)\big]\big), \mu(M_n)\big), \\ &\geq 0. \end{aligned}$$

Then by the definition of ξ we have

$$\xi\big(\mu\big(\big[f\big(M_n\big)\big]\big), \phi(\mu(M_n))\big) \leq \phi(\mu(M_n)) - \mu(M_{n+1}).$$

Using the properties of ϕ, we obtain

$$\mu(M_{n+1}) \leq \phi(\mu(M_n)) \leq \phi(\phi(\mu(M_{n-1}))) \leq ... \leq \phi(\mu(M_1)).$$

Letting $n \to \infty$, yields $l = 0$. □

Theorem 3.3. *Let $f : \overline{G} \longrightarrow X$ be a Z_μ or a Z'_μ-contraction mapping without a fixed point in ∂G. Then the fixed point index $i(f, G, X)$ is well defined.*

Proof. Let $\{M_n\}$ the sequence defined above, First, if $M_\infty = \emptyset$ we set

$$i(f, G, X) = 0 \tag{4}$$

Now, we suppose that $M_\infty = \bigcap_{n\geq 1} M_n \neq \emptyset$. By Proposition 3.2 M_∞ is a nonempty compact and convex subset of X. By the extension theorem, we obtain the existence of a completely continuous operator $f_1 : \overline{G} \to M_\infty$ such that $f_1x = fx$ for all $x \in M_\infty \cap \overline{G}$ and $f_1x \neq x \; \forall x \in \partial G$. Indeed, if we suppose that there exists $x \in \partial G$ such that $f_1x = x$, then $x \in M_\infty$ which implies that $fx = f_1x$ and contradicts the assumption that f has no fixed point in ∂G. Hence, $i(f_1, G, X)$ is well defined, then we set

$$i(f, G, X) = i(f_1, G, X). \tag{5}$$

Notice that the definition of fixed point index in (5) is independent of the choice of f_1. Indeed let $f_2 : \overline{G} \to X$ a completely continuous mapping such that $f_2x = fx$ for all $x \in M_\infty \cap \overline{G}$ and consider $H : [0, 1] \times \overline{G} \to X$ the completely continuous mapping defined by $H(t, x) = tf_1(x) + (1 - t)f_2(x)$. It easy to see that $\forall(t, x) \in [0, 1] \times \partial G$, $H(t, x) \neq x$. Indeed, if this is not the case, that is, there exists $t_0 \in [0, 1]$ and $x_0 \in \partial G$ such that $x_0 = t_0f_1(x_0) + (1 - t_0)f_2(x_0)$, then by the convexity of M_∞ we have $x_0 \in M_\infty$ leading to the contradiction $f(x_0) = x_0$. Hence our claim follows from the homotopy property of fixed point index of completely continuous. The proof is complete. □

Consequently, we have: The following result extends the properties of the fixed point index for completely continuous mapping to that of Z_μ-contraction mapping.

Theorem 3.4. *The fixed point index for Z_μ or Z'_μ-contraction mapping defined in the Theorem 3.3 satisfies the following properties*

1. *Normalization: $i(f, G, X) = 1$, whenever $fx = x_0 \in G$.*

2. *Additivity:* $i(f,G,X) = i(f,G_1,X) + i(f,G_2,X)$ *where* G_i *are disjoint open subsets of* G *and* f *has no fixed point in* $G \setminus G_1 \cup G_2$.

3. *Excision: If* U *is an open subset of* G *such that* f *has no fixed point in* $G \setminus U$*, then we have* $i(f,G,X) = i(f,U,X)$.

4. *Homotopy property: Let* $H : [0,1] \times \overline{G} \to X$ *be a continuous function and assume that* H *is a* Z_μ*-contraction in the following sense*

$$\xi[\mu\big(f([0,1]\times B)\big), \mu(B)]\big] \geq 0, \quad for\, all\, B \subset \mathcal{B}(E) \cap 2^{\overline{G}},$$

then if $h(t,x) \neq x$ *for all* $t \in [0,1]$ *and* $x \in \partial G$ *we have that the fixed point index* $i(H,G,X)$ *is independent of* t.

5. *Solution property: If* $i(f,G,X) \neq 0$*, then* f *has at least one fixed point in* G.

Proof. By equality (5) there exists $f_1 : \overline{G} \to M_\infty$ a completely continuous mapping such that $i(f,G,X) = i(f_1,G,X)$, so the properties $(1),(2),(3)$ and (5) follow directly from the corresponding properties of the fixed point index for completely continuous mapping. Thus, it remains to prove the homotopy property. Let

$$K_1 = \overline{Con}[f([0,1]\times\overline{G})], \quad K_{n+1} = \overline{Con}\big[f\big([0,1]\times(\overline{G}\cap K_n)\big)\big], \qquad (6)$$

and let $K_\infty = \bigcap_{n\geq 1} K_n$, arguing as in the proof of Proposition 3.1, we show that K_∞ is a compact convex subset of X. Obviously, if $K_\infty = \emptyset$, then $i(H,G,X) = 0$. Suppose that $K_\infty \neq \emptyset$, then there exists a completely continuous mapping

$$h_1 : [0,1]\times\overline{G} \to K_\infty$$

such that $h_1(t,x) = h(t,x)$, for all $t \in [0,1]$ and $x \in K_\infty \cap \overline{G}$. Notice that $h_1(t,x)$ has no fixed point in $[0,1]\times\partial G$. Indeed, if there exists $t_0 \in [0,1]$ and $x_0 \in \partial G$ such that $h_1(t_0,x_0) = x_0$, then $x_0 \in K_\infty \cap \overline{G}$ leading to the contradiction $h(t_0,x_0) = x_0$. Therefore, the Homotopy property of fixed point index for completely continuous mapping leads to our claim. □

3.2 Fixed point index computation

In what follows, we let G a bounded open subset of K and $f : \overline{G} \to K$ is a Z_μ-contraction (or Z'_μ-contraction) mapping having no fixed point in ∂G.

Lemma 3.5. *Assume that* $0 \in G$ *and*

$$fx \neq \lambda x, \forall x \in K \cap \partial G,\ \forall\, \lambda \geq 1, \qquad (7)$$

then

$$i(f, \overline{G}, K) = 1.$$

Proof. Consider the homotopy $H : [0,1] \times (\overline{G}) \to K$ defined by $H(t,x) = tfx$. Notice that H is continuous, and Hypothesis (7) implies that H has no fixed point in $[0,1] \times \partial G$. Now, we need to prove that H is Z_μ contraction indeed. To this aim let $B \subset K \cap \overline{G}$ be a bounded set, we have then $H\big([0,1] \times B\big) \subset con\big(f(B) \cup \{0\}\big)$ is a bounded set and

$$\mu([0,1] \times B) \leq \mu\big[Con(f(B) \cup \{0\})\big] = \mu\big(f(B) \cup \{0\}\big) \quad = \mu(f(B)),$$

Taking in account that f is Z_μ-contraction (or Z_μ-contraction), we find

$$\xi(\mu(H([0,1] \times B), \mu(B)\big) \geq \xi\big(\mu(f(B)), \mu(B)\big) \geq 0.$$

Hence, $i(H, [0,1] \times G, K)$ is well defined and by the homotopy invariance and normality of the fixed point index, we obtain

$$i(f, G, K) = i(0, G, K) = 1.$$

□

As an immediate consequence of the above lemma, we have:

Corollary 3.6. *Assume that $0 \in G$ and f has no fixed point in ∂G. If either*

- *$fx \ngeq x$ for all $x \in \partial G$. or*
- *$\| fx \| \leq \| x \|$, for all $x \in \partial G$.*

Then $i(f, G, K) = 1$.

Lemma 3.7. *Assume that $0 \in G$ and the MNC μ satisfies Conditions (C_6) and (C_7) . If f has no a fixed point in ∂G and there exists $e > 0_E$ such that*

$$x \neq fx + te \text{ for all } t \geq 0 \text{ and all } \ x \in \partial G. \tag{8}$$

Then, $i(f, G, K) = 0$.

Proof. To the contrary, suppose $i(f, K \cap g, K) \neq 0$. Since $f(\overline{G})$ is bounded we can choose $\lambda_0 > 0$ such that

$$\lambda_0 > \| e \|^{-1} \sup\{\| x \| + \| fx \|\}. \tag{9}$$

Thus, let $H(t,x) = fx + \lambda_0 t\ e$ and notice that because K is convex we have $H(t,x) \in K$ for all $(t,x) \in [0,1] \times \overline{G}$. Since for all $B \in \mathcal{B}$

$$\begin{aligned} H([0,1] \times B) &= \big\{t(f(B) + \lambda_0 e) + (1-t)f(B), \quad t \in [0,1]\big\}, \\ &\subset Con\big(f(B) + \lambda_0 e \cup f(B)\big), \end{aligned}$$

we have

$$\begin{aligned} \mu(H([0,1] \times B)) &\leq \mu\big[Con\big(\{f(B) + \lambda_0 e\} \bigcup \{f(B)\}\big)\big], \\ &= \mu\big[\{f(B) + \lambda_0 e\} \bigcup \{f(B)\}\big], \\ &= \max\{\mu(f(B) + \lambda_0 e), \mu(f(B))\}, \end{aligned}$$

leading to $\mu(f(B)+\lambda_0 e) = \mu(f(B))$, and finally to $\mu(H([0,1]\times B)) \leq \mu(f(B))$. Thus, if f is Z_μ contraction, we get

$$\xi\big(\mu(H([0,1] \times B)), \mu(B)\big) = \xi\big(\mu(f(B)), \mu(B)\big) \geq 0.$$

and if f is Z'_μ contraction, we obtain

$$\xi\big(\mu(H([0,1] \times B)), \phi(\mu(B))\big) = \xi\big(\mu(f(B)), \phi(\mu(B))\big) \geq 0.$$

Proving that H is Z_μ contraction.

At the end, since (8) implies that $H(t,x) \neq x$ for all $x \in \partial G$ and $t \in [0,1]$, we obtain by using the invariance homotopy of the fixed point index

$$i(f + \lambda_0 e, G, K) = i(f, G, K) \neq 0.$$

Therefore there exits $x_0 \in G$ such that $x_0 = fx_0 + \lambda_0 e$ and this leads to the contradiction

$$\lambda_0 \leq \| e \|^{-1} \sup\{\| x \| + \| fx \|\}.$$

The proof is complete. □

As a consequence of the above lemma, we have:

Corollary 3.8. *Assume that* $0 \in G$ *and* f *has no fixed point in* ∂G*. If*

$$fx \nleq x \quad \forall x \in \partial G.$$

Then, $i(T, G, K) = 0$.

3.3 Fixed point theorems

The main goal of this subsection is to prove positive fixed point theorems for Z_u and Z'_u contraction mapping. The first result obtained is a Schauder-type fixed theorem, the second and the third results are variants of the cone compression and expansion principle, for more details on this principle, we refer to see [11], [11], [19]. All these results will be proved by means the fixed point index theory.

Theorem 3.9. *Let G be a closed convex subset of E and let $T : \overline{U} \to E$ be a continuous mapping, where $U \subset G$ be a bounded open subset. Assume that the mapping has no fixed point ∂U and T is a Z'_μ-contraction map such that there exists $\xi \in Z'$ satisfying*

$$\xi(\mu(T(B)), \mu(B)) \geq 0,$$

for any nonempty subset B of U, where μ is an arbitrary MNC. and without a fixed point in and $I - T$ is ψ expansive and $i(T, U, C) \neq 0$ then T has a fixed point in U.

Proof. We set for all $n \in N$, $T_n = (1 - \frac{1}{n})T$, then $T_n : \overline{U} \to X$ is Z_μ-contraction. Since $\sup_{x \in \overline{U}} ||Tx|| < \infty$ we have

$$||Tx - T_n x|| = ||Tx - (1 - \frac{1}{n})Tx|| = \frac{1}{n}||Tx|| \to 0 \text{ when } n \to \infty$$

Using the definition of limit, there is N, such that for every $n > N$

$$\| Tx - T_n x \| < \delta \text{ were } 0 < \delta < \inf_{x \in \partial U} \| x - Tx \|$$

By the Definition 2.22 of fixed point index we have

$$i(T, U, C) = i(T_n, U, C) \neq 0, \ \forall n > N. \tag{10}$$

Then, the solvability property of fixed point index for Z_μ-contraction map implies that

$$\forall n > N, \quad \exists x_n \in U, \quad x_n = T_n x_n.$$

Hence for every $n > N$

$$\| x_n - Tx_n \| = \| x_n - T_n x_n - Tx_n + T_n x_n \| = \| T_n x_n - Tx \| \to 0 (n \to \infty).$$

So (x_n) is an almost fixed point then from Lemma 2.24, T admit a fixed point. □

For the following fixed point theorems we assume that μ is a regular MNC, that is μ satisfies all the conditions $(C_1) - (C_7)$.

Theorem 3.10. *Let G_1, G_2 be two bounded and open subsets in E such that $0 \in G_1$ and $\overline{G_1} \subset G_2$, and let $f : K \cap \overline{G_2} \to K$ be a $Z_\mu$contraction (or Z'_μcontraction). If either*

(H_1) *$fx \ngeq x, \forall x \in K \cap \partial G_1$ and $fx \nleq x, \forall x \in K \cap \partial G_2$, or*

(H_2) *$fx \nleq x, \forall x \in K \cap \partial G_1$ and $fx \ngeq x, \forall x \in K \cap \partial G_2$,* *then f has at least one fixed point in $K \cap (\overline{G_2} \backslash G_1)$.*

Proof. We present the proof in the case where (H_1) holds, the other case is checked similarly. Suppose that (H_1) is satisfied, so by Corollaries 3.7 and 3.8 one has

$$i(f, G_1 \cap K, K) = 1 \text{ and } i(f, G_2 \cap K, K) = 0.$$

Then additivity property of the fixed point index implies

$$i(f, K \cap (G_2 \setminus G_1), K) = i(f, G_2 \cap K, K) - i(f, G_1 \cap K, K) = 0 - 1 = -1.$$

This means that f has at least one fixed point in $K \cap (\overline{G_2} \backslash G_1)$. □

In the following we give a fixed point result for Z_u or Z'_u contraction which is an analogue result given by the authors in Theorem 3.8 (see [2]) and Theorem 2.3 (see [8]). Let us recall first the following result, known as Krein-Rutman theorem.

Lemma 3.11. *([19]) Assume that K is a total cone in E and let $L : E \to E$ be a compact linear operator with $L(K) \subset K$ having a positive spectral radius $r(L)$. Then there exists $e \in K$ and $e \neq 0_E$ such that $Le = r(L)e$.*

The statement of the following theorem needs to introduce the following notations Let ζ be the set of all positive compact linear operators L having a positive spectral radius, and denote by K_r the set $K \cap B_E(0_E, r)$ where $B(0_E, r)$ is the open ball of radius r and centered at 0_E and by ∂K_r its boundary. Next, we .

Theorem 3.12. *Assume that the cone K is normal and total and let $f : K \to K$ be Z_u or Z'_u contraction mapping satisfying*

$$fu \leq \frac{Lu}{A} + b, \quad \forall u \in K, \tag{11}$$

$$fu \geq \frac{Mu}{r(M)} \quad \forall u \in \partial K_\rho, \tag{12}$$

where $M, L \in \zeta$, $b \in K$ and A, ρ are two positive reals numbers with $A > r(L)$. Then f has a fixed point $u \in K \setminus \{0_E\}$.

Proof. At first Lemma 3.11 ensures the existence of $e > 0_E$ such that $Me = r(M)e$. Then arguing as in the proof of Theorem 2.3 in [8] we get that for all $\lambda \geq 0$ and for all $u \in K_\rho$ we have $u \neq fu + \lambda e$. Thus, using Lemma 3.7 we get that

$$i(f, K_\rho, K) = 0. \tag{13}$$

Again, arguing as in the proof of Theorem 3.8 in [2], we show that there exists a R large such that $fu \neq \lambda u$ for all $u \in \partial K_R$ and $\lambda \geq 1$. Hence, by lemma 3.5 we have

$$i(f, K_R, K) = 1.$$

By the additivity property of the fixed point index we conclude that

$$i(f, K \cap B(0,R) \setminus \overline{B(0,\rho)}, K) = i(f, K_R, K) - i(f, K_\rho, K) = 1 - 0 = 1$$

and f has a fixed point u such that $\rho \leq \| u \| \leq R$. □

4 Example of Application

By means of Theorem 3.12, we solve in this section the integral equation

$$u(t) = \int_0^1 K(t,s) f(s, u(s)) ds + g(t, u(t)), \tag{14}$$

where the Kernel $K : [0,1] \times [0,1] \to \mathbb{R}^+$ is continuous and does not vanish identically and $f, g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions. Here, $\mathbb{R}^+$ denotes the interval $[0, +\infty)$

In all what follows we assume that the functions f and g satisfy the following conditions.

$$(H_1) \begin{cases} \text{there is two continuous and nondecreasing functions } \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+ \\ \text{such that } \phi^{-1}(0) = \psi^{-1}(0) = 0 \text{ and } , \psi\big(|g(t,y) - g(t,x)\big)| \leq \phi\big(|y-x|\big), \\ \text{with } \phi(t) < \psi(t), \forall t \in \mathbb{R}^+. \end{cases}$$

Let E denotes the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm $\|\cdot\|$, and let Q be the cone in E of all nonnegative functions.

Let $T, N, L : E \to E$ the operators defined on E by :

$$Nu(t) = \int_0^1 K(t,s) f(s,u(s)) ds, \;\; Lu(t) = \int_0^1 K(t,s) u(s) ds, \;\; Gu(t) = g(t, u(t))$$

and

$$Tu(t) = Nu(t) + Gu(t).$$

Obviously, N is completely continuous and G is a $\phi\psi-$ contractive mapping, i.e.

$$\psi\big(\|G(x) - G(y)\|\big) \leq \phi\big(\|x - y\|\big), \quad for\, all\, x, y \in E. \tag{15}$$

Lemma 4.1. *(Proposition 3.3 [4]) Let Ω be a nonempty closed, and bounded subset of a Banach algebra X with $0 \in \Omega$ and let $A, C : X \to X$ and $B : \Omega \to X$ three operators such that* *(a) A and C are respectively ψ, ϕ_A and $\psi - \phi_C$ contractive mappings, where $\phi_A, \phi_C, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous and nondecreasing functions with ψ semi-linear and invertible.*

(b) B is completely continuous.

(c)

$$\phi(r) = M\phi_A(r) + \phi_C(r) < r \leq \psi(r), \quad \forall r > 0 \tag{16}$$

where $M = \sup\{\|B(x)\|, x \in G\}$.

Then the mapping $T = A \cdot B + C : \Omega \to X$ satisfies

$$\psi(\alpha(T(U))) \leq \phi(\alpha(U))$$

for all bounded subset $U \subset \Omega$ and α denotes the Kuratowskii MNC.

Consider the function $\xi(t,s) = \phi(t) - \psi(s)$, $s, t \in \mathbb{R}^+$ with $\phi(t) < t \leq \psi(t)$, we get that ξ is a simulation maps so the operator T is a $Z_\alpha-$contraction mapping where α is Kuratovski's measure of non-compactness.

Lemma 4.2 (Theorem 4.1, [8])**.** *Suppose that*

(H_2) there is a subset J of the interval $[0,1]$ with positive measure such that $\inf_{t \in J} K(t,s) = c(s) > 0$.

Then, the linear operator L is positive and compact having a positive spectral radius.

Set

$$f_\infty = \lim\sup_{u \to \infty} \left(\sup_{s \in [0,1]} \frac{f(s,u)}{u} \right) \quad f_0 = \lim\inf_{u \to 0} \left(\inf_{s \in [0,1]} \frac{f(s,u)}{u} \right).$$

Theorem 4.3. *Assume that Hypotheses (H_1) and (H_2) hold and*

$$f_\infty < \frac{1}{r(L)} < f_0. \tag{17}$$

Then Equation (14) admits at least one positive solution.

Proof. First, since it is a sum of a compact operator N and $Z_\alpha-$contraction mapping, by using Lemma 4.1, it is easy to see that the operator T is a $Z_\alpha-$contraction mapping on Q.

Then, it follows from (17) that there exists $\delta > 0$ such that for all $u \in [0, \delta]$ we have $f(t,u) > \frac{u}{r(L)}$. Therefore, for any $\rho \in (0, \delta)$ and all $u \in Q \cap \partial B(0, \rho)$, we have

$$Tu = Nu + G(u) \geq Nu \geq \frac{Lu}{r(L)}. \tag{18}$$

•

Also, for $\epsilon > 0$ with $f_\infty + \epsilon \leq \frac{1}{r(L)}$, we conclude from (17) that there exists $C_1 > 0$ such that for all $u \geq 0$ we have

$$f(t,u) \leq (f_\infty + \epsilon)u + C_1. \tag{19}$$

Finally, from (19) we obtain that for all $u \in Q$

$$\begin{aligned} Tu(t) &= \textstyle\int_0^1 K(t,s)f(t,u(t))ds + g(t,u(t)), \\ &\leq (f_\infty + \epsilon)Lu + \textstyle\int_0^1 K(t,s)C_1 ds + g(t,0) + |g(t,u(t)) - g(t,0)|, \\ &\leq (f_\infty + \epsilon)Lu + \textstyle\int_0^1 K(t,s)C_1 ds + g(t,0) + \psi^{-1}(\phi(|u(t)|)), \\ &\leq b(t) + (f_\infty + \epsilon)Lu, \end{aligned}$$

where

$$b(t) = g(t,0) + \int_0^1 K(t,s)C_1 ds + \sup_{t\in[0,1]} \psi^{-1}(\phi(u(t))).$$

Therefore we have

$$Tu \leq \frac{Lu}{A} + b \quad \text{for all } u \in Q. \tag{20}$$

where $A = (f^\infty + \epsilon)^{-1}$. [1]

Thus, All hypotheses of Theorem 3.12 hold and Equation (14) admits a positive solution. □

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