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On Spatial Quaternionic b-lift Curves

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Abstract

This study is based on the discovered relationships between the quaternionic slant helix and the quaternionic general helix. In this direction, we first examined quaternions, spatial quaternionic curves and b-lift curves. Furthermore, we defined the spatial quaternionic b-lift curve and characterized of Frenet fields. Afterward, we found the curvatures of the b-lift curve and using them we obtained a result between the quaternionic slant helix and quaternionic general helix. Finally, we consolidated our results with an example and visualized our curves with the MATLAB program.

1 Introduction

In literature, quaternions were first described by Irish mathematician W. R. Hamilton in [1] and generalized complex numbers to 3-dimensional space. Looking at the algebraic structure, quaternions differ from complex numbers in that they are not commutative according to the defined multiplication operation. In today's world, quaternions are very useful in the representation of rotational motion, so they have applications such as robotics, analysis of DNA structure, astrophysics, navigation systems, etc.

The theory of curves is one of the most fundamental topics in differential geometry. The Serret-Frenet formulas are expressed using the derivative in the theory of curves. The natural lift curve described in Thorpe's book "Elementary Topics in Differential Geometry" in 1979. Thorpe obtained [2] this curve

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by combining the endpoints of the tangent vector at each point of a curve. In $\mathbb{R}^3$, the properties and Frenet apparatus of natural lift curve were examined in [18, 19]. Curves are characterized by the state of the curvature κ and the torsion τ. In 1802, M. A. Lancret [21] proved that for a curve to be a general helix:

$$\frac{\tau}{\kappa} = constant.$$

Many authors have studied helices and different types of helices. S. Izumiya and N. Takeuchi defined slant helices as the principal normal vector makes a constant angle with a fixed direction. They also proved that the necessary and sufficient condition for a curve to be a slant helix is the following relation [20]:

$$\frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}}(\frac{\tau}{\kappa})' = constant.$$

K. Bharathi and M. Nagaraj described [3] the quaternionic curve in 3-dimensional and 4-dimensional spaces and they introduced the Frenet apparatus of this curve. Afterward, differential geometry of quaternions and quaternionic curves attracted the attention of many authors. Some of these studies are [4–13]. In [5], Kocayigit and Pekacar studied the characterizations of quaternionic slant helices in E^3 and E^4. In [6], Sahiner discovered the Frenet frames of the spatial quaternionic direction curve. In [7], Senyurt et al. calculated the principal curvature and torsion of the spatial quaternionic involute curve according to unit Darboux vector and normal vector ot the Smarandache curve.

The main plan of this study is to define the spatial quaternionic b-lift curves and to obtain the Frenet operators of this curve. In this direction, the main lines of the study are as follows: In Section 2, we give some basic definitions and theorems about spatial quaternions and b-lift curves. In Section 3, we denote spatial quaternionic b-lift curve and examine their properties. In the last section, we discuss the results shortly.

2 Preliminaries

The set of quaternions is

$$\mathbb{H} = \{q = a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

This set is a 4-dimensional vector space on $\mathbb{R}$. A quaternion can be written as [16]:

$$q = S_q + V_q,$$

where $S_q = a \in \mathbb{R}$ is scalar part and $V_q = bi + cj + dk \in \mathbb{R}^3$ is vector part of q respectively. The conjugate of a quaternion defined as [16]:

$$\overline{q} = S_q - V_q.$$

The quaternion inner product can be defined as [16]:

$$h : \mathbb{H} \times \mathbb{H} \to \mathbb{R}, \; h(q,p) = \frac{1}{2}(q \times \overline{p} + p \times \overline{q}).$$

The norm of q is as follows [16]:

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

If $q + \overline{q} = 0$ equality is satisfied, then the quaternion q is called the spatial quaternion. For any given quaternions $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$, the addition is [16]:

$$q_1 + q_2 = S_{q_1+q_2} + V_{q_1+q_2}$$

and the product of q_1 and q_1 defined as [16]:

$$\begin{aligned} q_1 \times q_2 &= (a_1 + b_1 i + c_1 j + d_1 k) \times (a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (c_1 d_2 + b_1 a_2 - d_1 c_2 + c_1 d_2) i \\ &+ (c_1 a_2 + a_1 c_2 - b_1 d_2 + d_1 b_2) j + (d_1 a_2 + a_1 d_2 - c_1 b_2 + b_1 c_2) k. \end{aligned}$$

Definition 2.1. Let $\gamma : I \to \mathrm{M} \subset \mathbb{R}^3$ be a unit speed curve, then $\gamma_b : I \to \mathrm{TM}$ is called the b-lift curve and ensures the following equation:

$$\gamma_b(s) = (\gamma(s), b(s)) = b(s)|_{\gamma(s)}.$$

where b is the binormal vector of the curve γ [17].

Definition 2.2. The 3-dimensional Euclidean space can be defined with the space of spatial quaternions $\mathbb{Q}_{\mathbb{H}} = \{q \in \mathbb{H} : q + \overline{q} = 0\}$ in an obvious manner. Let $I = [0,1]$ be an interval in the real line $\mathbb{R}$ and $s \in I$ be a parameter along the curve

$$\gamma : [0,1] \to \mathbb{Q}_{\mathbb{H}}, \; \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i \quad (1 \leq i \leq 3),$$

then the curve γ is called as a spatial quaternionic curve, where $\gamma^{'} = t$ is unit tangent vector , i.e. $\|t\| = 1$ for all $s \in I$ [3].

Let γ be a differentiable spatial quaternions curve with arc-length parameter s. Then the Frenet vectors and curvatures of the curve $\gamma(s)$ can be given, respectively, as follows [3]:

$$t(s) = \gamma^{'}(s),\ n(s) = \frac{\gamma^{''}(s)}{\|\gamma''(s)\|},\ b(s) = t(s) \times n(s)$$

and

$$\kappa(s) = \|\gamma^{'}(s) \times \gamma^{''}(s)\|,\ \tau(s) = \frac{h(\gamma^{'}(s) \times \gamma^{''}(s), \gamma^{'''}(s))}{\|\gamma'(s) \times \gamma''(s)\|^2}$$

Additionally, the following relationship holds [4]:

$$\begin{aligned} t(s) \times t(s) = n(s) \times n(s) = b(s) \times b(s) = -1, \\ t(s) \times n(s) = b(s) = -n(s) \times t(s), \\ n(s) \times b(s) = t(s) = -b(s) \times n(s), \\ b(s) \times t(s) = n(s) = -t(s) \times b(s). \end{aligned}$$

Let γ be a unit speed spatial quaternionic curve in $\mathbb{H}$ and $\{t(s), n(s), b(s)\}$ be the Frenet frame of the curve γ . Then the Frenet equations are given as [3]:

$$\begin{aligned} t^{'}(s) &= \kappa(s)n(s) \\ n^{'}(s) &= -\kappa(s)t(s) + \tau(s)b(s) \\ b^{'}(s) &= -\tau(s)n(s) \end{aligned}$$

where κ and τ are curvature and torsion of the curve γ, respectively.

Definition 2.3. A spatial quaternionic curve γ is called a spatial quaternionic helix if its unit tangent vector t makes a constant angle with a fixed unit quaternion U [4].

Theorem 2.4. *Let γ be a spatial quaternionic curve with nonzero curvatures. Then γ is a spatial quaternionic helix if and only if the following applies [4]:*

$$\frac{\tau}{\kappa} = constant.$$

Definition 2.5. A spatial quaternionic curve γ is called the spatial quaternionic slant helix if its unit normal vector n makes a constant angle with a fixed unit quaternion U [4].

Theorem 2.6. *Let γ be a spatial quaternionic curve with nonzero curvatures. Then γ is a spatial quaternionic slant helix if and only if the following applies [4]:*

$$\frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'} = constant.$$

3 On Spatial Quaternionic b-lift Curves

In this section, we describe the spatial quaternionic b-lift curve and we also obtain the Frenet operators of these curves. Moreover, we examine the case of the spatial quaternionic b-lift curve according to whether the main curve is slant helix and we give an example of these situations.

Definition 3.1. For any unit speed spatial quaternionic curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$, $\gamma_b : I \subset \mathbb{R} \to \mathbb{R}^3$ is called spatial quaternionic b-lift curve of γ which provides the following equation:

$$\gamma_b(s) = (\gamma(s), b(s)) = b(s)|_{\gamma(s)} \tag{3.1}$$

where b is the binormal vector of the curve γ.

Theorem 3.2. *Let γ_b be the b-lift curve of a spatial quaternionic curve γ. Then the following equations are provided:*

$$\begin{aligned} t_b(s) &= -n(s), \\ n_b(s) &= \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}} t(s) - \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}} b(s), \\ b_b(s) &= \frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}} t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}} b(s) \end{aligned}$$

where $\{t(s), n(s), b(s)\}$ and $\{t_b(s), n_b(s), b_b(s)\}$ are the Frenet vectors of the curve γ and its b-lift curve, respectively. Furthermore, κ is the curvature, τ is the torsion of the curve γ. (Specially torsion will be taken greater than zero.)

Proof. Let γ_b be b-lift curve of the spatial quaternionic γ, then we can write:

$$\gamma_b = b \quad , \quad \gamma_b' = -\tau n$$

$$t_b(s) = \frac{\gamma_b'}{\|\gamma_b'\|} = \frac{-\tau n}{|\tau|} = -n \quad (\tau > 0), \tag{3.2}$$

$$\gamma_b'' = -\tau' n - \tau(-\kappa t + \tau b),$$

$$\gamma_b'' = \kappa\tau t - \tau' n - \tau^2 b, \tag{3.3}$$

$$\gamma_b' \times \gamma_b'' = \tau^3 t + \kappa\tau^2 b, \tag{3.4}$$

$$\|\gamma_b^{'} \times \gamma_b^{''}\| \quad = \quad \tau^2\sqrt{\kappa^2+\tau^2}. \tag{3.5}$$

From $b_b(s) = \frac{\gamma_b^{'} \times \gamma_b^{''}}{\|\gamma_b^{'} \times \gamma_b^{''}\|}$, the equations (3.4) and (3.5) we get

$$b_b(s) \quad = \quad \frac{\tau}{\sqrt{\kappa^2+\tau^2}}t + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}b. \tag{3.6}$$

Using (3.2) and (3.6), we have

$$n_b(s) \quad = \quad b_b(s) \times t_b(s) = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}t - \frac{\tau}{\sqrt{\kappa^2+\tau^2}}b. \tag{3.7}$$

From the equations (3.2), (3.6) and (3.7), the proof is completed. □

Theorem 3.3. *Let γ_b be a b-lift curve of a spatial quaternionic curve γ. Then, we have the following formulas:*

$$\kappa_b(s) = \frac{\sqrt{\kappa^2+\tau^2}}{\tau} \quad , \quad \tau_b(s) = \frac{\kappa^{'}\tau - \kappa\tau^{'}}{\tau(\kappa^2+\tau^2)},$$

where κ_b and τ_b are curvature and torsion of γ_b, respectively.

Proof. From (3.5), we know

$$\|\gamma_b^{'} \times \gamma_b^{''}\| \quad = \quad \tau\sqrt{\kappa^2+\tau^2} \quad , \quad \|\gamma_b^{'}\| = \tau. \tag{3.8}$$

Since $\kappa_b = \frac{\|\gamma_b^{'} \times \gamma_b^{''}\|}{\|\gamma_b^{'}\|^3}$ is provided, we obtain the following equation:

$$\kappa_b(s) = \frac{\sqrt{\kappa^2+\tau^2}}{\tau}. \tag{3.9}$$

The torsion of γ_b is given as

$$\tau_b = \frac{h(\gamma_b^{'} \times \gamma_b^{''}, \gamma_b^{'''})}{\|\gamma_b^{'} \times \gamma_b^{''}\|^2}. \tag{3.10}$$

Using (3.3), we get

$$\gamma_b^{'''} \quad = \quad (\kappa^{'}\tau + 2\kappa\tau^{'})t + (\kappa^2\tau - \tau^{''} + \tau^3)n - 3\tau\tau^{'}b. \tag{3.11}$$

From (3.4), (3.7) and (3.11), we have

$$\tau_b(s) = \frac{\kappa^{'}\tau - \kappa\tau^{'}}{\tau(\kappa^2+\tau^2)}. \tag{3.12}$$

□

Theorem 3.4. *$\gamma : I \to \mathbb{R}^3$ is a spatial quaternionic slant helix if and only if γ_b is a spatial quaternionic general helix.*

Proof. Assume that γ is a spatial quaternionic slant helix. Then, we have

$$\sigma(s) = \frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'(s) = constant,$$

where κ and τ are curvature and torsion of the curve γ. We have to show if γ_b is a spatial quaternionic general helix. From last theorem, we can have

$$\frac{\tau_b}{\kappa_b} = -\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'(s) = -\sigma(s) = constant. \tag{3.13}$$

Then, γ_b is a spatial quaternionic general helix. Conversely, let γ_b be a spatial quaternionic general helix. Then we can write

$$\frac{\tau_b}{\kappa_b} = -\sigma(s) = constant.$$

Since σ(s)=constant, the curve γ is a spatial quaternionic slant helix. □

Example. Assume that the spatial quaternionic slant helix is given as

$$\gamma(s) = (\frac{1}{6}sin2s + \frac{2}{3}sins)e_1 + (\frac{1}{6}cos2s + \frac{2}{3}coss)e_2 + \frac{4\sqrt{2}}{3}cos\frac{s}{2}e_3.$$

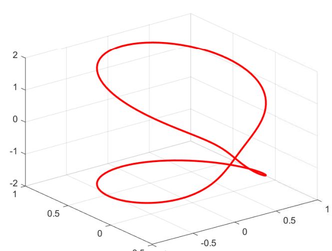

Figure 1: The spatial quaternionic slant helix $\gamma(s)$

Then the Frenet vectors of the curve γ are given as follows:

$$\begin{aligned}
t(s) &= (\frac{1}{3}cos2s + \frac{2}{3}coss, -\frac{1}{3}sin2s - \frac{2}{3}sins, -\frac{2\sqrt{2}}{3}sin\frac{s}{2}), \\
n(s) &= (-\frac{4}{3\sqrt{2}}sin\frac{3s}{2}, -\frac{4}{3\sqrt{2}}cos\frac{3s}{2}, -\frac{1}{3}), \\
b(s) &= (-\frac{1}{3}sin2s + \frac{2}{3}sins, -\frac{1}{3}cos2s + \frac{2}{3}coss, -\frac{4}{3\sqrt{2}}cos\frac{s}{2}).
\end{aligned}$$

Since γ_b(s) = b(s), we have the following equations:

$$\begin{aligned}
\gamma_b^{'}(s) \times \gamma_b^{''}(s) &= \left(\frac{2\sqrt{2}}{3} sin^2 \frac{s}{2} sin \frac{3s}{2}, \frac{2\sqrt{2}}{3} sin^2 \frac{s}{2} cos \frac{3s}{2}, -\frac{8}{3} sin^2 \frac{s}{2}\right), \\
\kappa_b(s) &= \frac{\|\gamma_b^{'} \times \gamma_b^{''}\|}{\|\gamma_b^{'}\|^3} = \frac{1}{sin\frac{s}{2}}, \\
\tau_b(s) &= \frac{< \gamma_b^{'} \times \gamma_b^{''}, \gamma_b^{'''} >}{\|\gamma_b^{'} \times \gamma_b^{''}\|^2} = -\frac{1}{2\sqrt{2}} \frac{1}{sin\frac{s}{2}}.
\end{aligned}$$

Therefore, we obtain

$$\frac{\tau_b}{\kappa_b} = -\frac{1}{2\sqrt{2}} = constant.$$

Since the ratio of the curvatures are constant, the curve γ_b is a spatial quaternionic general helix.

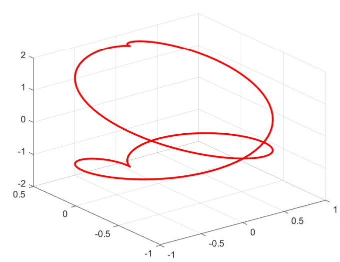

Figure 2: The spatial quaternionic general helix $\gamma_b(s)$

4 Conclusions

We can briefly summarize the results obtained in this study as follows:

1. We denoted the spatial quaternionic b-lift curve using the binormal vector of the principal curve.

2. With the help of the Frenet frames and curvatures of the main curve, we discoreved the Frenet apparatus of the b-lift curve.

3. Using quaternionic slant helix and quaternionic general helix theorems, we proved that the necessary and sufficient condition for the principal curve

to be a quaternionic slant helix is that the b-lift curve must be a quaternionic general helix.

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