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Framed general helix and framed ζ_3-slant helix in $\mathbb{R}^4$

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Abstract

In this paper, we focus on general and ζ_3-slant helices with any singular points in four-dimensional Euclidean space, which are called framed general and ζ_3-slant helices, respectively. Then, we state and prove the conditions of necessity and sufficiency for any framed curves to be general helices or ζ_3-slant helices in $\mathbb{R}^4$. Also, we give some characterizations for framed helices.

1 Introduction

It should be noted that the tangents at each point of the general helices form a constant angle with a constant vector. Such curves have always attracted attention since they exist in nature and have applications in various disciplines. Not only applications in real life but also elegant theoretical results are notable. In this regard, these types of curves have been investigated not only in 3-dimensional Euclidean space but also in higher-dimensional and/or non-Euclidean spaces. In the last quarter century, some new attempts have been seen to form helix-like curves based on the modified definitions of a constant vector that makes a constant angle with any vector in the Frenet frame instead of the tangent vector of the curve. Some analogous results have been obtained for these new curves and helices. This shows us that the subject is interesting. M. A. Lancret [7] asserted the main characterization of the general helices, and B. S. Venant [11] proved this. In a similar vein, S. Izumiya and N.

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Takeuchi [3] presented a new notion called slant helix such that the normals at each point of the curve have a constant angle with a constant vector. Within that period, A. Mağden obtained an integral characterization of the helix in $\mathbb{R}^4$ [8]. M. Önder et al. investigated a new variant of the slant helices in $\mathbb{R}^4$ [4]. In recent years, many researchers have studied helices with different aspects in different spaces [5, 6, 7, 10, 14].

The invention of the framed curves, introduced by M. Takahashi and S. Honda, attracted researchers since it solves the inefficiency of Frenet frames of the curves with singular points [2]. Subsequently, the rectifying curves with singular points in Euclidean $4-$space were analyzed by a relatively new frame called moving adapted frame presented by Y. Wang et al., [12]. Then, the adapted frame of framed curves was established, and the interrelations between the adapted frames and the Frenet-type frames of the framed curves were determined in Euclidean $4-$space by M. Akyiğit and Ö. G. Yıldız [1].

The main subjects of this paper are framed general and ζ_3-slant helices based on the adapted frames in Euclidean 4-space, and some characterizations based on the framed curvature functions.

2 Preliminaries

Initially, let us recall some main definitions related to the framed curves and their adapted frames in Euclidean 4-space (for further information see also [1, 9, 2, 12, 13, 15]).

Definition 2.1. Let $\Delta_3 = \left\{\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4\right\}$ be a 6-dimensional smooth manifold such that $\langle \mu_i, \mu_j \rangle = \delta_{ij}$, $i, j \in \{1, 2, 3\}$ and $v = \mu_1 \times \mu_2 \times \mu_3$ be a vector such that $\det(v, \mu_1, \mu_2, \mu_3) = 1$. Then $(\gamma, \mu) : I \to \mathbb{R}^4 \times \Delta_3$ is called framed curve provided that $\langle \gamma'(s), \mu_i(s) \rangle = 0$ for all $s \in I$ and $i \in \{1, 2, 3\}$. Additionally, $\gamma : I \to \mathbb{R}^4$ is also called a framed base curve provided that there exists $\mu : I \to \Delta_3$ where (γ, μ) is a framed curve, [2].

Let $\gamma(s)$ be a framed base curve and $\{v(s), \mu(s)\}$ be a moving frame along this curve, the Frenet-Serret type formula of γ is given by

$$\begin{bmatrix} {\mu_1}'(s) \\ {\mu_2}'(s) \\ {\mu_3}'(s) \\ v'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) & \kappa_3(s) \\ -\kappa_1(s) & 0 & \kappa_4(s) & \kappa_5(s) \\ -\kappa_2(s) & -\kappa_4(s) & 0 & \kappa_6(s) \\ -\kappa_3(s) & -\kappa_5(s) & -\kappa_6(s) & 0 \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \\ v(s) \end{bmatrix}.$$

Here $\kappa_1(s), \kappa_2(s), \kappa_3(s), \kappa_4(s), \kappa_5(s),$ and $\kappa_6(s)$ are any smooth curvature functions. $\gamma'(s) = \alpha(s)\, v(s)$ is considered by a smooth function $\alpha : I \to \mathbb{R}$. $(\kappa_1(s), \kappa_2(s), \kappa_3(s), \kappa_4(s), \kappa_5(s), \kappa_6(s), \alpha(s))$ are called curvatures of γ at $\gamma(s)$. $\alpha(s_0) = 0$ iff $\gamma(s_0)$ is a singular point of γ.

Theorem 2.2. Let $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve whose associated curvature is $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \alpha)$ [2].

Assume that $(\gamma, \mu) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed curve with $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \alpha)$. By using Euler angles, $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \Delta_3$ is given as follows

$$\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix} = \begin{bmatrix}\cos\Phi\cos\Psi & -\cos\Omega\sin\Psi + \sin\Omega\cos\Psi\sin\Phi & \sin\Omega\sin\Psi + \cos\Omega\cos\Psi\sin\Phi\\ \cos\Phi\sin\Psi & \cos\Omega\cos\Psi + \sin\Omega\sin\Psi\sin\Phi & -\sin\Omega\cos\Psi + \cos\Omega\sin\Psi\sin\Phi\\ -\sin\Phi & \sin\Omega\cos\Phi & \cos\Omega\cos\Phi\end{bmatrix}\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}.$$

Here Φ, Ω and Ψ are smooth functions. With simple calculations, there are the relations

$$\tilde{v} = \zeta_1 \times \zeta_2 \times \zeta_3 = \mu_1 \times \mu_2 \times \mu_3 = v.$$

Hence, $(\gamma, \zeta) :\to \mathbb{R}^4 \times \Delta_3$ is also a framed curve. Suppose that

$$\frac{\tan\Psi}{\cos\Phi} = \kappa_6 \sin\Omega - \kappa_5 \cos\Omega,$$

$$\kappa_3 = \cot\Phi\,(\kappa_6\cos\Omega + \kappa_5\sin\Omega)$$

hold for given smooth functions Φ, Ω and Ψ (These angles are called Euler angles.) and the adapted frame along $\gamma(s)$ is given as follows

$$\begin{bmatrix}v'(s)\\ \zeta'_1(s)\\ \zeta'_2(s)\\ \zeta'_3(s)\end{bmatrix} = \begin{bmatrix}0 & p(s) & 0 & 0\\ -p(s) & 0 & z(s) & 0\\ 0 & -z(s) & 0 & w(s)\\ 0 & 0 & -w(s) & 0\end{bmatrix}\begin{bmatrix}v(s)\\ \zeta_1(s)\\ \zeta_2(s)\\ \zeta_3(s)\end{bmatrix} \tag{1}$$

where $(p(s), z(s), w(s), \alpha(s))$ are framed curvature of $\gamma(s)$ and their expression are

$$\begin{aligned} p &= -\kappa_3\sec\Phi\sec\Psi,\\ z &= -(\kappa_4 - \Omega')\sin\Phi - \Psi',\\ w &= \frac{\cos\Phi}{\cos\Psi}(\kappa_4 - \Omega') \end{aligned}$$

and the following equalities

$$\kappa_1 = -\sin\Omega\left(\Phi' - w\sin\Psi\right),$$
$$\kappa_2 = -\cos\Omega\left(\Phi' - w\sin\Psi\right),$$
$$\kappa_4 = w\frac{\cos\Psi}{\cos\Phi} + \Phi'$$

are satisfied. v, ζ_1, ζ_2 , and ζ_3 are, respectively, called generalized tangent, principal normal, first binormal, and second binormal vectors of the framed curve [1].

3 Framed General Helix in Euclidean 4-Space

In this section, we define framed general helix and give the necessary and sufficient conditions for framed curves to be general helices.

Definition 3.1. The framed curve (γ, ζ) is called framed general helix if its generalized tangent vector v makes a constant angle Φ with a fixed direction N; that is $\langle v, N\rangle = \cos\Phi$ is a constant along the framed base curve.

Teorem 3.2. Let $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with nonzero generalized curvatures p, z, w. Then (γ, ζ) is a framed general helix iff the following equation holds,

$$\frac{p^2}{z^2} + \left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)^2 = \tan^2\Phi, \tag{2}$$

where Φ is the constant angle between the generalized tangent vector v and the constant unit vector N.

Proof. Let's assume that $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed general helix with nonzero curvatures and its axis be the unit vector N. In that case, we know that $\langle v, N\rangle =$ constant along the framed base curve. If we differentiate this equation and use (1), we find

$$p\langle \zeta_1, N\rangle = 0.$$

From there, the unit vector N can be chosen to be follows

$$N = a_1 v + a_2\zeta_2 + a_3\zeta_3, \tag{3}$$

where $a_1 = \langle v, N\rangle =$ constant, $a_2 = \langle \zeta_2, N\rangle$, $a_3 = \langle \zeta_3, N\rangle$, and ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$ since N is a unit vector. The differentiation of (3) gives

$$N' = (a_1 p - a_2 z)\,\zeta_1 + ({a_2}' - a_3 w)\,\zeta_2 + (a_2 w + {a_3}')\,\zeta_3 = 0, \tag{4}$$

and from (4), we find

$$a_2 = \frac{a_1}{z}p = -\frac{-{a_3}'}{w}, \tag{5}$$

$${a_2}' = a_3 w. \tag{6}$$

By using the differentiation of (5) in (6), we find the ODE for a_3 as follows

$${a_3}'' - \frac{w'}{w}{a_3}' + w^2 a_3 = 0. \tag{7}$$

If we change variables in (7) as $u = \int\limits_0^s w ds$, then the equation becomes

$$\frac{d^2 a_3}{du^2} + a_3 = 0. \tag{8}$$

For some constants K and L, the solution of (8) is

$$a_3 = K\cos u\,(s) + L\sin u\,(s)\,. \tag{9}$$

From equations (5) and (9), we obtain

$$\begin{aligned} a_2 &= \frac{p}{z}a_1 = K\sin u\,(s) - L\cos u\,(s)\,, \\ a_3 &= \left(\frac{p}{z}\right)' a_1 \frac{1}{w} = K\cos u\,(s) + L\sin u\,(s)\,. \end{aligned} \tag{10}$$

By straightforward calculations on (10), we obtain

$$K^2 + L^2 = {a_1}^2\left[\left(\frac{p}{z}\right)^2 + \left(\left(\frac{p}{z}\right)'\frac{1}{w}\right)^2\right].$$

Hence, we have

$$\left(\frac{p}{z}\right)^2 + \left(\left(\frac{p}{z}\right)'\frac{1}{w}\right)^2 = \text{constant}.$$

Contrariwise, let's assume that equation (2) is satisfied. In this case, there can always be a constant vector N such that $\langle v, N\rangle =$ constant. Let's take this constant vector as

$$N = v + \frac{p}{z}\zeta_2 + \frac{1}{w}\left(\frac{p}{z}\right)'\zeta_3. \tag{11}$$

By substituting the differentiation of (2) in (11), we obtain $N' = 0$. So, N is a unit constant vector. Consequently, (γ, ζ) is a framed general helix.

Theorem 3.3. Let $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with nonzero generalized curvatures p, z, w. (γ, ζ) is a framed general helix iff there exists a C^2-function h satisfying

$$wh(s) = \left(\frac{p}{z}\right)', \; h'(s) = -w\left(\frac{p}{z}\right). \tag{12}$$

Proof. Assume that $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed general helix with nonzero generalized curvatures. The differentiation of (2) gives

$$\left(\frac{p}{z}\right)\left(\frac{p}{z}\right)' + \left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)\left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)' = 0.$$

Hence, we obtain

$$\frac{1}{w}\left(\frac{p}{z}\right)' = -\frac{\frac{p}{z}\left(\frac{p}{z}\right)'}{\left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)'}. \tag{13}$$

If we call the right side of equality with a function h as follows

$$h(s) = -\frac{\frac{p}{z}\left(\frac{p}{z}\right)'}{\left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)'},$$

then, (13) becomes

$$\frac{1}{w}\left(\frac{p}{z}\right)' = h(s). \tag{14}$$

From (13), we obtain

$$\left(\frac{1}{w}\left(\frac{p}{z}\right)'\right)' = -\left(\frac{p}{z}\right)w. \tag{15}$$

If we substitute (14) in (15), we obtain

$$h'(s) = -\left(\frac{p}{z}\right)w. \tag{16}$$

Conversely, if the equation (12) holds, we can write a unit constant vector N such that

$$N = v + \frac{p}{z}\zeta_2 + h\zeta_3.$$

Then, $\langle v, N\rangle = 1$. Consequently (γ, ζ) is a framed general helix.

Theorem 3.4. Let $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with nonzero generalized curvatures p, z, w. (γ, ζ) is a general helix iff

$$\frac{p}{z} = C_1 \cos u + C_2 \sin u \tag{17}$$

is satisfied where C_1, C_2 are constants and $u(s) = \int\limits_0^s w ds$.

Proof. Assume that $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed general helix with nonzero curvatures. By using Theorem 3.2, we can define the C^2-function $u(s)$ and the C^1-functions $\rho_1(s)$ and $\rho_2(s)$ as follows

$$u(s) = \int\limits_0^s w ds, \tag{18}$$

$$\rho_1 = \frac{p}{z} \cos u - h(s) \sin u, \tag{19}$$

$$\rho_2 = \frac{p}{z} \sin u + h(s) \cos u. \tag{20}$$

If we differentiate (19) and (20) with respect to s and consider (18), (14), and (16), then we find $\rho_1{}' = \rho_2{}' = 0$. So, $\rho_1 = C_1$ and $\rho_2 = C_2$ are constants. Hence, from (19) and (20), we obtain

$$C_1 \cos u + C_2 \sin u = \frac{p}{z}.$$

Conversely, assume that (17) is satisfied. Then from (19) and (20), we have

$$C_2 \cos u - C_1 \sin u = h(s),$$

which proves Theorem 3.3. Consequently (γ, ζ) is a framed general helix .

4 Framed ζ_3-Slant Helix in Euclidean 4-Space

In this section, we define framed ζ_3-slant helix and give the necessary and sufficient conditions for framed curves to be ζ_3-slant helices.

Definition 4.1. The framed curve (γ, ζ) is called framed ζ_3-slant helix if its generalized second binormal vector ζ_3 makes a constant angle Φ with a fixed direction N, i.e. $\langle \zeta_3, N \rangle = \cos\Phi$ is a constant along the framed base curve.

Theorem 4.2. Let $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with nonzero generalized curvatures p, z, w. Then (γ, ζ) is a ζ_3-slant helix iff the following equation holds,

$$\left(\frac{w}{z}\right)^2 + \left(\frac{1}{p}\left(\frac{w}{z}\right)'\right)^2 = \tan^2\Phi, \tag{21}$$

where Φ is the constant angle between the generalized second binormal vector ζ_3 and a constant unit vector N.

Proof. Let $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ be a framed ζ_3-slant helix with nonzero curvatures and its axis be the unit vector N. Then,

$$\langle \zeta_3, N \rangle = \cos\Phi = \text{constant} \tag{22}$$

along the framed base curve. By differentiating (22) and using (1), we find

$$-w\langle \zeta_2, N \rangle = 0.$$

From there, the unit vector N can be chosen to be follows

$$N = a_1 v + a_2\zeta_1 + a_3\zeta_3, \tag{23}$$

where $a_1 = \langle v, N \rangle$, $a_2 = \langle \zeta_1, N \rangle$, and $a_3 = \langle \zeta_3, N \rangle =$ constant. Seeing that N is a unit vector, we find ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$. The differentiation of (23) gives

$$N' = ({a_1}' - a_2 p)\, v + (a_1 p + {a_2}')\, \zeta_1 + (a_2 z - a_3 w)\, \zeta_2 + {a_3}'\zeta_3 = 0, \tag{24}$$

and from (24), we find

$$a_2 = \frac{1}{p}\frac{da_1}{ds} = a_3\frac{w}{z}, \tag{25}$$

$$\frac{da_2}{ds} = -a_1 p. \tag{26}$$

By using the differentiation of (25) in (26), we find the ODE for a_1 as follows

$${a_1}'' - \frac{p'}{p}{a_1}' - a_1 p^2 = 0. \tag{27}$$

If we change the variable in (27) as $u = \int\limits_0^s p\,(s)\, ds$, then the equation becomes

$$\frac{d^2 a_1}{du^2} + a_1 = 0.$$

The solution of the above equation is

$$a_1 = K \cos u(s) + L \sin u(s), \tag{28}$$

where K and L are constants. By substituting (28) in (25) and (26), we find

$$\begin{aligned} a_2 &= \frac{w}{z} a_3 = -K \sin u(s) + L \cos u(s), \\ a_1 &= -\frac{1}{p}\left(\frac{w}{z}\right)' a_3 = K \cos u(s) + L \sin u(s). \end{aligned} \tag{29}$$

By straightforward calculations on (29), we obtain

$$K^2 + L^2 = \left(\frac{w}{z}\right)^2 + \left(\left(\frac{w}{z}\right)' \frac{1}{p}\right)^2.$$

Hence, we have

$$\left(\frac{w}{z}\right)^2 + \left(\left(\frac{w}{z}\right)' \frac{1}{p}\right)^2 = \text{constant}.$$

Contrariwise, let's assume that equation (21) is satisfied. In this case, there can always be a constant vector N such that $\langle \zeta_3, N \rangle = \text{constant}$.

Let's take this constant vector as

$$N = -\frac{1}{p}\left(\frac{w}{z}\right)' v + \frac{w}{z}\zeta_1 + \zeta_3. \tag{30}$$

By considering the differentiation of (21) and (30), we obtain $N' = 0$. So, N is a constant vector. Consequently, (γ, ζ) is a framed ζ_3-slant helix.

Theorem 4.3. A framed curve $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ with nonzero generalized curvatures p, z, w. (γ, ζ) is a framed ζ_3-slant helix iff there exists a C^2-function such that the following equations

$$h(s)\, p = \left(\frac{w}{z}\right)', \; h'(s) = -p\frac{w}{z}$$

hold.

Proof. Assume that $(\gamma, \zeta) : I \to \mathbb{R}^4 \times \Delta_3$ is a framed ζ_3-slant helix with nonzero generalized curvatures. The differentiation of (21) gives

$$\left(\frac{w}{z}\right)\left(\frac{w}{z}\right)' + \frac{1}{p}\left(\frac{w}{z}\right)'\left(\frac{1}{p}\left(\frac{w}{z}\right)'\right)' = 0,$$

and with the help of the last equation, we obtain

$$\frac{1}{p}\left(\frac{w}{z}\right)' = -\frac{\left(\frac{w}{z}\right)\left(\frac{w}{z}\right)'}{\left(\frac{1}{p}\left(\frac{w}{z}\right)'\right)'}. \tag{31}$$

If we call the right side of the equality with a function h as

$$h(s) = -\frac{\left(\frac{w}{z}\right)\left(\frac{w}{z}\right)'}{\left(\frac{1}{p}\left(\frac{w}{z}\right)'\right)'},$$

then, (31) becomes

$$\left(\frac{w}{z}\right)' = p\,h(s). \tag{32}$$

From (31), we obtain

$$\left(\frac{1}{p}\left(\frac{w}{z}\right)'\right)' = -p\frac{w}{z}. \tag{33}$$

By using equation (32) in (33), we find

$$h'(s) = -p\frac{w}{z}. \tag{34}$$

Contrariwise, let's assume that equation (32) is satisfied. Then the unit vector N can be written as

$$N = \cos\Phi\left(-h(s)\,v + \frac{w}{z}\zeta_1 + \zeta_3\right).$$

Then, we find that $\langle\zeta_3, N\rangle = \cos\Phi$ is a constant. Consequently, (γ,ζ) is a framed ζ_3-slant helix.

Theorem 4.4. A framed curve $(\gamma,\zeta): I \to \mathbb{R}^4 \times \Delta_3$ with nonzero generalized curvatures p, z, w is a framed ζ_3-slant helix iff the following equation is satisfied

$$\frac{w}{z} = C_1\cos\beta + C_2\sin\beta, \tag{35}$$

where C_1, C_2 are constants and $\beta = \int\limits_0^s p(s)\,ds$.

Proof. Assume that the framed curve $(\gamma,\zeta): I \to \mathbb{R}^4 \times \Delta_3$ with nonzero curvatures is a framed ζ_3-slant helix. By using Theorem 4.2, we can define the C^2-function $\beta(s)$ and the C^1-functions $\rho_1(s)$ and $\rho_2(s)$ as follows

$$\beta(s) = \int_0^s p(s)\, ds, \tag{36}$$

$$\rho_1(s) = \frac{w}{z}\cos\beta - h(s)\sin\beta, \tag{37}$$

$$\rho_2(s) = \frac{w}{z}\sin\beta + h(s)\cos\beta. \tag{38}$$

If we differentiate (37) and (38) with respect to s and consider (32), (34), and (36), then we find ${\rho_1}' = {\rho_2}' = 0$. So, $\rho_1(s) = C_1,\ \rho_2(s) = C_2$ are constants. Hence, from (37) and (38), we obtain

$$C_1\cos\beta + C_2\sin\beta = \frac{w}{z}.$$

Conversely, assume that (35) is satisfied. Then from (37) and (38), we get

$$h = -C_1\sin\beta + C_2\cos\beta,$$

which proves Theorem 4.3. Then, (γ, ζ) is a framed ζ_3-slant helix.

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