DOI: 10.2478/auom-2023-0042
An. Şt. Univ. Ovidius Constanţa **Vol. 31(3),2023, 245–253**

A characterization of operators via Berezin symbol and related questions

Ulaş Yamancı

Abstract

In this paper, we characterize the hyponormal operators with regard to Berezin symbol and reproducing kernel. Also, we demonstrate several Berezin number inequalities for bounded linear operators.

1 Introduction

Recall that a reproducing kernel Hilbert space, denoted by RKHS, is the Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ comprised of complex-valued functions on a set Θ with properties that:

(a) the point evaluation at $\mu \in \Theta$ is a continuous linear functional;

(b) there is $g_\mu \in \mathcal{H}$ for any $\mu \in \Theta$ with property that $g_\mu(\mu) \neq 0$.

The classical Riesz representation theorem certifies that there is only an element $k_\mu \in \mathcal{H}$ for each $\mu \in \Theta$ with property that $\langle g, k_\mu \rangle = g(\mu)$ for every $g \in \mathcal{H}$. The element k_μ is said to be a reproducing kernel of the RKHS $\mathcal{H}$ at μ. If a sequence $\{e_n\}_{n\geq 0}$ is an orthonormal basis of the space $\mathcal{H}(\Theta)$, then k_μ is calculated by (see [1, 13])

$$k_\mu(z) = \sum_{n=0}^{\infty} \overline{e_n(\mu)} e_n(z)$$

Key Words: Berezin number, Berezin norm, Berezin symbol, hyponormal operator, reproducing kernel Hilbert space, inequalities.
2010 Mathematics Subject Classification: Primary 47A63, Secondary 47B32.
Received: 07.09.2022
Accepted: 22.01.2023

The normalized reproducing kernel at μ is represented by $\widehat{k}_\mu = \dfrac{k_\mu}{\|k_\mu\|_{\mathcal{H}}}$ (we definitely have $k_\mu \neq 0$ from (b)). The Berezin symbol for a bounded linear operator A on the RKHS $\mathcal{H}$, denoted by $\widetilde{A}$, is a function such that (see [2]) for $\mu \in \Theta$

$$\widetilde{A}(\mu) := \left\langle A\widehat{k}_\mu, \widehat{k}_\mu \right\rangle_{\mathcal{H}}.$$

Since

$$\left|\widetilde{A}(\mu)\right| \leq \left\|A\widehat{k}_\mu\right\| \left\|\widehat{k}_\mu\right\| \leq \|A\|,$$

the Berezin symbol $\widetilde{A}$ is a bounded function by norm of the operator. The Berezin symbol provides us remarkable information about operators such as uniqueness of operators and boundary behavior of operators. That is, $A_1 = A_2$ if and only if $\widetilde{A}_1 = \widetilde{A}_2$. More informations about Berezin symbols and its applications, can be found in Garayev et al. [7], Saltan [14], and Yamancı [18].

Karaev in [9, 10] defined the Berezin number and the Berezin set, respectively

$$ber\,(A) := \sup_{\mu\in\Theta} \left|\widetilde{A}\,(\mu)\right|$$

and

$$Ber\,(A) := \left\{\widetilde{A}\,(\mu) : \mu \in \Theta\right\}.$$

The $ber\,(A)$ holds the following features:

(a) $ber\,(\beta A) = |\beta|\, ber\,(A)$ for every $\beta \in \mathbb{C}$.

(b) $ber\,(A) \leq \|A\|$.

(c) $ber\,(A+B) \leq ber\,(A) + ber\,(B)$.

The Berezin number of an operator does not normally define a norm. But, $ber\,(A)$ defines a norm on $\mathcal{B}\,(\mathcal{H}\,(\mathbb{D}))$ when $\mathcal{H}$ is a RKHS of analytic functions, (such as on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) (see, for example, Zhu [19]).

The Berezin norm of the operator A on the RKHS $\mathcal{H} = \mathcal{H}(\Theta)$ is given by formula

$$\|A\|_{ber} = \sup_{\mu\in\Theta} \left\|A\widehat{k}_\mu\right\|_{\mathcal{H}}.$$

Obviously, $\|A\|_{ber}$ holds the features (a)-(c) with $ber\,(A)$. Moreover, $\|A\|_{ber} = 0$ if and only if $A = 0$ because the family $\{k_\mu : \mu \in \Theta\}$ is complete in $\mathcal{H}$. So, all of these features imply that $\|A\|_{ber}$ is a norm in $\mathcal{B}\,(\mathcal{H})$. Obviously, for any $A \in \mathcal{B}\,(\mathcal{H})$, $ber\,(A) \leq \|A\|_{ber}$.

Recently, by using the well-known inequalities and discrete Hardy-Hilbert type inequalities, remarkable consequences for the Berezin number inequalities were obtained in [3, 6, 8, 12, 15, 16, 17].

Note that the numerical radius and the numerical range of the operator A, represented by $w(A)$ and $W(A)$, are defined by respectively

$$w(A) := \sup_{\|g\|_{\mathcal{H}}=1} |\langle Ag, g\rangle| .$$

and

$$W(A) := \{\langle Ag, g\rangle : \|g\|_{\mathcal{H}} = 1\}$$

One can easily see that

$$ber(A) \leq w(A) \leq \|A\| \text{ and } Ber(A) \subset W(A)$$

for any $A \in \mathcal{B}(\mathcal{H})$.

In this paper, we characterize the hyponormal operators with regard to Berezin symbol and reproducing kernel. Also, we demonstrate several Berezin number inequalities for bounded linear operators.

2 Berezin symbol and hyponormal operator

Following result gives the characterization of hyponormal operators via reproducing kernel and Berezin symbol. Recall that A is a hyponormal operator if $A^*A \geq AA^*$.

Theorem 1. *Let A be a bounded operator on a RKHS $\mathcal{H}(\Theta)$. Then A is a hyponormal operator if and only if*

$$\inf_{\phi\in\mathbb{C}} \left\| A\widehat{k}_\mu - \phi\widehat{k}_\mu \right\| \geq \inf_{\phi\in\mathbb{C}} \left\| A^*\widehat{k}_\mu - \overline{\phi}\widehat{k}_\mu \right\|$$

for all $\mu \in \Theta$.

Proof. From property of $\widetilde{A}(\mu)\widehat{k}_\mu \perp A\widehat{k}_\mu - \widetilde{A}(\mu)\widehat{k}_\mu$, we get

$$\left\| A\widehat{k}_\mu - \widetilde{A}(\mu)\widehat{k}_\mu \right\|^2 + \left| \widetilde{A}(\mu) \right|^2 = \left\| A\widehat{k}_\mu \right\|^2 \tag{1}$$

for all $\mu \in \Theta$. Using the Lemma 2.3 in [5], we obtain from (1)

$$\left\| A\widehat{k}_\mu - \widetilde{A}(\mu)\widehat{k}_\mu \right\| = \inf_{\phi\in\mathbb{C}} \left\| A\widehat{k}_\mu - \phi\widehat{k}_\mu \right\|$$

and

$$\left\| A^*\widehat{k}_\mu - \widetilde{A^*}(\mu)\widehat{k}_\mu \right\| = \inf_{\phi\in\mathbb{C}} \left\| A^*\widehat{k}_\mu - \overline{\phi}\widehat{k}_\mu \right\|$$

for all $\mu \in \Theta$. Since $\left\|A\widehat{k}_\mu\right\|^2 = \widetilde{A^*A}(\mu)$ and $\left\|A^*\widehat{k}_\mu\right\|^2 = \widetilde{AA^*}(\mu)$, we immediately have from above equalities that

$$\inf_{\phi\in\mathbb{C}} \left\|A\widehat{k}_\mu - \phi\widehat{k}_\mu\right\|^2 + \left|\widetilde{A}(\mu)\right|^2 = \widetilde{A^*A}(\mu)$$

and

$$\inf_{\phi\in\mathbb{C}} \left\|A^*\widehat{k}_\mu - \overline{\phi}\widehat{k}_\mu\right\|^2 + \left|\widetilde{A}(\mu)\right|^2 = \widetilde{AA^*}(\mu).$$

Since the operator is uniquely determined by the Berezin symbol, we have from the last two formulas $A^*A \geq AA^*$ if and only if

$$\inf_{\phi\in\mathbb{C}} \left\|A\widehat{k}_\mu - \phi\widehat{k}_\mu\right\| \geq \inf_{\phi\in\mathbb{C}} \left\|A^*\widehat{k}_\mu - \overline{\phi}\widehat{k}_\mu\right\|$$

which gives the desired result. □

3 Estimations for Berezin number of operators

The following lemmas help us to prove our main result:

Lemma 1 ([11])**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then, for unit vector $x \in \mathcal{H}$, the operator Jensen's inequality*

$$\langle Ax, x\rangle^k \leq \left\langle A^k x, x\right\rangle, k \geq 1$$

and

$$\left\langle A^k x, x\right\rangle \leq \langle Ax, x\rangle^k, k \in [0,1]$$

Now, we demonstrate several Berezin number inequalities for bounded linear operators. Note that the below inequality is known as the Buzano inequality (see [4]).

Lemma 2. *Let $a, b, e \in \mathcal{H}$. Then*

$$\frac{1}{2}\left(|\langle a,b\rangle| + \|a\|\,\|b\|\right) \geq |\langle a,e\rangle\,\langle e,b\rangle|,$$

where $\|e\| = 1$.

Theorem 2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$ber^2(A) \leq \frac{1}{2}\left[\|A\|^2\left(\min_{k\in[0,1]} \|kA^*A + (1-k)AA^*\|_{ber}\right) \right.$$
$$\left. + ber^2\left(A^2\right) + ber\left(A^2\right)\|A^*A + AA^*\|\right]^{\frac{1}{2}}_{ber}.$$

Proof. Utilising the Lemma 1, we get

$$\begin{aligned}
&\left|\widetilde{A}(\mu)\right|^2 \\
&=\left|\widetilde{A}(\mu)\right|\left|\widetilde{A}(\mu)\right| \\
&\leq \frac{1}{2}\left(\left|\left\langle A\widehat{k}_\mu, A^*\widehat{k}_\mu\right\rangle\right| + \left\|A\widehat{k}_\mu\right\|\left\|A^*\widehat{k}_\mu\right\|\right) \\
&=\frac{1}{2}\Bigg(\left|\left\langle A\widehat{k}_\mu, A^*\widehat{k}_\mu\right\rangle\right|^2 + \left\|A\widehat{k}_\mu\right\|^2\left\|A^*\widehat{k}_\mu\right\|^2 \\
&+2\left|\left\langle A\widehat{k}_\mu, A^*\widehat{k}_\mu\right\rangle\right|\left\|A\widehat{k}_\mu\right\|\left\|A^*\widehat{k}_\mu\right\|\Bigg)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg(\left|\widetilde{A^2}(\mu)\right|^2 + \left\langle A^*A\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\left\langle AA^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \\
&+\left|\widetilde{A^2}(\mu)\right|\left\langle\left(AA^*+A^*A\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\Bigg)^{\frac{1}{2}} \\
&=\frac{1}{2}(\left|\widetilde{A^2}(\mu)\right|^2 + \left\langle A^*A\widehat{k}_\mu, \widehat{k}_\mu\right\rangle^k.\left\langle AA^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle^{1-k}. \\
&\left\langle A^*A\widehat{k}_\mu, \widehat{k}_\mu\right\rangle^{1-k}.\left\langle AA^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle^k \\
&+\left|\widetilde{A^2}(\mu)\right|\left\langle\left(AA^*+A^*A\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg(\left|\widetilde{A^2}(\mu)\right|^2 + \left\langle\left(kA^*A+(1-k)AA^*\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\left\langle\left((1-k)A^*A+kAA^*\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \\
&+\left|\widetilde{A^2}(\mu)\right|\left\langle\left(AA^*+A^*A\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\Bigg)^{\frac{1}{2}} \text{ (by the McCarthy inequality)} \\
&\leq \frac{1}{2}\Bigg(\left|\widetilde{A^2}(\mu)\right|^2 + \left\|kA^*A+(1-k)AA^*\right\|_{ber}\left\|(1-k)A^*A+kAA^*\right\|_{ber} \\
&+\left|\widetilde{A^2}(\mu)\right|\left\langle\left(AA^*+A^*A\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\Bigg)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg(\left|\widetilde{A^2}(\mu)\right|^2 + \left\|kA^*A+(1-k)AA^*\right\|_{ber}\left\|A\right\|_{ber}^2 \\
&+\left|\widetilde{A^2}(\mu)\right|\left\langle\left(AA^*+A^*A\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\Bigg)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Big(ber^2\left(A^2\right) + \left\|kA^*A+(1-k)AA^*\right\|_{ber}\left\|A\right\|_{ber}^2 \\
&+ber\left(A^2\right)\left\|AA^*+A^*A\right\|_{ber}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Taking supremum over $\mu \in \Theta$, we have

$$ber^2(A) \leq \frac{1}{2}\left[\|A\|^2 \left(\|kA^*A + (1-k)AA^*\|_{ber}\right) + ber^2\left(A^2\right) + ber\left(A^2\right)\|A^*A + AA^*\|\right]_{ber}^{\frac{1}{2}}$$

for all $k \in [0,1]$. So, considering minimum over $k \in [0,1]$, we get

$$ber^2(A) \leq \frac{1}{2}\left[\|A\|^2 \left(\min_{k\in[0,1]} \|kA^*A + (1-k)AA^*\|_{ber}\right) + ber^2\left(A^2\right) + ber\left(A^2\right)\|A^*A + AA^*\|\right]_{ber}^{\frac{1}{2}}.$$

as desired. □

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$4ber^4(A) \leq ber^2\left(A^2\right) + \frac{1}{4}\left\|(A^*A)^2 + (AA^*)^2\right\|_{ber} + \frac{1}{2}ber\left(A^*A^2A^*\right) + ber\left(A^2\right)\|A^*A + AA^*\|_{ber}$$

Proof. Let $\widehat{k}_\mu$ be a reproducing kernel at $\mu \in \Theta$. Then we get from Lemma 1 that

$$\begin{aligned}
&\widetilde{A^*A}(\mu)\,\widetilde{AA^*}(\mu) \\
&= \widetilde{A^*A}(\mu)\left\langle \widehat{k}_\mu, AA^*\widehat{k}_\mu\right\rangle \\
&\leq \frac{1}{2}\left(\left\|A^*A\widehat{k}_\mu\right\|_{ber}\left\|AA^*\widehat{k}_\mu\right\|_{ber} + \left|\left\langle AA^*\widehat{k}_\mu, A^*A\widehat{k}_\mu\right\rangle\right|\right) \\
&\leq \frac{1}{4}\left(\left\|A^*A\widehat{k}_\mu\right\|_{ber}^2 + \left\|AA^*\widehat{k}_\mu\right\|_{ber}^2\right) + \frac{1}{2}\left|\left\langle A^*A^2A^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| \\
&= \frac{1}{4}\left\langle\left((A^*A)^2 + (AA^*)^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \frac{1}{2}\left|\left\langle A^*A^2A^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| \\
&\leq \frac{1}{4}\left\|(A^*A)^2 + (AA^*)^2\right\|_{ber} + \frac{1}{2}ber\left(A^*A^2A^*\right).
\end{aligned}$$

Following the proof of above Theorem, we get that

$$\begin{aligned}
&\left|\widetilde{A}(\mu)\right|^4 \\
&\leq \frac{1}{4}\left(\left|\widetilde{A^2}(\mu)\right|^2 + \widetilde{A^*A}(\mu)\,\widetilde{AA^*}(\mu) + \left|\widetilde{A^2}(\mu)\right|\left\langle (AA^* + A^*A)\,\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right) \\
&\leq \frac{1}{4}\left(ber^2\left(A^2\right) + \widetilde{A^*A}(\mu)\,\widetilde{AA^*}(\mu) + ber\left(A^2\right)\|A^*A + AA^*\|_{ber}\right) \\
&\leq \frac{1}{4}\left[ber^2\left(A^2\right) + \frac{1}{4}\left\|(A^*A)^2 + (AA^*)^2\right\|\right. \\
&\left. + \frac{1}{2}ber\left(A^*A^2A^*\right) + ber\left(A^2\right)\|A^*A + AA^*\|_{ber}\right]
\end{aligned}$$

for all $\mu \in \Theta$. Taking supremum over $\mu \in \Theta$, we arrive at desired consequence. □

Acknowledgement. The author thanks the referee and editor for their precious comments and suggestions.

References

[1] N. Aronzajn, Theory of reproducing kernels, *Transactions of the American Mathematical Society,* **68**(1950), 337-404.

[2] F.A. Berezin, Covariant and contravariant symbols for operators, *Mathematics of the USSR-Izvestiya,* **6**(1972): 1117-1151.

[3] M. Bakherad M, M. Hajmohamadi, R. Lashkaripour, S. Sahoo, Some extensions of Berezin number inequalities on operators, *Rocky Mountain Journal of Mathematics*, **51**(2021), 1941-1951.

[4] M.L. Buzano, Generalizzazione della disiguaglianza di Cauchy-Schwaz, *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, **31**(1971), 405-409 (in Italian).

[5] S.S. Dragomir, Some Grüss type inequalities in inner product spaces, *J. Inequal. Pure Appl. Math.*, 4(2)(2003), Article 42.

[6] M. Hajmohamadi, R. Lashkaripour and M. Bakherad, Improvements of Berezin number inequalities, *Linear and Multilinear Algebra*, **68**(2020), 1218-1229.

[7] M.T. Garayev, H. Guediri, H. Sadraoui, Applications of reproducing kernels and Berezin symbols, *New York J. Math.*, **22**(2016), 583–604.

[8] M.T. Garayev, M.W. Alomari , Inequalities for the Berezin number of operators and related questions, *Complex Analysis and Operator Theory*, **15**(2021), 1-30.

[9] M.T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, *Journal of Functional Analysis*, **238**(2006), 181-192.

[10] M.T. Karaev, Reproducing kernels and Berezin symbols techniques in various questions of operator theory, *Complex Analysis and Operator Theory*, **7**(2013), 983-1018.

[11] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publications of the Research Institute for Mathematical Sciences*, **24**(1988), 283-293.

[12] S. Sahoo, N. Das and N.C. Rout, On Berezin number inequalities for operator matrices, *Acta Mathematica Sinica English Series*, **37**(2021), 873-892.

[13] S. Saitoh and Y. Sawano, *Theory of Reproducing Kernels and Applications*, Singapore: Springer, 2016.

[14] S. Saltan, On the weak limit of compact operators on the reproducing kernel Hilbert space and related questions, *Analele Universitatii Ovidius Constanta. Seria Matematica*, **24**(2016), 253-260.

[15] R. Tapdigoglu, New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space, O*perators and Matrices*, **15**(2021), 1031-1043.

[16] U. Yamancı, R. Tunç and M. Gürdal, Berezin number, Grüss-type inequalities and their applications, *Bulletin of the Malaysian Mathematical Sciences Society*, **43**(2020), 2287-2296.

[17] U. Yamancı and İ.M. Karlı, Further refinements of the Berezin number inequalities on operators, *Linear and Multilinear Algebra*, **70**(2022), 5237-5246.

[18] U. Yamancı, On the summability methods of logarithmic type and the Berezin symbol, *Turkish Journal of Mathematics*, **42**(2018), 2417-2422.

[19] K. Zhu, *Operator theory in function spaces*. Second ed., Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, Providence, R.I. 2007.

Ulaş Yamancı,
Department of Statistics,
Suleyman Demirel University,
32260, Isparta, Turkey.
Email: ulasyamanci@sdu.edu.tr