DOI: 10.2478/auom-2023-0038
An. Şt. Univ. Ovidius Constanţa **Vol. 31(3),2023, 177–206**

Generalized Rectifying Ruled Surfaces of Special Singular Curves

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Abstract

In this study, generalized rectifying ruled surfaces of Frenet-type framed base curves in the three-dimensional Euclidean space are introduced. These surfaces are a generalization of not only the tangent and binormal surfaces of Frenet-type framed base curves, but also the tangent and binormal surfaces of regular curves. Additionally, we present some geometric characterizations and properties of these surfaces. Then, the singular point classes of the surface are scrutinized and the conditions for being a cross-cap surface are stated. Moreover, generalized rectifying surfaces are examined as framed surfaces by using the framed surface theory, and we investigate the basic invariants and curvatures of them. Then, several illustrative examples with figures are given to support the theorems and results.

1 Introduction

From past to present, surface theory has a highly important place and wide applications in several disciplines such as differential geometry, architecture, engineering, computer graphics, etc. Ruled surfaces which are famous and interesting examples of surfaces were investigated by Gaspard Monge [28]. Since this type of surface is suitable geometrically in order to use in the architecture area, it has been used over the centuries and still is used. A ruled surface is

Key Words: Framed surfaces, Frenet-type framed base curves, generalized rectifying ruled surfaces.
2010 Mathematics Subject Classification: Primary 53A05; Secondary 57R45, 58K05.
Received: 25.09.2022
Accepted: 22.12.2022

determined as a surface such that through each of its points passes a straight line contained in this surface. Because of the fact that the ruled surfaces are tremendously relevant geometrical and architectural structures that are also attractive regarding cost and duration, it has been used in many architectural structures [5]. Furthermore, ruled surfaces are examined widely in the existing literature, and there are several studies with respect to the ruled surfaces, their properties, and classification from exploration of them to until now. It can be said that a type of ruled surface is a developable surface. In addition to these, singularities of the tangent developable of a regular space curve are studied [4]. Also, the singularities of tangent surfaces and generalized frontals are examined in [15]. In the study [15], classifications of singularities such as cuspidal edge, swallowtail, folded umbrella, embedded cuspidal edge, etc. A great deal of investigation can be found in the existing literature, and also we can refer to the studies [16–20] with respect to the ruled surfaces.

The theory of curves, which is a great deal of fundamental topics of differential geometry, also has attracted much attention by many researchers. It is well known that, Frenet frame is constructed for only regular curves with the condition non-zero curvature. Although this frame has been a long time since its discovery, it has been an interesting subject studied by many researchers due to the fact that this frame is substantially convenient in order to examine, evaluate and interpret the regular curves with respect to several geometric properties. By using the Frenet frames for regular space curves, there are lots of studies have been done especially in classical differential geometry. Nevertheless, provided that a curve has singular points, then we can not construct the Frenet frame of it. In that case, the concept of the framed curve and framed base curve for the purpose of constructing the Frenet frame for non-regular curves are needed [11]. It has attracted the notice of several researchers since it supplies the occasion to establish a moving frame on curves with singular points. Framed curves which were determined by Honda and Takahashi, are smooth curves with a moving frame that have singular points [11]. These authors discovered a new type of moving frame and introduced the framed curves which are curves with singular points. These new structure and notions have made a great contribution to the literature and attracted several researchers who study especially differential geometry. Besides, the framed curves in E^3 as special case are given detailed and examined in this study. Framed curves have many importance on the singularity theory and several researchers examined this special type curve in the existing literature, and are ongoing [1, 7, 10, 11, 29–33].

Frenet-type framed curves are a special type of framed curves. Tangent vectors of a non-regular curve vanish at singular points, for constructing the Frenet frame, a regular spherical curve which these curves are named Frenet-type framed base curves is considered [10]. Since framed curves allow constructing a moving frame on curves with singular points, lots of researchers are interested in this subject, and many studies have been done and are ongoing in the literature. The existence conditions of framed curves [7] and the evolutes of framed immersions were examined [12]. Additionally, the following special type framed curves were studied by using Frenet-type framed curves: framed rectifying curves [29, 30], framed normal curves [31], and framed helices [10, 29]. Also, some characterizations of framed curves were studied in the four-dimensional Euclidean space [1, 32].

Then, Fukunaga and Takahashi defined the framed surfaces in E^3, and also the curvatures of framed surfaces and fundamental invariants of framed surfaces were examined [8]. This new attractive structure, which is named framed surface, constructs a new working area for researchers and has contributed to the literature. Because the normal vector of a surface with singular points can not be established, framed surfaces can be treated as smooth surfaces determined via a moving frame with singular points. Indeed, framed surfaces, which might have singularities, are defined as smooth surfaces in E^3 by utilizing a moving frame. For more detailed information with respect to the framed surfaces, we refer to the study [8]. Moreover, the principal normal and binormal surfaces constructed via singular curves which are smooth curves with singular points in E^3 [14] and the tangent developables and Draboux developables of framed curves [25] were examined. Tubular surfaces associated with framed base curves and developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in Euclidean 3-space were scrutinized [6, 24].

Generalized rectifying ruled surfaces of regular curves in E^3 were investigated, and also some properties of this new surface examined such as cases when they are asymptotic, geodesic, developable, cylindrical, helix, minimal, etc. [26]. With the same logic, the generalized normal and osculating-type ruled surfaces of regular curves [22, 23] are scrutinized in E^3. Moreover, some characterizations and geometric properties of these types of surfaces were presented giving supporting examples in these two studies. In addition to these, the rectifying developable surfaces of special singular curves were also studied [10]. Inspired by the study [26], we intend to examine the generalized rectifying ruled surface through utilizing special singular curves which can be defined as the smooth curves with singular points in E^3.

In this article, we investigate a new type surface which is called as the generalized rectifying ruled surface of Frenet-type framed base curve in E^3. Some basic and required information that is used throughout this paper is recalled in Section 2. After that, we define and scrutinize this new type surface that we are sure will make new contributions to the field of differential geometry and singularity theory in Section 3. These surfaces are a generalization of both the tangent surfaces and binormal surfaces of Frenet-type framed base curves, and the tangent and binormal surfaces of regular curves. Furthermore, geometric properties and some characterizations of generalized rectifying ruled surfaces of singular curves are examined. For instance, these new surfaces are examined regarding these notions cylindrical, developable, and striction curve. Moreover, singularity types are scrutinized with the help of the basic singularity theory of curves and the conditions for being a cross-cap surface are expressed. Then, we analyze the generalized rectifying ruled surface as framed surface, and basic invariants, curvatures, some classifications and properties of them are investigated. Afterward, some examinations with respect to the Gaussian and mean curvatures are given, and we present the necessary conditions for these type surfaces being flat and minimal surfaces. In addition to these, we study on the surfaces in terms of conditions of being immersion, Legendre immersion and framed immersion. In Section 4, some examples are given in order to facilitate understanding the given theorems and results, and these examples are supported by figures. Once and for all, we present conclusions in Section 5.

2 Basic concepts

We recall the required general notions and notations needed throughout the study in this part of the paper. Let us remember the ruled surface, framed curve, Frenet-type framed base curve, framed surface, and properties of them, respectively.

Let $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3), \boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3$ is given. Then, the inner product $\langle \mathbf{\Gamma}, \boldsymbol{\rho} \rangle = \Gamma_1\rho_1 + \Gamma_2\rho_2 + \Gamma_3\rho_3$ is defined and the norm of $\mathbf{\Gamma}$ is given as $\|\mathbf{\Gamma}\| = \sqrt{\langle \mathbf{\Gamma}, \mathbf{\Gamma} \rangle}$. Also, the vector product of $\mathbf{\Gamma}$ and $\boldsymbol{\rho}$ is given as

$$\mathbf{\Gamma} \wedge \boldsymbol{\rho} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \rho_1 & \rho_2 & \rho_3 \end{vmatrix}$$

where $\boldsymbol{e}_i$ for $i = 1, 2, 3$ are canonical bases on $\mathbb{R}^3$ [8].

It should be noted that the symbol prime $'$ denotes the derivative throughout this study.

Definition 2.1. *Let J be an open interval or a unit circle $\mathbb{S}^1$. Then, $\varphi : J \to \mathbb{R}^3$ and $v : J \to \mathbb{R}^3 - \{0\}$ be given as smooth functions. A ruled surface in $\mathbb{R}^3$ is the mapping $\psi_{(\varphi,v)} : J \times \mathbb{R} \to \mathbb{R}^3$ determined as $\psi_{(\varphi,v)}(s, u) = \varphi(s) + uv(s)$ where φ is directrix and v is director curve. Additionally, the straight line $u \mapsto \varphi(s) + uv(s)$ is called as ruling [18].*

If $\frac{\partial \psi_{(\varphi,v)}(s,u)}{\partial s} \wedge \frac{\partial \psi_{(\varphi,v)}(s,u)}{\partial u} = 0$ at any points (s_0, u_0), these points are called as singular points of the surface $\psi_{(\varphi,a)}(s,u)$. Else they are called as regular points. Since developable surfaces are a type of ruled surfaces, the following classification can be given. The equation $\det(\varphi'(s), v(s), v'(s)) = 0$ holds if and only if a ruled surface is developable. Also, a ruled surface $\psi_{(\varphi,v)}(s,u)$ with $\|v(s)\| = 1$ is called as cylindrical if and only if $v'(s) = 0$, and also non-cylindrical if and only if $v'(s) \neq 0$. A curve $\sigma(s)$ lying on $\psi_{(\varphi,v)}(s,u)$ with the condition $\langle \sigma'(s), v'(s) \rangle = 0$ is striction curve of the surface $\psi_{(\varphi,v)}(s,u)$. The striction curve of the surface $\psi_{(\varphi,v)}(s,u)$

$$\sigma(s) = \varphi(s) - \frac{\langle \varphi'(s), v'(s) \rangle}{\langle v'(s), v'(s) \rangle} v(s)$$

can be written [10, 21–23, 26]. The theory of ruled surfaces can be examined in these books [9, 27], as well.

Definition 2.2. $(\gamma, \Phi_1, \Phi_2) : I \to \mathbb{R}^3 \times \Delta_2$ *is a framed curve if* $\langle \gamma'(s), \Phi_i(s) \rangle = 0$ *for every* $s \in I$ *and* $i = 1, 2$*, where the followsing set*

$$\Delta_2 = \{\Phi = (\Phi_1, \Phi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle \Phi_i(s), \Phi_j(s) \rangle = \delta_{ij}; i, j = 1, 2\}$$

is a three-dimensional smooth manifold. Then, $\gamma : I \to \mathbb{R}^3$ *is named as a framed base curve if there exists* $(\Phi_1, \Phi_2) : I \to \Delta_2$ *such that* (γ, Φ_1, Φ_2) *is a framed curve [11].*

The following derivative formulas is presented

$$\begin{cases} \nu'(s) = -m(s)\Phi_1(s) - n(s)\Phi_2(s), \\ \Phi_1'(s) = m(s)\nu(s) + l(s)\Phi_2(s), \\ \Phi_2'(s) = n(s)\nu(s) - l(s)\Phi_1(s), \end{cases}$$

where $\nu = \Phi_1 \wedge \Phi_2$ is a unit vector, namely $\nu \perp \Phi_1$ and $\nu \perp \Phi_2$. Also, s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$, and α characterizing the singular points is given by $\gamma'(s) = \alpha(s)\nu(s)$. Additionally, $\{\nu, \Phi_1, \Phi_2\}$ is a moving frame of γ and (l, m, n, α) is the framed curvature of γ, and also (l, m, n, α) is used to search the singular points [11]. In case γ has a singular point, Frenet frame can not be established along $\gamma(s)$. Notwithstanding, the Frenet-type frame is determined along $\gamma(s)$ under the condition $m^2 + n^2 \neq 0$. These type curves are named as Frenet-type framed base curves which are a special framed curve and are defined as follows.

Definition 2.3. $\gamma : I \to \mathbb{R}^3$ *is called a Frenet-type framed base curve if there exist a smooth mapping* $\alpha : I \to \mathbb{R}$ *and a regular spherical curve* $\mathcal{T} : I \to \mathbb{S}^2$

such that $\gamma'(s) = \alpha(s)\mathcal{T}(s)$ for every $s \in I$. Then, we can also said that $\mathcal{T}(s)$ is unit tangent vector and $\alpha(s)$ is called a speed function of $\gamma(s)$. Also, s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$. The unit principal normal vector $\mathcal{N}(s) = \mathcal{T}'(s)/\|\mathcal{T}'(s)\|$ and the unit binormal vector $\mathcal{B}(s) = \mathcal{T}(s) \wedge \mathcal{N}(s)$ of $\gamma(s)$ can be given. Then, $\{\mathcal{T}(s), \mathcal{N}(s), \mathcal{B}(s)\}$ is constructed and this frame, which is named as the Frenet-type frame along $\gamma(s)$, is orthonormal [10].

The following derivative formula is also given

$$\begin{cases} \mathcal{T}'(s) = \kappa(s)\mathcal{N}(s), \\ \mathcal{N}'(s) = -\kappa(s)\mathcal{T}(s) + \tau(s)\mathcal{B}(s), \\ \mathcal{B}'(s) = -\tau(s)\mathcal{N}(s), \end{cases} \tag{2.1}$$

where $\kappa(s) = \|\mathcal{T}'(s)\|$ is curvature and $\tau(s) = \frac{\langle \mathcal{T}(s) \wedge \mathcal{T}'(s), \mathcal{T}''(s)\rangle}{\|\mathcal{T}'(s)\|^2}$ is torsion of γ [10]. Then, the three-dimensional smooth manifold $\Delta = \{(\mathbf{\Gamma}, \boldsymbol{\rho}) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \mathbf{\Gamma}, \boldsymbol{\rho}\rangle = 0\}$ is given where $\mathbb{S}^2 = \{\mathbf{\Gamma} \in \mathbb{R}^3 : \|\mathbf{\Gamma}\| = 1\}$ is a unit sphere in $\mathbb{R}^3$. Now, let us recall the definition of the framed surface, and take U as a simply connected domain of $\mathbb{R}^2$ [8].

Definition 2.4. *$(\phi, \phi_1, \phi_2) : U \to \mathbb{R}^3 \times \Delta$ is called a framed surface if $\langle \phi_s(s,u), \phi_1(s,u)\rangle = 0$ and $\langle \phi_u(s,u), \phi_1(s,u)\rangle = 0$ for every $(s,u) \in U$ where $\phi_s(s,u) = \dfrac{\partial \phi}{\partial s}(s,u)$ and $\phi_u(s,u) = \dfrac{\partial \phi}{\partial u}(s,u)$. Also, $\phi : U \to \mathbb{R}^3$ is called a framed base surface if there exists $(\phi_1, \phi_2) : U \to \Delta$ such that (ϕ, ϕ_1, ϕ_2) framed surface [8].*

In addition to these, $\{\phi_1(s,u), \phi_2(s,u), \phi_3(s,u)\}$ is a moving frame along $\phi(s,u)$ where $\phi_3(s,u) = \phi_1(s,u) \wedge \phi_2(s,u)$. The basic invariant functions of the framed surface (ϕ, ϕ_1, ϕ_2) are presented as

$$\begin{pmatrix} \phi_s \\ \phi_u \end{pmatrix} = \begin{pmatrix} \eta_{01} & \eta_{02} \\ \eta_{03} & \eta_{04} \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} \phi_{1s} \\ \phi_{2s} \\ \phi_{3s} \end{pmatrix} = \begin{pmatrix} 0 & \eta_{11} & \eta_{12} \\ -\eta_{11} & 0 & \eta_{13} \\ -\eta_{12} & -\eta_{13} & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} \phi_{1u} \\ \phi_{2u} \\ \phi_{3u} \end{pmatrix} = \begin{pmatrix} 0 & \eta_{21} & \eta_{22} \\ -\eta_{21} & 0 & \eta_{23} \\ -\eta_{22} & -\eta_{23} & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.4}$$

where $\eta_{ij} : U \to \mathbb{R}, i = 0,1,2, j = 1,2,3,4$ are smooth functions. The above matrices which are seen in the equations (2.2)-(2.4) are denoted as $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$, respectively. Also, the matrices $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ are named as basic invariants of

(ϕ,ϕ_1,ϕ_2). It should be noted that (s_0,u_0) is a singular point of ϕ if and only if $\det \mathcal{F}_0(s_0,u_0)=0$. Via the integrability conditions of (ϕ,ϕ_1,ϕ_2), then $\eta_{01}\eta_{21}+\eta_{02}\eta_{22}=\eta_{03}\eta_{11}+\eta_{04}\eta_{12}$. The $C_\phi=(J_\phi,K_\phi,H_\phi):U\to\mathbb{R}^3$ is called a curvature of (ϕ,ϕ_1,ϕ_2) where

$$\begin{aligned} J_\phi &= \det\begin{pmatrix}\eta_{01} & \eta_{02}\\ \eta_{03} & \eta_{04}\end{pmatrix},\\ K_\phi &= \det\begin{pmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{pmatrix},\\ H_\phi &= -\frac{1}{2}\left\{\det\begin{pmatrix}\eta_{01} & \eta_{12}\\ \eta_{03} & \eta_{22}\end{pmatrix}-\det\begin{pmatrix}\eta_{02} & \eta_{11}\\ \eta_{04} & \eta_{21}\end{pmatrix}\right\}.\end{aligned}$$

Suppose that $\phi:U\to\mathbb{R}^3$ is a regular surface and there exists $(\phi_1,\phi_2):U\to\Delta$ such that (ϕ,ϕ_1,ϕ_2) is a framed surface. Also, the relationships between the first fundamental invariants (namely E,F,G), second fundamental invariants (namely L,M,N) and the basic invariants are given as:

$$\left\{\begin{array}{l} E=\eta_{01}^2+\eta_{02}^2,\\ F=\eta_{01}\eta_{02}+\eta_{03}\eta_{04},\\ G=\eta_{03}^2+\eta_{04}^2,\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l} L=-\eta_{01}\eta_{11}-\eta_{02}\eta_{12},\\ M=-\eta_{01}\eta_{21}-\eta_{02}\eta_{22},\\ N=-\eta_{03}\eta_{21}-\eta_{04}\eta_{22}.\end{array}\right.$$

Proposition 2.1. *Let $\phi:U\to\mathbb{R}^3$ be a regular surface. The following relationships are satisfied*

$$K=\frac{K_\phi}{J_\phi}\quad\textit{and}\quad H=\frac{H_\phi}{J_\phi}$$

where K is Gauss curvature of ϕ, H is the mean curvature of ϕ and the curvature of (ϕ,ϕ_1,ϕ_2) is $C_\phi=(J_\phi,K_\phi,H_\phi)$ [8].

Proposition 2.2. *Let (ϕ,ϕ_1,ϕ_2) is a framed surface and $\varrho\in U$. (ϕ,ϕ_1) is a Legendre immersion* around ϱ if and only if $C_\phi(\varrho)\neq 0$ [8].*

In addition to this, if the surface (ϕ,ϕ_1,ϕ_2) is an immersion, then (ϕ,ϕ_1,ϕ_2) is a framed immersion. Let $I_\phi:U\to\mathbb{R}^8$ defined as

$$I_\phi=\left(\begin{array}{l} C_\phi,\det\begin{pmatrix}\eta_{01} & \eta_{13}\\ \eta_{03} & \eta_{23}\end{pmatrix},\det\begin{pmatrix}\eta_{02} & \eta_{13}\\ \eta_{04} & \eta_{23}\end{pmatrix},\det\begin{pmatrix}\eta_{11} & \eta_{13}\\ \eta_{21} & \eta_{23}\end{pmatrix},\\ \det\begin{pmatrix}\eta_{12} & \eta_{13}\\ \eta_{22} & \eta_{23}\end{pmatrix},\det\begin{pmatrix}\eta_{01} & \eta_{11}\\ \eta_{03} & \eta_{21}\end{pmatrix}\end{array}\right)$$

* $(\phi,\phi_1):U\to\mathbb{R}^3\times\mathbb{S}^2$ is a Legendre surface (Legendre immersion) if $\langle\phi_s(s,u),\phi_1(s,u)\rangle=0$ and $\langle\phi_u(s,u),\phi_1(s,u)\rangle=0$ for every $(s,u)\in U$. Also, $\phi:U\to\mathbb{R}^3$ is a frontal (front) if there exists $\phi:U\to\mathbb{S}^2$ such that ϕ,ϕ_1 is a Legendre surface (Legendre immersion). For more detailed information, see [2,3,8].

The framed surface (ϕ, ϕ_1, ϕ_2) be given and $\varrho \in U$. Then, the surface (ϕ, ϕ_1, ϕ_2) is called as framed immersion around ϱ if and only if $I_\phi \neq 0$ [8] (see also [13]).

3 Generalized rectifying ruled surfaces of special singular curves

The aim of this section is to determine the generalized rectifying ruled surface by using special singular curves which are the smooth curves with singular points in E^3. According to this purpose of the part, we initially define these new type special surfaces. Additionally, some geometric characterizations, properties and results can be listed as set of singular points, cylindrical, developable, striction curve and the others are given. Furthermore, singularity types are examined by using the basic singularity theory of curves and the conditions with respect to being a cross-cap surface are given. Then, we introduce these type surface as framed surface via utilizing the theory of framed surface, and we present basic invariants, curvatures, some classifications and properties. Moreover, some examinations according to the Gaussian and mean curvatures are given. We establish the necessary conditions for these surfaces being flat and minimal surfaces. In addition to these, we scrutinize these surfaces with respect to the conditions of being immersion, Legendre immersion and framed immersion.

Definition 3.1. *Let $\gamma(s) : I \to \mathbb{R}^3$ be given as a Frenet-type framed base curve. The ruled surface $\phi_{(\gamma,a_n)}(s,u) : I \times \mathbb{R} \to \mathbb{R}^3$ determined as*

$$\phi_{(\gamma,a_n)}(s,u) = \gamma(s) + ua_n(s), \;\; a_n(s) = a_1(s)\mathcal{T}(s) + a_2(s)\mathcal{B}(s) \tag{3.1}$$

is called the generalized rectifying ruled surface of Frenet-type framed base curve $\gamma(s)$ where $a_1(s)$ and $a_2(s)$ are smooth functions and $a_1^2(s) + a_2^2(s) = 1$.

Corollary 3.1. *Let $\phi_{(\gamma,a_n)}(s,u)$ be a generalized rectifying ruled surface of Frenet-type framed base curve $\gamma(s)$. The following results can be given.*

(1) *If $a_1(s) = \frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$ and $a_2(s) = \frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau^2(s)}}$, then the surface $\phi_{(\gamma,a_n)}(s,u)$ is rectifying developable surface along the Frenet-type framed base curve $\gamma(s)$ (cf. [10]).*

(2) *If $a_1(s) = 0$ and $a_2(s) = \pm 1$, then the surface $\phi_{(\gamma,a_n)}(s,u)$ is binormal surface along the Frenet-type framed base curve $\gamma(s)$ (cf. [14]).*

(3) *If $a_1(s) = \pm 1$ and $a_2(s) = 0$, then $\phi_{(\gamma,a_n)}(s,u)$ is tangent developable surface along $\gamma(s)$ (cf. [25]).*

Theorem 3.1. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$. The surface $\phi_{(\gamma,a_n)}(s,u)$ is not regular surface if and only if*

$$u\left(a_1(s)\kappa(s) - a_2(s)\tau(s)\right) = 0$$

and

$$a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right) = 0.$$

Proof. By taking the partial derivatives of the equation (3.1) according to the s and u, and via equation (2.1), we get

$$\begin{aligned}\frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial s} &= (\alpha(s) + ua_1'(s))\mathcal{T}(s) + u(a_1(s)\kappa(s) - a_2(s)\tau(s))\mathcal{N}(s) \\ &\quad + ua_2'(s)\mathcal{B}(s),\end{aligned} \tag{3.2}$$

$$\frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial u} = a_1(s)\mathcal{T}(s) + a_2(s)\mathcal{B}(s). \tag{3.3}$$

From cross product of the equations (3.2) and (3.3), we have

$$\begin{aligned}\frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial s} \wedge \frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial u} =& u(a_1(s)\kappa(s) - a_2(s)\tau(s))(a_2(s)\mathcal{T}(s) - a_1(s)\mathcal{B}(s)) \\ & - \left(a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right)\right)\mathcal{N}(s).\end{aligned}$$

Hence, when $a_1(s)$ and $a_2(s)$ are not zero at the same time, we achieve $u\left(a_1(s)\kappa(s) - a_2(s)\tau(s)\right) = 0$ and $a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right) = 0$ if and only if $\frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial s} \wedge \frac{\partial\phi_{(\gamma,a_n)}(s,u)}{\partial u} = 0$. □

The Theorem 3.1 helps to establish the set of singular points of the surface $\phi_{(\gamma,a_n)}(s,u)$. Accordingly, the singular points of the surface establish the following set

$$S = \left\{ \begin{aligned} (s,u) \in I \times U \ :& \ u(a_1(s)\kappa(s) - a_2(s)\tau(s)) = 0, \\ & a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right) = 0 \end{aligned} \right\}.$$

In order to examine the types of singularity of the surface, we can divide the set S into two classes S_1 and S_2 as follows

$$S_1 = \{(s,0) \in I \times U : a_2(s)\alpha(s) = 0\}$$

and

$$S_2 = \left\{ \begin{aligned} (s,u) \in I \times U :& \, a_1(s)\kappa(s) - a_2(s)\tau(s) = 0, \\ & a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right) = 0, \quad u \neq 0 \end{aligned} \right\}$$

where

$$u = -\frac{\alpha(s)a_2(s)}{a_1'(s)a_2(s) - a_1(s)a_2'(s)}.$$

Corollary 3.2. *The followings can be given:*

(1) *If $a_1(s) = 0$ and $a_2(s) = 1$, the generalized rectifying ruled surface $\phi_{(\gamma,a_n)}(s,u)$ is a principal normal surface of Frenet-type framed curves $\gamma(s)$. Therefore, from the definition of the singularity sets S_1 and S_2, we get:*

$$S_1 = \{(s,0) \in I \times U : \quad \alpha(s) = 0\}$$

and

$$S_2 = \{(s,u) \in I \times U : \quad \tau(s) = 0, \quad \alpha(s) = 0\}.$$

Then, this particular case gives the results of the study [14].

(2) *If $a_1(s) = 1$ and $a_2(s) = 0$, generalized rectifying ruled surface $\phi_{(\gamma,a_n)}$ is a tangent surface of Frenet-type framed base curves. Since $a_2(s) = 0$, from definition set S_2 for $u \neq 0$, we have $\kappa(s) = 0$. This is a contradiction since $\kappa(s) \neq 0$ for Frenet-type framed base curves. Hence, there is no S_2 singularity for tangent surfaces.*

It can be seen that the points of S_1 are located on $\gamma(s)$. Now let us present an examination for the singularity set S_2 of the surface $\phi_{(\gamma,a_n)}(s,u)$. Suppose that

$$\begin{aligned} x(s) &= a_1(s)\kappa(s) - a_2(s)\tau(s), \\ y(s,u) &= a_2(s)\alpha(s) + u\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right). \end{aligned} \tag{3.4}$$

By using the definition of the singularity sets S_1 and S_2, we obtain the followings.

Proposition 3.1. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$. The locus of the singular points of the generalized rectifying surface $\phi_{(\gamma,a_n)}(s,u)$ of Frenet-type framed curve $\gamma(s)$ is given by:*

(1) *If the singular points of the surface $\phi_{(\gamma,a_n)}(s,u)$ belongs to the set S_1, the locus of the singular points of the surface $\phi_{(\gamma,a_n)}(s,u)$ is the Frenet-type framed base curve $\gamma(s)$.*

(2) *If the singular points of the surface $\phi_{(\gamma,a_n)}(s,u)$ belongs to the set S_2, the locus of the singular points of the surface $\phi_{(\gamma,a_n)}(s,u)$ is the curve*

$$\beta(s) = \gamma(s) + u a_n(s)$$

where $u = -\dfrac{\alpha(s)}{\left(\frac{\tau(s)}{\kappa(s)}\right)' a_2(s)}$ *and* $a_2(s) \neq 0$.

Proof. The first one is straightforward. Also, the definition of singular point set S_2, $x(s) = 0$ and $y(s,u) = 0$. Accordingly, we get $u = -\frac{\alpha(s)}{\left(\frac{\tau(s)}{\kappa(s)}\right)' a_2(s)}$ and $a_2(s) \neq 0$. □

Theorem 3.2. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$. Then, we give the followings:*

(1) *If $(s_1, 0) \in S_1$ and $a_2(s_1)\alpha'(s_1)x(s_1) \neq 0$, then $\phi_{(\gamma,a_n)}(s,u)$ is a cross-cap at the point $(s_1, 0)$.*

(2) *If $(s_2, u_2) \in S_2$ and $\alpha(s)(a_2(s)x(s))' \neq 0$, then $\phi_{(\gamma,a_n)}(s,u)$ is a cross-cap at the point (s_2, u_2).*

Proof. According to equations (3.2) and (3.3), we get the following second-order partial derivations of $\phi_{(\gamma,a_n)}(s,u)$.

$$\begin{aligned}
\frac{\partial^2 \phi_{(\gamma,a_n)}}{\partial u \partial s}(s,u) =& a_1'(s)\mathcal{T}(s) + (a_1(s)\kappa(s) - a_2(s)\tau(s))\mathcal{N}(s) + a_2'(s)\mathcal{B}(s)\\
\frac{\partial^2 \phi_{(\gamma,a_n)}}{\partial s^2}(s,u) =& [(\alpha(s) + ua_1'(s))' - ux(s)\kappa(s)]\mathcal{T}(s)\\
&+ [(\alpha(s) + ua_1'(s))\kappa(s) + u(x(s))' - ua_2'(s)\tau(s)]\mathcal{N}(s)\\
&+ [ux(s)\tau(s) + (ua_2'(s))']\mathcal{B}(s)
\end{aligned}$$

Then,

$$\begin{aligned}
&\det\left(\frac{\partial \phi}{\partial u}(s,u), \frac{\partial^2 \phi}{\partial u \partial s}(s,u), \frac{\partial^2 \phi}{\partial s^2}(s,u)\right)\\
&= a_1(s)\begin{bmatrix} a_1(s)\kappa(s)\tau(s)ux(s) + a_1(s)\kappa(s)(ua_2'(s))' - a_2(s)\tau^2(s)ux(s)\\ - a_2(s)\tau(s)(ua_2'(s))' - a_2'(s)\kappa(s)(\alpha(s) + ua_1'(s)) - a_2'(s)u(x(s))'\\ + (a_2'(s))^2 u\tau(s)\end{bmatrix}\\
&\quad + a_2(s)\begin{bmatrix} a_1'(s)\kappa(s)(\alpha(s) + ua_1'(s)) + a_1'(s)u(x(s))' - a_1'(s)a_2'(s)u\tau(s)\\ - a_1(s)\kappa(s)(\alpha(s) + ua_1'(s))' + a_2(s)\tau(s)(\alpha(s) + ua_1'(s))'\\ + a_1(s)\kappa^2(s)ux(s) - a_2(s)\tau(s)u\kappa(s)x(s))\end{bmatrix}.
\end{aligned}$$

If $(s_1, 0) \in S_1$, then

$$\det\left(\frac{\partial \phi_{(\gamma,a_n)}}{\partial u}(s_1,0), \frac{\partial^2 \phi_{(\gamma,a_n)}}{\partial u \partial s}(s_1,0), \frac{\partial^2 \phi_{(\gamma,a_n)}}{\partial s^2}(s_1,0)\right) = -a_2(s_1)\alpha'(s_1)x(s_1).$$

If $(s_2, u_2) \in S_2$, then

$$\det\left(\frac{\partial\phi_{(\gamma,a_n)}}{\partial u}(s_2,u_2), \frac{\partial^2\phi_{(\gamma,a_n)}}{\partial u\partial s}(s_2,u_2), \frac{\partial^2\phi_{(\gamma,a_n)}}{\partial s^2}(s_2,u_2)\right) = -\alpha(s)(a_2(s)x(s))'.$$

We attained the desired. □

Corollary 3.3. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$. If s_0 is a singular point of $\gamma(s)$, then $(s_0, 0)$ is a singular point of $\phi_{(\gamma,a_n)}$.*

Theorem 3.3. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$.*

(1) *The surface $\phi_{(\gamma,a_n)}(s,u)$ is cylindrical if and only if $a_1(s) = 0$, $a_2(s) = \pm 1$ and $\tau(s) = 0$ or $a_1(s)$, $a_2(s)$ are non-zero constants and $x(s) = 0$ for every $s \in I$.*

(2) *The surface $\phi_{(\gamma,a_n)}(s,u)$ is developable if and only if $\alpha(s)a_2(s)x(s) = 0$ for every $s \in I$.*

Proof. (1) The surface $\phi_{(\gamma,a_n)}(s,u)$ is cylindrical if and only if $a_n(s)$ is constant ($a_n'(s) = 0$).

(⇒) Since $a_n(s) = a_1(s)\mathcal{T}(s) + a_2(s)\mathcal{B}(s)$, we get:

$$a_n'(s) = a_1'(s)\mathcal{T}(s) + (a_1(s)\kappa(s) - a_2(s)\tau(s))\,\mathcal{N}(s) + a_2'\mathcal{B}(s).$$

Then, we have

$$\begin{aligned} a_1'(s) &= 0, \\ a_1(s)\kappa(s) - a_2(s)\tau(s) &= 0, \\ a_2'(s) &= 0. \end{aligned}$$

If $a_2(s) = 0$ and $a_1(s) = \pm 1$, we have $\kappa(s) = 0$, but this contradicts the existence of Frenet-type framed base curves. So, $a_2(s) \neq 0$. Therefore, $a_n'(s) = 0$ if and only if $a_1(s) = 0$, $a_2(s) = \pm 1$ and $\tau(s) = 0$ or $a_1(s)$, $a_2(s)$ are non-zero constants and $x(s) = 0$.

(⇐) It is clear.

(2) The surface $\phi_{(\gamma,a_n)}(s,u)$ is developable if and only if the equation $\det(\gamma'(s), a_n(s), a_n'(s)) = 0$ holds. Then, we get

($\Rightarrow$) Let the surface $\phi_{(\gamma,a_n)}(s,u)$ be a developable surface.

$$\begin{aligned}\det(\gamma'(s),a_n(s),a_n'(s)) =& \det(\alpha(s)\mathcal{T}(s),a_1(s)\mathcal{T}(s)+a_2(s)\mathcal{B}(s),\\ & a_1'(s)\mathcal{T}(s)+(a_1(s)\kappa(s)-a_2(s)\tau(s))\,\mathcal{N}(s)\\ & +a_2'\mathcal{B}(s))\\ =& -\alpha(s)a_2(s)x(s)\end{aligned}$$

($\Leftarrow$) It is straightforward.

Therefore, the desired results are achieved. □

Corollary 3.4. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$.*

1. *If generalized rectifying ruled surface $\phi_{(\gamma,a_n)}(s,u)$ is a tangent developable surface (i.e. $a_2(s)=0$), then $\phi_{(\gamma,a_n)}(s,u)$ is developable.*
2. *If Frenet-type framed curve $\gamma(s)$ is planar curve (i.e. $\tau(s)=0$) and $a_1(s)=0, a_2(s)=\pm1$, then $\phi_{(\gamma,a_n)}(s,u)$ is developable.*

Theorem 3.4. *Let $\gamma(s)$ is a Frenet-type framed base curve, $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$, and this surface be non-cylindrical. The base curve $\gamma(s)$ of surface $\phi_{(\gamma,a_n)}(s,u)$ is its striction curve if and only if $a_1(s)$ is a constant or $\alpha(s)=0$.*

Proof. The striction curve of the surface $\phi_{(\gamma,a_n)}(s,u)$ can be written as

$$\sigma(s)=\gamma(s)-\frac{\langle\gamma'(s),a_n'(s)\rangle}{\langle a_n'(s),a_n'(s)\rangle}a_n(s)$$

and we have

$$\begin{aligned}\langle\gamma'(s),a_n'(s)\rangle &= \langle\alpha(s)\mathcal{T}(s),a_1'(s)\mathcal{T}(s)+(a_1(s)\kappa(s)-a_2(s)\tau(s))\,\mathcal{N}(s)+a_2'(s)\mathcal{B}(s)\rangle\\ &=\alpha(s)a_1'(s).\end{aligned}$$

Via straightforward calculations, then we get

$$\sigma(s)=\gamma(s)-\frac{\alpha(s)a_1'(s)}{(a_1'(s))^2+(x(s))^2+(a_2'(s))^2}a_n(s). \tag{3.5}$$

Therefore, we can see that the base curve $\gamma(s)$ of surface $\phi_{(\gamma,a_n)(s,u)}$ is its striction curve if and only if $a_1(s)$ is a constant or $\alpha(s)=0$ by using the equation $\langle\sigma'(s),a_n'(s)\rangle=0$. □

Corollary 3.5. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$. If $(a_1'(s))^2+(x(s))^2+(a_2'(s))^2 = 0$ (namely cylindrical), then striction curve of the surface $\phi_{(\gamma,a_n)}(s,u)$ can not be constructed.*

Corollary 3.6. *Let $\gamma(s)$ is a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$ and the surface be non-cylindrical. The surface is conical if and only if $\alpha(s) = 0$.*

Proof. Assume that the surface $\phi_{(\gamma,a_n)}(s,u)$ be non-cylindrical. By using the condition $\sigma'(s) = 0$ for being conical surface (cf. this condition from [10]) and applying this, then we get desired. Therefore, the surface is conical if and only if $\alpha(s) = 0$. □

3.1 Generalized rectifying ruled surfaces as framed surfaces

Since the generalized rectifying ruled surfaces of Frenet-type framed base curves have singular points, then we intend to study and examine them as the framed surface. By using the theory of the framed surface which is examined in the study [8], we get our special type surface. Let's introduce and scrutinize this new surface, then give some properties and classifications such as invariants, curvatures. Also, conditions for surface being flat, minimal, immersion, Legendre immersion and framed immersion are examined.

Definition 3.2. *Let $\gamma(s)$ be a Frenet-type framed base curve and $\phi_{(\gamma,a_n)}(s,u)$ is the generalized rectifying ruled surface of $\gamma(s)$ with the condition $a_1^2(s) + a_2^2(s) = 1$. If there exist smooth functions $\Theta, \xi : I \times U \to \mathbb{R}^3$ such that $\langle \phi_s(s,u), \phi_1(s,u)\rangle = 0$ and $\langle \phi_u(s,u), \phi_1(s,u)\rangle = 0$ where*

$$\phi_1(s,u) = \cos\Theta(s,u)(a_2(s)\mathcal{T}(s) - a_1(s)\mathcal{B}(s)) + \sin\Theta(s,u)\mathcal{N}(s)$$

and

$$\begin{aligned}\phi_2(s,u) = &\cos\xi(s,u)(a_1(s)\mathcal{T}(s) + a_2(s)\mathcal{B}(s)) \\ &- \sin\xi(s,u)(\sin\Theta(s,u)(a_2(s)\mathcal{T}(s) - a_1(s)\mathcal{B}(s)) - \cos\Theta(s,u)\mathcal{N}(s))\end{aligned}$$

then, we obtain the framed surface $(\phi, \phi_1, \phi_2) : I \times U \to \mathbb{R}^3 \times \Delta$ where

$$\Delta = \{(\phi_1, \phi_2) \in \mathbb{S}^2 \times \mathbb{S}^2, \langle \phi_1, \phi_2\rangle = 0\}.$$

We can get $\phi_3(s,u) = \phi_1(s,u) \wedge \phi_2(s,u)$ as follows:

$$\begin{aligned}\phi_3(s,u) =&(a_2(s)\cos\xi(s,u)\sin\Theta(s,u) + a_1(s)\sin\xi(s,u))\mathcal{T}(s) \\ &- \cos\xi(s,u)\cos\Theta(s,u)\mathcal{N}(s) \\ &+ (a_2(s)\sin\xi(s,u) - a_1(s)\cos\xi(s,u)\sin\Theta(s,u))\mathcal{B}(s).\end{aligned}$$

By using the integrability conditions of the framed surface, the following equation is obtained

$$\begin{aligned}&\Theta_u(s,u)[-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)]\\&=-x(s)\sin\Theta(s,u)-(a_1'(s)a_2(s)-a_1(s)a_2'(s))\cos\Theta(s,u).\end{aligned}$$

By taking into the equations (2.2), (2.3) and (2.4), then the basic invariants of the surface (ϕ,ϕ_1,ϕ_2) are presented by

$$\begin{cases}\eta_{01}=\sin\xi(s,u)\left[-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right],\\ \eta_{02}=-\cos\xi(s,u)\left[-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right],\\ \eta_{03}=\cos\xi(s,u),\\ \eta_{04}=\sin\xi(s,u),\end{cases}$$

$$\begin{cases}\eta_{11}=-\cos\xi(s,u)\left[\cos\Theta(s,u)\left(a_1'(s)a_2(s)-a_1(s)a_2'(s)\right)+x(s)\sin\Theta(s,u)\right]\\ \qquad+\sin\xi(s,u)\left(\Theta_s(s,u)+a_2(s)\kappa(s)+a_1(s)\tau(s)\right),\\ \eta_{12}=-\sin\xi(s,u)\left[\cos\Theta(s,u)\left(a_1'(s)a_2(s)-a_1(s)a_2'(s)\right)+x(s)\sin\Theta(s,u)\right]\\ \qquad-\cos\xi(s,u)\left(\Theta_s(s,u)+a_2(s)\kappa(s)+a_1(s)\tau(s)\right),\\ \eta_{13}=-\xi_s(s,u)+x(s)\cos\Theta(s,u)+(a_1'(s)a_2(s)-a_1(s)a_2'(s))\sin\Theta(s,u),\end{cases}$$

and

$$\begin{cases}\eta_{21}=\Theta_u(s,u)\sin\xi(s,u),\\ \eta_{22}=-\Theta_u(s,u)\cos\xi(s,u),\\ \eta_{23}=-\xi_u(s,u),\end{cases}$$

where $x(s)$ and $y(s,u)$ can be seen in the equation (3.4).

Corollary 3.7. *Let the surface $\phi_{(\gamma,a_n)}:U\to\mathbb{R}^3$ is given. Then, the curvature $C_\phi=(J_\phi,K_\phi,H_\phi)$ of the framed surface $\phi_{(\gamma,a_n)}(s,u)$ is presented as*

$$\begin{aligned}J_\phi&=-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u),\\ K_\phi&=\Theta_u(s,u)\left[\cos\Theta(s,u)\left(a_1'(s)a_2(s)-a_1(s)a_2'(s)\right)+x(s)\sin\Theta(s,u)\right],\\ H_\phi&=-\frac{\Theta_s(s,u)+a_2(s)\kappa(s)+a_1(s)\tau(s)}{2}.\end{aligned}$$

Corollary 3.8. *Let $\phi_{(\gamma,a_n)}:U\to\mathbb{R}^3$ is a regular surface. Then, the following relationships between the first fundamental invariants, second fundamental invariants and the basic invariants are given:*

$$\begin{aligned}&E=\left(-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right)^2,\\ &F=-\sin\xi(s,u)\cos\xi(s,u)\left[\left(-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right)^2-1\right],\\ &G=1,\\ &L=-\left(-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right)\left(\Theta_s(s,u)+a_2(s)\kappa(s)+a_1(s)\tau(s)\right),\\ &M=-\Theta_u(s,u)\left(-y(s,u)\sin\Theta(s,u)+ux(s)\cos\Theta(s,u)\right),\\ &N=0.\end{aligned}$$

Theorem 3.5. *Let $\phi_{(\gamma,a_n)} : U \to \mathbb{R}^3$ is a regular surface. The Gauss and mean curvature of $\phi_{(\gamma,a_n)}$ are given as follows:*

$$K = \frac{\Theta_u(s,u)\left[\cos\Theta(s,u)\left(a_1'(s)a_2(s) - a_1(s)a_2'(s)\right) + x(s)\sin\Theta(s,u)\right]}{-y(s,u)\sin\Theta(s,u) + ux(s)\cos\Theta(s,u)}$$

and

$$H = -\frac{\Theta_s(s,u) + a_2(s)\kappa(s) + a_1(s)\tau(s)}{2\left(-y(s,u)\sin\Theta(s,u) + ux(s)\cos\Theta(s,u)\right)},$$

respectively.

Proof. By using the Proposition 2.1 and Corollary 3.7, then the proof is completed. □

Corollary 3.9. *Let $\phi_{(\gamma,a_n)} : U \to \mathbb{R}^3$ be a regular surface. The followings are yielded.*

(1) *The regular surface $\phi_{(\gamma,a_n)}$ is flat (developable) if and only if*

$$\Theta_u(s,u) = 0 \quad or \quad \cot\Theta(s,u) = -\frac{x(s)}{a_1'(s)a_2(s) - a_1(s)a_2'(s)}.$$

(2) *The regular surface $\phi_{(\gamma,a_n)}$ is a minimal surface if and only if*

$$\Theta_s(s,u) = -a_2(s)\kappa(s) - a_1(s)\tau(s).$$

Corollary 3.10. *Let $(\phi,\phi_1,\phi_2) : U \to \mathbb{R}^3$ be a framed surface. The followings are satisfied.*

(1) *If $\varrho \in S_1$ or $\varrho \in S_2$,*
   - (i) *ϕ is not an immersion (namely, not a regular surface) at ϱ since $J_\phi(\varrho) = 0$.*
   - (ii) *(ϕ,ϕ_1) is a Legendre immersion at ϱ if and only if $H_\phi \neq 0$.*

(2) *If $\varrho \notin S_1$ and $\varrho \notin S_2$,*
   - (i) *ϕ is an immersion (a regular surface) around ϱ.*
   - (ii) *(ϕ,ϕ_1) is a Legendre immersion around ϱ.*

(3) *Let $\varrho \in U$,*
   - (i) *(ϕ,ϕ_1,ϕ_2) is a framed immersion around ϱ if and only if $I_\phi(\varrho) \neq 0$.*

(ii) *Let* $(\eta_{13}, \eta_{23}) \neq (0,0)$ *around* ϱ. *If*

$$\det\begin{pmatrix} \eta_{01} & \eta_{13} \\ \eta_{03} & \eta_{23} \end{pmatrix} = \det\begin{pmatrix} \eta_{02} & \eta_{13} \\ \eta_{04} & \eta_{23} \end{pmatrix} = \det\begin{pmatrix} \eta_{11} & \eta_{13} \\ \eta_{21} & \eta_{23} \end{pmatrix} = 0$$

$$\det\begin{pmatrix} \eta_{12} & \eta_{13} \\ \eta_{22} & \eta_{23} \end{pmatrix} = 0$$

around ϱ, *then* $I_\phi(\varrho) = 0$.

(iii) *Let* $(\eta_{13}, \eta_{23}) = (0,0)$ *around* ϱ. *If* $C_\phi(\varrho) = 0$, *then* $I_\phi(\varrho) = 0$.

4 Applications

In this section, we present several numerical and explanatory examples for supporting attained materials which are in this study.

Example 4.1. *Let* $\phi_{(\gamma,a_n)}$ *be presented by the parametrization*

$$\phi_{(\gamma,a_n)}(s,u) = \gamma(s) + u a_n(s)$$

where

$$\begin{aligned} \gamma : [0, 2\pi) &\to \mathbb{R}^3 \\ s \mapsto \gamma(s) &= (\sin^3 s, \cos^3 s, -\cos 2s) \end{aligned} \tag{4.1}$$

The singular points of γ *are* $s_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, *and* γ *is a Frenet-type framed base curve. Also,* $\gamma(s)$ *can be seen in the following Figure 1.*

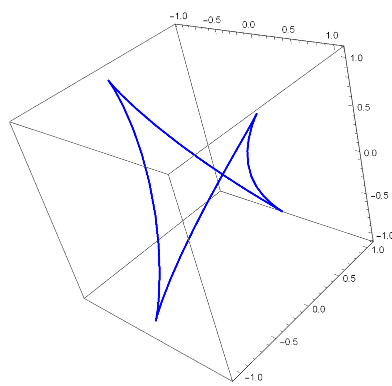

Figure 1: $\gamma(s)$ in the equation (4.1)

Then, if necessary calculations are completed, the followings are calculated:

$$\begin{cases} \mathcal{T}(s) = \left(\frac{3}{5}\sin s, -\frac{3}{5}\cos s, \frac{4}{5}\right), \\ \mathcal{N}(s) = (\cos s, \sin s, 0), \\ \mathcal{B}(s) = \left(-\frac{4}{5}\sin s, \frac{4}{5}\cos s, \frac{3}{5}\right), \end{cases}$$

and

$$\begin{cases} \kappa(s) = \frac{3}{5}, \\ \tau(s) = \frac{4}{5}, \\ \alpha(s) = 5\sin s\cos s. \end{cases}$$

- *Let us take* $a_1(s) = \dfrac{4}{5}$ *and* $a_2(s) = \dfrac{3}{5}$*. Then, the followings are obtained*

$$a_n(s) = (0, 0, 1)$$

and

$$\phi_{(\gamma,a_n)}(s,u) = (\sin^3 s, \cos^3 s, -\cos 2s + u). \tag{4.2}$$

We get,

$$x(s) = 0$$

and

$$y(s,u) = 3\sin s\cos s.$$

Then, we can say that the surface $\phi_{(\gamma,a_n)}(s,u)$ *given in the equation* (4.2) *is cylindrical and developable.*

Now, we shall give the followings about the singularity set:

(1) *There exist points* $\alpha(s_0) = 0$ *for* $s_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ *where* $u = 0$*, so the set of singular points is* S_1*.*

(2) *If* $x(s_0) = 0$*,* $u \neq 0$ *and* $3\sin(s_0)\cos(s_0) = 0$ *at the points* s_0*, the set of singular point is* S_2*.*

The graph of the surface $\phi_{(\gamma,a_n)}(s,u)$ *generated with* $\gamma(s)$ *can be seen in the following Figure 2.*

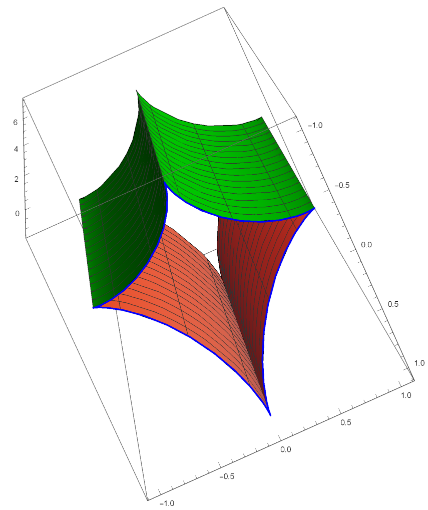

Figure 2: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s,u)$ in the equation (4.2)

- *Let us take* $a_1(s) = \sin s$ *and* $a_2(s) = \cos s$. *Then, we have*

$$a_n(s) = \left(\frac{3}{5}\sin^2 s - \frac{4}{5}\sin s\cos s, -\frac{3}{5}\sin s\cos s + \frac{4}{5}\cos^2 s, \frac{4}{5}\sin s + \frac{3}{5}\cos s\right)$$

and

$$\phi_{(\gamma,a_n)}(s,u) = \left(\sin^3 s, \cos^3 s, -\cos 2s\right) + u\left(\frac{3}{5}\sin^2 s - \frac{4}{5}\sin s\cos s, -\frac{3}{5}\sin s\cos s + \frac{4}{5}\cos^2 s, \frac{4}{5}\sin s + \frac{3}{5}\cos s\right). \tag{4.3}$$

Also, we obtain

$$x(s) = \frac{3}{5}\sin s - \frac{4}{5}\cos s$$

and

$$y(s,u) = 5\cos^2 s\sin s + u.$$

Therefore, the surface $\phi_{(\gamma,a_n)}(s,u)$ *given in the equation* (4.3) *is non-cylindrical and non-developable.*

The following expressions can be given for singularity sets:

(1) *If* $u = 0$ *and* $\alpha(s_0) = 0$, *then* $(s_0, 0) \in S_1$ *is a singular point of the surface. Then,* $\alpha(s_0) = 0$ *for* $s_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ *where* $u = 0$, *so* S_1 *is the set of singular points.*

(2) *If* $\tan(s_0) = \frac{4}{3}$ *and* $u = -5\cos^2(s_0)\sin(s_0) \neq 0$*, then* S_2 *is the set of singular points.*

The graph of the surface $\phi_{(\gamma,a_n)}(s,u)$ *generated with* $\gamma(s)$ *can be seen in the following Figure 3.*

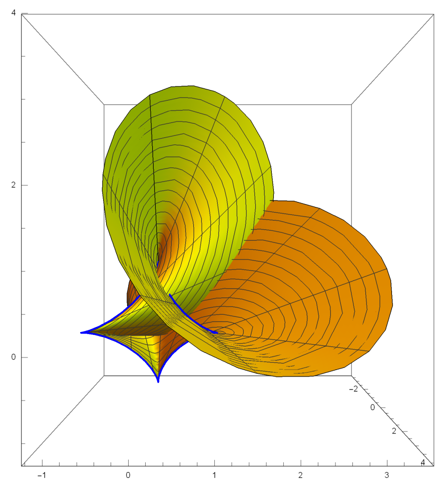

Figure 3: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s,u)$ in equation (4.3)

Example 4.2. *Let* $\phi_{(\gamma,a_n)}$ *be presented by the parametrization*

$$\phi_{(\gamma,a_n)}(s,u) = \gamma(s) + ua_n(s)$$

where

$$\begin{aligned} \gamma &: [0,2\pi) \to \mathbb{R}^3 \\ s &\mapsto \gamma(s) = \left(\frac{3}{2}\sin s - \frac{1}{2}\sin 3s, \frac{3}{2}\cos s - \frac{1}{2}\cos 3s, \sqrt{3}\cos s\right) \end{aligned} \tag{4.4}$$

γ *is a Frenet-type framed base curve with singular points* $s_0 = 0, \pi$*. In the following Figure 4,* $\gamma(s)$ *can be seen.*

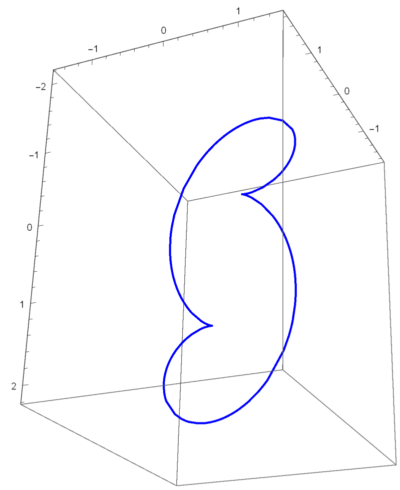

Figure 4: $\gamma(s)$ in the equation (4.4)

Then, the followings are calculated as

$$\begin{cases} \mathcal{T}(s) = \left(\frac{\sqrt{3}}{2}\sin 2s, \frac{\sqrt{3}}{2}\cos 2s, -\frac{1}{2}\right), \\ \mathcal{N}(s) = (\cos 2s, -\sin 2s, 0), \\ \mathcal{B}(s) = \left(-\frac{1}{2}\sin 2s, -\frac{1}{2}\cos 2s, -\frac{\sqrt{3}}{2}\right), \end{cases}$$

and

$$\begin{cases} \kappa(s) = \sqrt{3}, \\ \tau(s) = 1, \\ \alpha(s) = 2\sqrt{3}\sin s. \end{cases}$$

- *Let us choose* $a_1(s) = \cos s$ *and* $a_2(s) = \sin s$. *Then, the followings are given*

$$a_n(s) = \left(\frac{\sqrt{3}}{2}\cos s\sin 2s - \frac{1}{2}\sin s\sin 2s, \frac{\sqrt{3}}{2}\cos s\cos 2s - \frac{1}{2}\sin s\cos 2s, -\frac{1}{2}\cos s - \frac{\sqrt{3}}{2}\sin s\right)$$

and

$$\begin{aligned}&\phi_{(\gamma,a_n)}(s,u)\\&=\left(\frac{3}{2}\sin s-\frac{1}{2}\sin 3s,\frac{3}{2}\cos s-\frac{1}{2}\cos 3s,\sqrt{3}\cos s\right)\\&+u\left(\frac{\sqrt{3}}{2}\cos s\sin 2s-\frac{1}{2}\sin s\sin 2s,\frac{\sqrt{3}}{2}\cos s\cos 2s-\frac{1}{2}\sin s\cos 2s,-\frac{1}{2}\cos s-\frac{\sqrt{3}}{2}\sin s\right).\end{aligned}\tag{4.5}$$

Hence, we get

$$x(s)=\sqrt{3}\cos s-\sin s$$

and

$$y(s,u)=2\sqrt{3}\sin^2 s-u.$$

The surface $\phi_{(\gamma,a_n)}(s,u)$ given in the equation (4.5) is non-cylindrical and non-developable.

Let us give some examinations with respect to the singularity sets:

(1) *For $u=0$ and $\alpha(s_0)=0$ for $s_0=k\pi$ where $k\in\mathbb{Z}$, then the set of singular points is S_1.*

(2) *If $u=2\sqrt{3}\sin^2(s_0)\neq 0$ and $\tan s_0=\sqrt{3}$, then the singularity set of this surface is S_2.*

The graph of the surface $\phi_{(\gamma,a_n)}$ generated with $\gamma(s)$ is given by the following Figure 5.

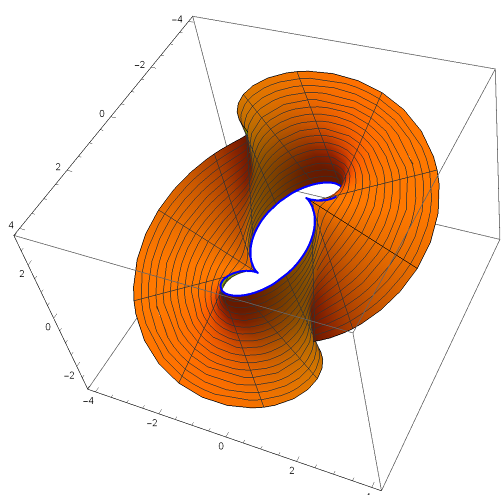

Figure 5: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s,u)$ in the equation (4.5)

- *Let us take* $a_1(s) = \frac{1}{2}$ *and* $a_2(s) = \frac{\sqrt{3}}{2}$*. Therefore, we obtain*

$$a_n(s) = (0, 0, -1)$$

and

$$\phi_{(\gamma,a_n)}(s,u) = \left(\frac{3}{2}\sin s - \frac{1}{2}\sin 3s, \frac{3}{2}\cos s - \frac{1}{2}\cos 3s, \sqrt{3}\cos s - u\right). \quad (4.6)$$

Besides, we attain

$$x(s) = 0$$

and

$$y(s,u) = 3\sin s.$$

Then, the surface $\phi_{(\gamma,a_n)}$ *given in the equation* (4.6) *is developable and cylindrical.*

Additionally, we give the following examinations, as well:

(1) *For* $u = 0$ *and* $\alpha(s_0) = 0$*,* $s_0 = k\pi$ *where* $k \in \mathbb{Z}$*, then the set of singular points is* S_1*.*

(2) *If* $u \neq 0, s_0 = k\pi$ *where* $k \in \mathbb{Z}$*, then the singularity set of this surface is* S_2*.*

The graph of the surface $\phi_{(\gamma,a_n)}$ *generated with* $\gamma(s)$ *is given by Figure 6.*

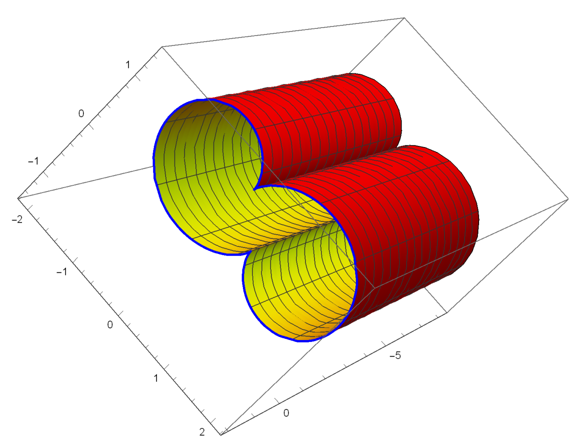

Figure 6: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s,u)$ in equation (4.6)

Example 4.3. *Let $\phi_{(\gamma,a_n)}$ be presented by the parametrization*

$$\phi_{(\gamma,a_n)}(s,u) = \gamma(s) + ua_n(s)$$

where

$$\begin{aligned} \gamma &: (-1,1) \to \mathbb{R}^3 \\ s &\mapsto \gamma(s) = \left(\frac{s^2}{2}, \frac{\sqrt{2}s^3}{3}, \frac{s^4}{4}\right) \end{aligned} \tag{4.7}$$

γ is a Frenet-type framed base curve with singular point $s_0 = 0$. In the Figure 7, $\gamma(s)$ can be seen.

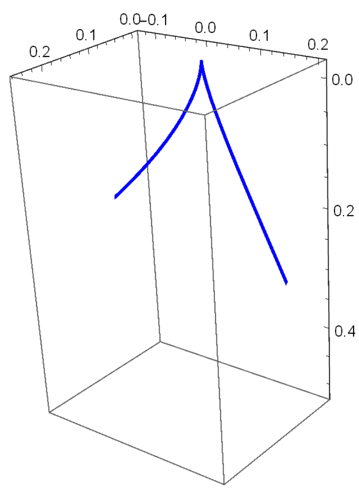

Figure 7: $\gamma(s)$ in the equation (4.7)

Then, the followings are calculated as

$$\begin{cases} \mathcal{T}(s) = \left(\dfrac{1}{s^2+1}, \dfrac{\sqrt{2}s}{s^2+1}, \dfrac{s^2}{s^2+1}\right), \\ \mathcal{N}(s) = \left(\dfrac{-\sqrt{2}s}{s^2+1}, \dfrac{1-s^2}{s^2+1}, \dfrac{\sqrt{2}s}{s^2+1}\right), \\ \mathcal{B}(s) = \left(\dfrac{s^2}{s^2+1}, \dfrac{-\sqrt{2}s}{s^2+1}, \dfrac{1}{s^2+1}\right), \end{cases}$$

and

$$\begin{cases} \kappa(s) = \dfrac{\sqrt{2}}{s^2+1}, \\ \tau(s) = \dfrac{\sqrt{2}}{s^2+1}, \\ \alpha(s) = s^3 + s. \end{cases}$$

- *Let us take $a_1(s) = \sqrt{1-s^2}$ and $a_2(s) = s$. Hence, we obtain*

$$a_n(s) = \left(\frac{\sqrt{1-s^2}+s^3}{s^2+1}, \frac{\sqrt{2}s\sqrt{1-s^2}-\sqrt{2}s^2}{s^2+1}, \frac{s^2\sqrt{1-s^2}+s}{s^2+1} \right)$$

and

$$\phi_{(\gamma,a_n)}(s,u) = \left(\frac{s^2}{2}, \frac{\sqrt{2}s^3}{3}, \frac{s^4}{4} \right) + u \left(\frac{\sqrt{1-s^2}+s^3}{s^2+1}, \frac{\sqrt{2}s\sqrt{1-s^2}-\sqrt{2}s^2}{s^2+1}, \frac{s^2\sqrt{1-s^2}+s}{s^2+1} \right) \tag{4.8}$$

Also, we get

$$x(s) = \frac{\sqrt{2(1-s^2)} - s\sqrt{2}}{s^2+1}$$

and

$$y(s,u) = \frac{(s^4+s^2)\sqrt{1-s^2} - u}{\sqrt{1-s^2}}.$$

It is easy to see that, the surface $\phi_{(\gamma,a_n)}(s,u)$ given in the equation (4.8) is non-cylindrical and non-developable.

Let us give the followings about the singularity set:

(1) *Since $\alpha(0) = 0$ where $u = 0$, then $(0,0) \in S_1$.*

(2) *Since $x(s_0) \neq 0$, S_2 is not the set of singular points.*

The graph of the surface $\phi_{(\gamma,a_n)}$ generated with $\gamma(s)$ is given by Figure 8.

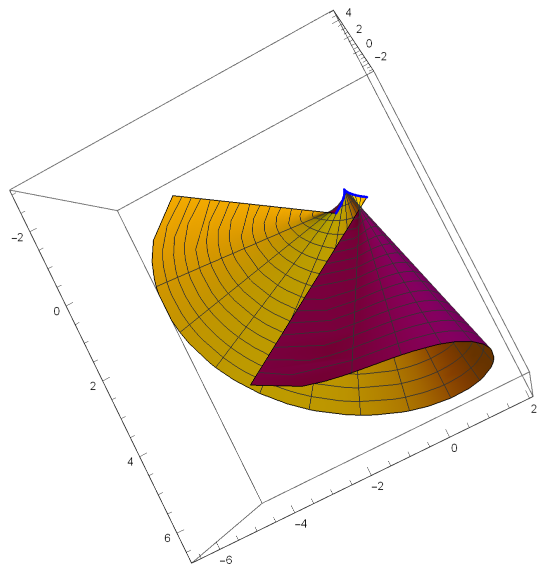

Figure 8: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s,u)$ in the equation (4.8)

- *Let us take* $a_1(s) = \dfrac{\sqrt{2}}{2}$ *and* $a_2(s) = \dfrac{\sqrt{2}}{2}$. *Then, we get*

$$a_n(s) = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$$

and

$$\phi_{(\gamma,a_n)}(s,u) = \left(\frac{s^2}{2} + \frac{\sqrt{2}}{2}u, \frac{\sqrt{2}s^3}{3}, \frac{s^4}{4} + \frac{\sqrt{2}}{2}u\right) \tag{4.9}$$

Additionally,

$$x(s) = 0$$

and

$$y(s,u) = \frac{\sqrt{2}}{2}\left(s^3 + s\right).$$

The surface $\phi_{(\gamma,a_n)}(s,u)$ *given in the equation* (4.9) *is developable and cylindrical.*

The followings can be given for the singularity sets:

(1) *If* $u = 0$ *and* $\alpha(s_0) = 0$*, the* $(s_0, u_0) \in S_1$ *is a singular point of surface. Since* $\alpha(0) = 0$*, the* $(0, 0) \in S_1$ *is sigular point of the surface.*

(2) *Since* $x(s_0) = 0$*, if* $u \neq 0$ *and* $y(s_0, u_0) = 0$*, then* S_2 *is the set of singular points.*

The graph of the surface $\phi_{(\gamma,a_n)}(s, u)$ *generated with* $\gamma(s)$ *can be seen in the Figure 9.*

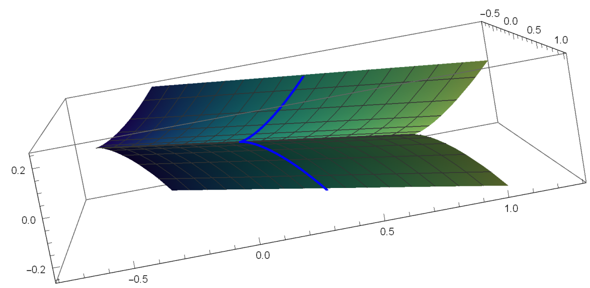

Figure 9: $\gamma(s)$ and $\phi_{(\gamma,a_n)}(s, u)$ in the equation (4.9)

5 Conclusions

The main aims of this study are to determine and examine the concept of the generalized rectifying ruled surfaces with the help of smooth curves with singular points (Frenet-type framed base curves). We obtained some geometric properties and characterization for these new type surfaces. For example, we scrutinized these surfaces with respect to these notions cylindrical, developable, striction curve. In addition to these, singularity types were presented by utilizing the basic singularity theory of curves and the conditions for being a cross-cap surface were scrutinized. Then, we also examined this surfaces as framed surface, and basic invariants and some classifications were found. Also, we gave examinations according to the Gaussian and mean curvatures. Then, we constructed the necessary conditions for these surfaces being flat and minimal surfaces. We investigated the conditions of being immersion, Legendre immersion and framed immersion of them. Moreover, we presented some examples with illustrative figures.

We do not doubt that this study will lead to a new perspective for future studies with respect to the surface theory and singularity theory which have various applications in several and different disciplines. In future studies, we intend to investigate the other types of generalized ruled surfaces of special singular curves.

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