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A New Filled Function for Global Optimization

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Abstract

The filled function method has recently become very popular in optimization theory, as it is an efficient and effective method for finding the global minimizer of multimodal functions. However, the fact that the existing filled functions in the literature generally have exponential or logarithmic terms and/or parameter sensitivity reduces the effectiveness of this method. In this study, we propose a new non parameter and without exponential/logarithmic terms filled function, which is numerically stable, and is successfully used to solve global optimization problems. Furthermore, we have demonstrated how successful this new filled function method in terms of efficiency with numerical experiments and comparisons.

1 Introduction

Today's technology and science age is witnessing that many problems in real-world can be modeled as optimization problems. Especially, thanks to the rapidly developing theoretical and algorithmic infrastructure of global optimization, the many problems that require globally optimal solutions in science and engineering can be solved ([3], [6], [7], [11] [17]). In recent years, some effective methods are proposed in global optimization theory. In the literature, it is known how effective methods such as Newton's method, conjugate gradient method, and the steepest regular method in solving optimization

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problems involving only one local minimum (see for details [4], [5]). However, these methods do not work for problems with more than one local minimum. Because in optimization problems involving more than one local minimum, existing methods can get stuck in any existing local minimum or find a local minimum worse than the existing local minimum. In this sense, in solving global optimization problems involving multiple local minima, two major questions must be answered;

1) How to escape current local local minimum?

2) Is the current optimum solution global or not?

In order to deal with the fundamental difficulties in these questions, Ge [8] proposed the filled function method for the first time.

In general, there are mainly two approaches to solve global optimization problems; probabilistic and deterministic approaches. The filled function method, which can be considered an effective deterministic approach in the literature due to is easy to construct and convergence is faster than other methods, have some disadvantages and shortcomings. It is generally known the most of the existing conventional filled function methods have computational weaknesses because the sensitivity to parameters and the exponential/logarithmic term. For example, the first filled function defined by Ge in the equation (2.3) has some disadvantages as it contains parameters that are very difficult to set for any unconstrained global optimization problem and it causes overflows in calculations due to the exponential term it contains. In this sense, there are many valuable studies in the literature in order to improve the filled function method and make it more efficient.

In this work, our main motivation is to give a new filled function without parameter, exponential-logarithmic terms, which is numerically stable, and is successfully used to solve global optimization problems.

2 Preliminaries

An unconstrained global minimization problem can be briefly represented following:

$$\begin{array}{ll} \min & g(\xi) \\ s.t. & \xi \in \mathbb{R}^n \end{array}, \tag{2.1}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Suppose that $g(\xi)$ is global convex, that is $g(\xi) \to +\infty$, when $\|\xi\| \to +\infty$. Therefore, there is a closed and bounded region $\Omega \subset \mathbb{R}^n$ containing all the minimizers of $g(x)$. Otherwise, unbounded global optimization problem over an unbounded region would not be solved. Thus, the problem (2.1) can be reduced as

$$\begin{array}{ll} \min & g(\xi) \\ s.t. & \xi \in \Omega \end{array} . \tag{2.2}$$

For the solution of the unconstrained global optimization problem in (2.2), a method called " filled function method " was developed by Ge in 1987 (see for details [8],[9]). The first filled function to be used in this method was given as in (2.3);

$$F(\xi, \xi_k^*, \lambda, \mu) = \frac{1}{\lambda + g(\xi)} \exp\left(-\frac{\|\xi - \xi_k^*\|}{\mu^2}\right). \tag{2.3}$$

However, it is quite difficult to adjust the parameters of this filled function according to the problem (2.2). In this sense, in order to eliminate this disadvantage and to improve this method, researchers have not only defined new filled functions (see [12], [15], [16]), but also updated and revised the Definition 2.1. In the literature, the most widely used filled function definition is given by Yang and Shang in [18] as follows:

Definition 2.1. Suppose that ξ_k^* is a local minimizer of an objective function $g(\xi)$. The filled function $F : \mathbb{R}^n \to \mathbb{R}$ is a function (called filled function of $g(\xi)$ at ξ_k^*), if following three conditions hold:

(1) ξ_k^* is a strictly local maximizer of $F(\xi, \xi_k^*)$,

(2) For any $\xi \in \Omega_1$, $\nabla F(\xi, \xi_k^*) \neq 0$ in the region

$$\Omega_1 = \{\xi \in \Omega : g(\xi_k^*) \leq g(\xi),\ \xi \in \Omega - \{\xi_k^*\}\},$$

(3) If ξ_k^* is not a global minizer and $\Omega_2 = \{\xi \in \Omega : g(\xi) < g(\xi_k^*)\} \neq \emptyset$, then there exist a point $\xi_k' \in \Omega_2$ such that ξ_k' is a local minimum of $F(\xi, \xi_k^*)$.

In order to understand the operating logic of the filled function (method) finding the global minimizer, the general strategy of the filled function (**ff**) method can be briefly summarized with the following.

Filled Function Algorithm;

Step 1 : Find a local minimizer ξ_k^* of the objective function $g(\xi)$,

Step 2 : Construct the filled function $F(\xi, \xi_k^*)$ that takes its maximum value at ξ_k^*,

Step 3 : Find a local minimizer ξ_k' of the filled function ,

Step 4 : If $g(\xi_k') \leq g(\xi_k^*)$, then go to *Step 1* and take ξ_k' as a initial minimizer of $g(\xi)$. Otherwise ξ_k^* is a global minimum point of $g(\xi)$.

It is clear that in the (**ff**) algorithm, the (**ff**) plays a vital role in obtaining the global minimizer of the objective function. So, it is very important to define effective and efficient filled functions. For example, the (**ff**) based on one parameter that can be adjusted according to the global optimization problem is defined in [10] as follows

$$F(\xi, \xi_k^*, \lambda) = -\left(g(\xi) - g(\xi_k^*)\right) \exp\left[\lambda \|\xi - \xi_k^*\|^2\right].$$

However, the global minimizer of objective function may be lost since the case of λ being too large lead to overflow in the calculations. To overcome this shortcoming based on exponential terms, a new (**ff**) without exponential terms is given as

$$F(\xi, \xi_k^*, \lambda) = -\frac{1}{In\left[1 - g(\xi) - g(\xi_k^*)\right]} - \lambda \|\xi - \xi_k^*\|^2$$

in [14]. But, this filled function is not defined at ξ such that $g(\xi) \leq 1 - g(\xi_k^*)$. This limitation can decrease the efficient of this function.

In light of the aforementioned disadvantages and limitations, our main motivation in this article is to define a new (**ff**) to solve an unconstrained global optimization problem. The numerical comparisons we made on several test examples in Section 5 reveal that the proposed (**ff**) is more efficient than some of the most frequently cited filled functions in the literature.

3 A New Auxiliary Function and Its Properties

To define a new (**ff**), we first introduce the following univariate function based on the bezier curve $v : \mathbb{R} \to \mathbb{R}$ is defined by

$$v(t) = \begin{cases} 1 & , \quad t \leq -2\varepsilon \\ \dfrac{(t+\varepsilon)^2}{\varepsilon^3}\left(3\varepsilon + 2(t+\varepsilon)\right) & , \quad -2\varepsilon \leq t < -\varepsilon \\ \dfrac{(t+\varepsilon)^2}{\varepsilon^3}\left(3\varepsilon - 2(t+\varepsilon)\right) & , \quad -\varepsilon \leq t < 0 \\ 1 & , \quad t \geq 0 \end{cases},$$

where ε is a positive real constant very close to zero ($v(t, \varepsilon)$ notation will be used instead of $v(t)$ in the next part of the article to emphasize "ε").

Now, we define the new filled function $S : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with regard to $v(t, \varepsilon)$

$$S(\xi, \xi_1^*) = c\left(-\frac{2}{\pi}\arctan\|\xi - \xi_1^*\|^2 + 1\right) v(\xi, \varepsilon), \quad (\ c \text{ is positive constant }), \tag{3.1}$$

where the set Ω is the domain of objective function g including all minima, and ξ_1^* is the existing local minimizer of $g(\xi)$.

In the next theorems, it will be proved that the function $S(\xi, \xi_1^*)$ satisfies the conditions given in Definition 2.1, and that it is a (**ff**).

Theorem 3.1. *Let ξ_1^* be a local minimizer of the objective function $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Then $S(\xi, \xi_1^*)$ has a strictly local maximizer at the ξ_1^*.*

Proof. Since g has a minimizer at ξ_1^*, there exists a neighborhood $N(\xi_1^*, \varepsilon)$ with $\varepsilon > 0$ such that $g(\xi) \geq g(\xi_1^*)$ for each x in the neighborhood $N(\xi_1^*, \varepsilon)$. So, for the same neighborhood $N(\xi_1^*, \varepsilon)$, and $\xi \neq \xi_1^*$, it is obtained that

$$S(\xi, \xi_1^*) = c\left(-\frac{2}{\pi}\arctan\|\xi - \xi_1^*\|^2 + 1\right) \leq S(\xi_1^*, \xi_1^*) = c.$$

This means ξ_1^* is strictly local maximizer of $S(\xi, \xi_1^*)$. □

Theorem 3.2. *If $g(\xi) - g(\xi_1^*) > 0$, then $\nabla S(\xi, \xi_1^*) \neq 0$. That is, in higher basins of $S(\xi, \xi_1^*)$ have neither a stationary point nor a minimizer.*

Proof. Let $g(\xi) - g(\xi_1^*) > 0$ and $\xi \neq \xi_1^*$. Then it is obtained that

$$\nabla S(\xi, \xi_1^*) = -\frac{4c}{\pi}\frac{\xi - \xi_1^*}{1 + \|\xi - \xi_1^*\|^4} \neq 0.$$

This proves the theorem. □

Theorem 3.3. *Let $S_2 = \{\xi \in \Omega \subset \mathbb{R}^n : g(\xi) - g(\xi_1^*) < 0\}$ be not empty set. Then the function $S(\xi, \xi_1^*)$ has a minimizer $\widetilde{\xi}$ in the set S_2.*

Proof. Let $|g(\xi) - g(\xi_1^*)| = \varepsilon < \min_{i \neq j}\left|g(\xi_i^*) - g\left(\xi_j^*\right)\right|$, where $g(\xi_i^*)$, $g\left(\xi_j^*\right)$ are local minimizers of g with different values. We know that $v(t, \varepsilon)$ has a minimizer at $t = -\varepsilon$ and $v(-\varepsilon, \varepsilon) = 0$. Let $\widetilde{\xi}$ be any point such that $g\left(\widetilde{\xi}\right) - g(\xi_1^*) = -\varepsilon$. Then, it is obtained that

$$\nabla S\left(\widetilde{\xi}, \xi_1^*\right) = \left(-\frac{4c}{\pi}\frac{\widetilde{\xi}-\xi_1^*}{1+\left\|\widetilde{\xi}-\xi_1^*\right\|^4}\right) v\left(g\left(\widetilde{\xi}\right)-g(\xi_1^*)\right) + v'\left(g\left(\widetilde{\xi}\right)-g(\xi_1^*)\right)\nabla g\left(\widetilde{\xi}\right) = 0.$$

So, the point $\widetilde{\xi}$ is a stationary point of the function $S(\xi, \xi_1^*)$. We also have $S\left(\widetilde{\xi}, \xi_1^*\right) = 0$. On the other hand, since $S(\xi, \xi_1^*) \geq 0$ for each $\xi \in \Omega$. This implies that there exist some $\varepsilon > 0$ such that $S\left(\widetilde{\xi}, \xi_1^*\right) \leq S(\xi, \xi_1^*)$ for $\xi \in N\left(\widetilde{\xi}, \varepsilon\right)$, where $N\left(\widetilde{\xi}, \varepsilon\right)$ is an $\varepsilon-$ neighborhood of $\widetilde{\xi}$. □

4 A New Filled Function Algorithm

With regard to the theorems and explanations in the previous section, a new filled function algorithm similar to the algorithm in reference [16] to find the global minimum of the objective function $g(\xi)$ is as follows:

Algorithm

Step 0. – Set $k = 1$, $\alpha = 10^{-2}$, $\beta = 10^{-4}$, $R = 2$, the maximum number of directions N, upper bound M for the parameter α, the directions d_i for $i = 1, 2, \ldots, N$ and determine boundary of Ω.

Step 1. Find the local minimizer ξ_k^* of the objective function $g(\xi)$ starting from the point ξ_0.

Step 2. Construct the function $S(\xi, \xi_1^*)$ and set $i = 1$.

Step 3. Use $\xi_0 = \xi_k^* + \alpha d_i$ as a starting point and find the minimizer of $S(\xi, \xi_1^*)$ and denote it as x_s.

Step 4. If $\xi_s \in \Omega$, then go to Step 5; otherwise, go to Step 6.

Step 5. Take $\xi_0 = \xi_s$ and go to Step 1.

Step 6. If $i < N$, set $i = i + 1$ $(d_i \to d_{i+1})$ and go to Step 3; otherwise go to Step 7.

Step 7. If $\alpha \geq M$ or the gradient of $S(\xi, \xi_1^*)$ vanishes outside of the search domain or $|g_k^* - g_{k-1}^*| \leq \beta$, stop the algorithm and take the global minimizer $\xi^* = \xi_k^*$; otherwise take $\alpha = \alpha R$ go to Step 2.

5 Numerical Experiments

In this section, our new proposed filled method has implemented on following 16 bencmark test problems , which are the most widely used in the literature. Afterward, we have compared the proposed (**ff**) with filled functions in [1], [2], [13], [19]. The proposed method is run 10 times independently on a computer with an Intel(R) Core(TM) i7 (2.81 GHz) in Matlab R2015a. Table 1 shows the performance results of our proposed (**ff**) algorithm on the below test functions. For comparison, some (**ff**) algorithms according to these test functions and the numerical results of our proposed algorithm are given in Table 2-3. The abbreviations in these tables are explained below;

No	:	the problem number,
n	:	the dimension of the test problems,
$iterm$	:	the iteration number,
$fbest$	:	the best function value in 10 runs,
$fmean$	:	the mean of the best function value,
$feval$	:	the total number of the function and gradient evaluations of $g(\xi)$ and $S(\xi, \xi_1^*)$,
sr	:	the successful rate in 10 runs.
$-$	:	the data is unavailable for that algorithm,

The proposed algorithm is tested on following benchmark problems;

Problem 1 (Two-dimeonal function)

$$\begin{aligned} &\min \quad g(a,b) = [1 - 2b + c\sin(4\pi b) - a]^2 + [b - 0.5\sin(2\pi b)]^2, \\ &s.t. \quad a, b \in [0, 10]. \end{aligned}$$

where $c = 0.2,\ 0.5$ and 0.05. The global minimmum value is $g(a,b) = 0$ for all c.

Problem 2 (Three-hump back camel function)

$$\begin{aligned} &\min \quad g(a,b) = 2a^2 - 1.05a^4 + \tfrac{1}{6}a^6 - ab + b^2, \\ &s.t. \quad a, b \in [-3, 3]. \end{aligned}$$

The global minimizer is $(a,b) = (0,0)^T$ and $g(a,b) = 0$.

Problem 3 (Six-hump back camel function)

$$\begin{aligned} &\min \quad g(a,b) = 4a^2 - 2.1a^4 + \tfrac{1}{3}a^6 - ab - 4b^2 + 4b^4, \\ &s.t. \quad a, b \in [-3, 3]. \end{aligned}$$

The global minimizer is $(a,b) = (-0.0898, -0.7127)^T$ or $(a,b) = (0.0898, 0.7127)^T$ and $g(a,b) = -1.0316$.

Problem 4 (Treccani function)

$$\begin{aligned} &\min \quad g(a,b) = a^4 + 4a^3 + 4a^2 + b^2, \\ &s.t. \quad a, b \in [-3, 3]. \end{aligned}$$

The global minimizers are $(a,b) = (0,0)^T$ and $(a,b) = (-2,0)^T$ and $g(a,b) = 0$.

Problem 5 (Goldstein and Price function)

$$\begin{aligned} \min \quad g(a,b) &= 1 + (a+b+1)^2 \left(19 - 14a + 3a^2 - 14b + 6ab + 3b^2\right) \\ &\times \left[30 + (2a-3b)^2 \left(18 - 32a + 12a^2 - 48b - 36ab + 27b^2\right)\right], \\ s.t. \quad & a, b \in [-3, 3]. \end{aligned}$$

The global minimizer is $(a,b) = (0,-1)^T$ and $g(a,b) = 3$.

Problem 6 (Beale function)

$$\begin{aligned} &\min \quad g(a,b) = (1.5 - a + ab)^2 + \left(2.25 - a + ab^2\right)^2 + \left(2.625 - a + ab^3\right)^2 \\ &s.t. \quad a, b \in [-4.5, 4.5]. \end{aligned}$$

The global minimizer is $(a,b) = (3, 0.5)^T$ and $g(a,b) = 0$.

Problem 7 (Bohachevsky-1 function)

$$\begin{aligned} &\min \quad g(a,b) = a^2 + 2b^2 - 0.3\cos(3\pi a) - 0.4\cos(4\pi b) + 0.7, \\ &s.t. \quad a, b \in [-100, 100]. \end{aligned}$$

The global minimizer is $x^* = (0,0)^T$ and $g(x^*) = 0$.

Problem 8 (Bohachevsky-2 function)

$$\begin{aligned} &\min \quad g(a,b) = a^2 + 2b^2 - 0.3\cos(3\pi a)\cos(4\pi b) + 0.3, \\ &s.t. \quad a, b \in [-100, 100]. \end{aligned}$$

The global minimizer is $(a,b) = (0,0)^T$ and $g(a,b) = 0$.

Problem 9 (Bohachevsky-3 function)

$$\begin{aligned} &\min \quad g(a,b) = a^2 + 2b^2 - 0.3\cos(3\pi a + 4\pi b) + 0.3, \\ &s.t. \quad a, b \in [-100, 100]. \end{aligned}$$

The global minimizer is $(a,b) = (0,0)^T$ and $g(a,b) = 0$.

Problem 10 (Booth function)

$$\begin{aligned} &\min \quad g(a,b) = (a + 2b - 7)^2 + (2a + b - 5)^2, \\ &s.t. \quad a, b \in [-10, 10]. \end{aligned}$$

The global minimizer is $(a,b) = (1,3)^T$ and $g(a,b) = 0$.

Problem 11 (Brain function)

$$\begin{aligned} \min \quad & g(a,b) = \left(b - 1.275a^2/\pi^2 + 5a/\pi - 6\right)^2 + 10\left(1 - 0.125/\pi\right)\cos(a) + 10, \\ s.t. \quad & a,b \in [-5,15]. \end{aligned}$$

The global minimizer is $(a,b) = (-\pi, 12.275)^T$ and $g(a,b) = -3.3979$.

Problem 12 (Matyas function)

$$\begin{aligned} \min \quad & g(a,b) = 0.26\left(a^2 + b^2\right) - 0.48ab, \\ s.t. \quad & a,b \in [-10,10]. \end{aligned}$$

The global minimizer is $(a,b) = (0,0)^T$ and $g(a,b) = 0$.

Problem 13 (Ackley function)

$$\begin{aligned} \min \quad & g(x) = -20\exp\left(-0.2\sqrt{\frac{1}{d}\sum_{i=1}^{d} x_i^2}\right) - \exp\left(\frac{1}{d}\sum_{i=1}^{d}\cos(2\pi x_i)\right) + 20 + e, \\ s.t. \quad & x_i \in [-15,15]. \end{aligned}$$

The global minimizer is $x^* = (0,0)^T$ and $g(x^*) = 0$ for the dimensions d.

One can see from Table 1 that our proposed method can easily find global optimal solutions for above problems. In addition, it can be seen that 13 of the 16 success rate values in the last column of Table 1 are 100%, 2 of them are 90% and 1 of them is 30%. Obviously, the success rate for the 1st test problem ($c = 0.05$) can be considered low. However, the method we propose to find the global optimal solutions of this problem does not use more iterations and the total number of function evaluations is quite low compared to other methods. Moreover, even for the 50-dimensional function 13, the success rate is quite high at 90%. These indicates that the effectiveness and stable of proposed method.

The comparisons of proposed algorithm about the method in [13] and [19] are summarized in Table 2. It should be noted here that the filled functions in [13] and [19] are functions that are frequently cited filled functions in the literature. As can be seen in Table 2, the most striking capability of our proposed algorithm that makes it superior to the algorithms frequently cited in the literature is that our method finds the global optimum solution for all test problems with very small iterations compared to other methods. From

No	n	$iterm$	$fbest$	$fmean$	$feval$	$fevalm$	sr
1 $(c = 0.5)$	2	1.5	$1.4692e-15$	0.0019613	34	241.9	1
1 $(c = 0.2)$	2	1.4	$7.3507e-16$	0.035469	35	250.2	1
1 $(c = 0.05)$	2	1.3333	$1.331e-15$	$1.9309e-15$	35	43	0.3
2	2	1.4	$3.0312e-16$	$3.211e-14$	25	263.6	1
3	2	1.3333	-1.0316	-1.0316	30	34.667	0.9
4	2	1	$6.5032e-18$	$2.2513e-14$	23	229.1	1
5	2	1.5	3	3	46	276.2	1
6	2	1.4	$1.9859e-14$	$2.0403e-13$	48	278.3	1
7	2	1.6	$1.1102e-16$	$1.9873e-15$	42	253.9	1
8	2	1.6	$2.0539e-15$	$2.0539e-15$	42	250	1
9	2	1.6	0	$1.0503e-14$	36	249.6	1
10	2	1	$4.0739e-15$	$9.0649e-15$	23	220.2	1
11	2	1	0.39789	0.39789	28	223.6	1
12	2	1	$2.0234e-15$	$6.7244e-14$	27	224.4	1
13	2	1.6	$1.3295e-11$	$1.1277e-10$	38	287.6	1
13	50	2.3333	$1.281e-10$	$1.281e-10$	306	52287	0.9

Table 1: The numerical results obtained by our (**ff**) algorithm on above problems

		Our filled function		Wei's [19]		Liu's [13]	
No	n	$iterm$	$feval$	$iterm$	$feval$	$iterm$	$feval$
1 ($c = 0.5$)	2	1.5	34	2	297	2.7	455.8
1 ($c = 0.2$)	2	1.4	35	3	255	3	635
1 ($c = 0.05$)	2	1.3333	35	1	374	2.9	619.8
2	2	1.4	25	1	74	2	379
3	2	1.3333	30	2	74	2	224.7
4	2	1	23	2	72	2	214.3
5	2	1.5	46	3	80	3	321.2
6	2	1.4	48	–	102	–	–
7	2	1.6	42	–	–	–	–
8	2	1.6	42	–	226	–	–
9	2	1.6	36	–	–	–	–
10	2	1	23	–	77	–	–
11	2	1	28	–	202	2	217.2
12	2	1	27	–	207	–	–
13	2	1.6	38	–	–	–	–
13	50	2.3333	306	–	–	–	–

Table 2: Numerical Comparison

Table2, comparing "*feval*" and "*iterm*", it is clear that our proposed algorithm performs better than the filled function algorithm in [13] and [19].

Comparisons of our proposed algorithm with the filled function methods given in [1] and [2] at 2021, which are two of the most recently determined methods in the literature, are summarized in Table 3. The proposed filled method uses fewer function evaluations than the algorithm in [2], except for only 1 out of 9 test functions. Although Ahmed's [2] algorithm, with only 2 wins out of 9 test functions, looks slightly better in terms of number of iterations than our new filled function algorithm, it suffers a major defeat in terms of function evaluation. From the above comparisons, it is clear that the proposed filled method is more efficient than the algorithms in [1] and [2].

Table 3 : Numerical Comparison

	Our filled function		The algorithm in [1]		The algorithm in [2]	
No	*iterm*	*feval*	*iterm*	*feval*	*iterm*	*feval*
1 ($c = 0.5$)	1.5	34	2	380	1.4	203.4
1 ($c = 0.2$)	1.4	35	2	581	2.3	278.5
1 ($c = 0.05$)	1.3333	35	2	588	2.9	319.7
2	1.4	25	2	282	2	264.7
3	1.3333	30	2	263	1.2	220.9
4	1	23	2	186	1	195.4
5	1.5	46	2	323	1	277.6
6	1.4	48	–	–	–	–
7	1.6	42	–	–	–	–
8	1.6	42	–	–	–	–
9	1.6	36	–	–	–	–
10	1	23	–	–	–	–
11	1	28	2	209	1	379.6
12	1	27	–	–	–	–
13 ($d = 2$)	1.6	38	–	–	–	–
13 ($d = 50$)	2.3333	306	–	–	–	–

6 Conclusions

The fact that the existing filled functions have some drawbacks such as containing more than one parameter and the exponential or logarithmic term, the sensitivity to parameters, ill-conditioning and so on. All of these limitations are undesirable in numerical calculations. In this paper, we present a new (**ff**), which could tackle the mentioned shortcomings. Also, a new algorithm are given with regard to the proposed (**ff**). Moreover, in the numerical experiments made in the third part and in the comparison tables made with other methods (see Table 2, Table 3), we show that our proposed method is effective, its performance is quite satisfactory, and numerically stable.

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