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# On Cauchy Products of $q$-Central Delannoy Numbers 

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#### Abstract

In this study, we have examined $q-$ central Delannoy numbers and their Cauchy products. We have given some related equalities using the properties of recurrence relations. Moreover, using quantum integers, we have obtained the fundamental identities provided by Cauchy products of central Delannoy numbers.


## 1 Introduction

Recurrence relations are used both for calculating and representing complex sequences. In particular, these relations are often used to obtain new integer sequences. One of these types of sequences is the Delannoy sequence. Delannoy numbers are defined by the following relation

$$
\begin{equation*}
d_{n_{1}, n_{2}}=d_{n_{1}-1, n_{2}}+d_{n_{1}, n_{2}-1}+d_{n_{1}-1, n_{2}-1} \tag{1}
\end{equation*}
$$

with $d_{0,0}=1$. When $n_{1}$ or $n_{2}$ is negative number $d_{n_{1}, n_{2}}=0$ is accepted. The numbers $d_{n_{1}, n_{2}}$ are typically also derived from recursive relation (1) or with generating functions. Moreover, these numbers are counted directly as follows [3].

$$
\begin{equation*}
d_{n_{1}, n_{2}}=D\left(n_{1}, n_{2}\right)=\sum_{k=0}^{n_{1}} 2^{k}\binom{n_{2}}{k}\binom{n_{1}}{k} \tag{2}
\end{equation*}
$$

[^0]To put it briefly, Delannoy numbers denoted by $d_{n_{1}, n_{2}}$ are integers that give the number of lattice paths that can be drawn using only steps $(1,0),(0,1)$ and $(1,1)$ from point $(0,0)$ to point $\left(n_{1}, n_{2}\right)$. From the equation (2), $d_{n_{1}, n_{2}}=$ $d_{n_{2}, n_{1}}$. The generating function belonging to these numbers is given with the help of the following equation.

$$
\begin{equation*}
G(x, y)=\frac{1}{1-x-y-x y}=\sum_{n_{1}, n_{2} \geq 0} d_{n_{1}, n_{2}} x^{n_{1}} y^{n_{2}} \tag{3}
\end{equation*}
$$

where $x, y$, and $x y$ represent the steps $(1,0),(0,1)$, and $(1,1)$ respectively. Also, for positive integers $n_{1}, n_{2}$ with property $n_{1} \leq n_{2}$ the following equality is satisfied [11].

$$
\begin{equation*}
d_{n_{1}, n_{2}}=\sum_{i=0}^{n_{1}}\binom{n_{2}}{n_{1}-i}\binom{n_{2}+i}{i} \tag{4}
\end{equation*}
$$

In the case of $n_{1}=n_{2}=n$, these numbers are reduced to the following numbers.

$$
\begin{equation*}
D\left(n_{1}, n_{2}\right)=D(n)=\sum_{k}\binom{n}{k}\binom{n+k}{k}=\sum_{k} \frac{2 n!}{k!k!(n-k)!} \tag{5}
\end{equation*}
$$

The numbers $D(n)$ are called central Delannoy numbers. These numbers have appeared as properties of lattice and posets. For some studies on these numbers can be looked at [4],[6],[9], [10], [12], [13], and [14]. In 2011, Sun studied the Schröder numbers and also provided a generalization of the $D(n)$ numbers, and the relationships between central Delannoy numbers, and the Schröder numbers [11]. Some elements of the sequence $D(n)$ are $1,3,13,63,321,1683, \ldots$, and their generating function is

$$
\begin{equation*}
G(x)=\frac{1}{1-6 x+x^{2}}=\sum_{n \geq 0} D(n) x^{n}=1+3 x+13 x^{2}+\ldots \tag{6}
\end{equation*}
$$

For the numbers $\sum_{l=0}^{n} D(l) D(n-l)$ called Cauchy products, $C$ - products. In [2], the authors gave the following equations [13].

$$
\begin{gather*}
\sum_{l=0}^{n} D(l) D(n-l)=(-1)^{n} \operatorname{det}(\operatorname{diag}(1,-6,1)), \quad n \geq 1  \tag{7}\\
\operatorname{det}(\operatorname{diag}(1,-6,1))=\frac{1}{6^{n}} \sum_{l=0}^{n}(-1)^{l} 6^{2 l}\binom{l}{n-l} \tag{8}
\end{gather*}
$$

Moreover, the authors of this study stated that if any complex number $c$ is written instead of the value -6 , the following equality is also valid. And they
gave the following equality using the tridiagonal matrix $M_{n}(c)$ which is of $n \times n$ type
$\operatorname{det}\left(M_{n}(c)\right)=\frac{(-1)^{n}}{c^{n}} \sum_{l=0}^{n}(-1)^{l} c^{2 l}\binom{l}{n-l}=c^{n} \sum_{m=0}^{n} \frac{(-1)^{m}}{c^{2 m}}\binom{n-m}{m}=D_{n}(c)$.
In [12], the authors gave generating function and iterative correlation for the numbers $D_{n}(c)$ obtained with the help of the last equation. It should be noted that the recursive relation of these numbers is

$$
\begin{equation*}
D_{n}(c)=c D_{n-1}(c)-D_{n-2}(c), \quad n \geq 2 \tag{10}
\end{equation*}
$$

with $D_{0}(c)=1, D_{1}(c)=c$. In addition to these, these authors gave the $C$ - products of these numbers and their relations, showing their relation to Chebyshev polynomials. Moreover, the following equations are provided by the elements of this sequence.

$$
\begin{gather*}
D_{n}(c)=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, \quad c \neq \pm 2 .  \tag{11}\\
D_{n}(c)=n+1, \quad c=2 ; \quad D_{n}(c)=(-1)^{n}(n+1), \quad c=-2 . \tag{12}
\end{gather*}
$$

Here the values of $\alpha, \beta$ are roots of the characteristic equation of the sequence $\left\{D_{n}(c)\right\}$. So, in order to write the sequence $\left\{D_{n}(c)\right\}$ explicitly, for $c \neq \pm 2$, we give

$$
\begin{equation*}
\left\{D_{n}(c)\right\}_{n \geq 0}=\left\{1, c, c^{2}-1, c^{3}-2 c, c^{4}-3 c^{2}+1, \ldots, D_{n}(c), \ldots\right\} \tag{13}
\end{equation*}
$$

The aim of our study is to examine some basic and essential properties of the sequence in question by using both recursion relations and quantum integers.

## 2 q- Analog Representation for $D_{n}(c)$

In our work in [5], we examined Cauchy products of central Delannoy numbers and derived some important identities, such as Cassini Catalan, using recursion relations.

In this section, we analyzed $q$-analog representation for elements of the sequence $\left\{D_{n}(c)\right\}$ by using the relations of the elements of the second-order integer sequences involving $q$-integers. It should be noted that quantum calculus, or $q$-calculus for short, which was systematically developed by F. H. Jackson, gained importance again with the transfer of structures known by
many mathematicians to this area. The general properties of quantum integers are studied in detail in [7] and [1]. A quantum integer is a polynomial in $q$ of the form

$$
\begin{equation*}
[n]_{q}=1+q+q^{2}+\ldots+q^{n-1}=[n] \tag{14}
\end{equation*}
$$

where $n$ is any natural number $[7]$. As $q$ goes to $1,[n]_{q}$ goes to $n$ and $[0]_{q}=0$ for $n=0$. Accordingly, the equation $[-n]_{q}=-q^{-n}[n]_{q}$ can also be written for negative real numbers. With the help of this definition, the $q$-factorial equation is

$$
[n]!=[n][n-1] \ldots[1]
$$

where $n$ is a positive integer. Thus, the $q$-binomial coefficients for nonnegative integers $n$ and $k$ are as follows [7]

$$
\binom{n}{k}_{q}=\frac{([n][n-1] \ldots[n-k+1])}{([k][k-1] \ldots[1])}
$$

where $\lim _{q \rightarrow 1^{-}}\binom{n}{k}_{q}=\binom{n}{k}$. With the help of these definitions, we obtain the $q-$ form of central Delannoy numbers, also called $q$-central Delannoy numbers. That is, for all $q \in C-\{1\}, n \in N$, we have

$$
\begin{equation*}
D_{q}(n)=\sum_{k=0}^{n}\binom{n}{k}_{q}\binom{n+k}{n}_{q} q^{\binom{k+1}{2}} \tag{15}
\end{equation*}
$$

For $n=3, D_{q}(3)=\left(1+q+2 q^{2}+3 q^{3}+2 q^{4}+2 q^{5}+q^{6}+q^{7}\right)$. When $q=1$, we get the known central Delannoy numbers.
Now, we have dealt with the identities provided by the elements of the sequence $\left\{D_{n}(c)\right\}$ using quantum calculus. In particular, we examine the elements of this sequence for the numbers $c \neq \pm 2$. For this purpose, we assume $i=\alpha \sqrt{-q}$ depending on the root $\alpha$, with $q \in C-\{1\}, n \in N$. Notice that we give all identities using only the numbers $\alpha$ and $q$.

Below, we give $q$ - Binet form, which gives the general term of the sequence $\left\{D_{n}(c)\right\}$.

Theorem 1. For the numbers $q \neq 1 ; q, c \in C$ and $n \geq 0$ the following equality is satisfied.

$$
\begin{equation*}
D_{n}(c)=\alpha^{n}[n+1]_{q} . \tag{16}
\end{equation*}
$$

Proof. If we use the equation $[n]_{q}=\frac{1-q^{n}}{1-q}$ and the definition of $q$, then we obtain

$$
D_{n}(c)=\frac{1}{\sqrt{c^{2}-4}}\left(\alpha^{n+1}-q^{n+1} \alpha^{n+1}\right)=\frac{\alpha^{n+1}\left(1-q^{n+1}\right)}{\alpha(1-q)}=\alpha^{n}[n+1]_{q}
$$

which completes the proof. Indeed, for $n=0,1,2$, one can get

$$
D_{0}(c)=[1]_{q}, \quad D_{1}(c)=\alpha[2]_{q}=c, \quad D_{2}(c)=\alpha^{2}[3]_{q}=c^{2}-1
$$

Corollary 1. The root $\alpha$ of the recursive relation related to the $\left\{D_{n}(c)\right\}$ provides the following equality.

$$
\begin{equation*}
\alpha^{n}=\alpha D_{n-1}(c)-D_{n-2}(c) . \tag{17}
\end{equation*}
$$

Proof. If the second side of the equation is considered with the help of Binet form and definition of $q$, then the correctness of the desired equation can be seen. So,

$$
\begin{gathered}
\alpha D_{n-1}(c)-D_{n-2}(c)=\alpha^{n}[n]_{q}-\alpha^{n-2}[n-1]_{q} \\
\alpha D_{n-1}(c)-D_{n-2}(c)=\frac{\alpha^{n}}{1-q}\left\{\left(1-q^{n}\right)-q\left(1-q^{n-1}\right)\right\}=\alpha^{n}
\end{gathered}
$$

is obtained. Thus, the claim is true.
Corollary 2. Successive elements of the sequence $\left\{D_{n}(c)\right\}$ are prime between them.

$$
\begin{equation*}
\left(\alpha^{n}[n+1]_{q}, \quad \alpha^{n+1}[n+2]_{q}\right)=1 \tag{18}
\end{equation*}
$$

Proof. We write

$$
\begin{align*}
\frac{D_{n+1}(c)}{D_{n}(c)} & =\alpha^{n}\left\{\frac{[n]_{q}+q^{n}(1+q)}{[n]_{q}+q^{n}}\right\}=\alpha^{n}\left\{\frac{1-q^{n}+q^{n}\left(1-q^{2}\right)}{1-q^{n+1}}\right\} \\
& =\alpha^{n}\left\{\frac{1-q^{n+2}}{1-q^{n+1}}\right\} . \tag{19}
\end{align*}
$$

So, the claim is true.
Below, we give Cassini's identity, one of the important identities provided by the elements of the sequence $\left\{D_{n}(c)\right\}$.

Theorem 2. For $\left\{D_{n}(c)\right\}$, we have

$$
\begin{equation*}
D_{n+1}(c) D_{n-1}(c)-D_{n}^{2}(c)=-q^{n} \alpha^{2 n} \tag{20}
\end{equation*}
$$

Proof. Using definition of the $n-t h$ term,

$$
D_{n+1}(c) D_{n-1}(c)-D_{n}^{2}(c)=\frac{\alpha^{2} n}{(1-q)^{2}}\left\{\left(1-q^{n}\right)\left(1-q^{n+2}\right)-\left(1-q^{n+1}\right)^{2}\right\}
$$

can be written. If necessary algebraic operations are done after this, then for the second side of equality

$$
\frac{\alpha^{2} n}{(1-q)^{2}}\left\{(1-q)\left(q^{n+1}-q^{n}\right)\right\}
$$

is written. Thus, we get

$$
D_{n+1}(c) D_{n-1}(c)-D_{n}^{2}(c)=\frac{\alpha^{2} n}{(1-q)^{2}}\left\{(-q)^{n}(1-q)^{2}\right\}=-q^{n} \alpha^{2 n}
$$

The last equality is the desired result.
Corollary 3. The sequence $\left\{D_{n}(c)\right\}$ is concave.
Proof. For $q$-analog recursive relation, we write the following equation.

$$
\begin{equation*}
\alpha^{n+1}[n+2]_{q}=c \alpha^{n}[n+1]_{q}-\alpha^{n-1}[n]_{q} \tag{21}
\end{equation*}
$$

To see the truth of the claim, the last equation and Cassini identity are used

$$
D_{n+1}(c) D_{n-1}(c)-D_{n}^{2}(c)=-q^{n} \alpha^{2 n}
$$

then the following inequality is obtained.

$$
\alpha^{2 n}[n+1]_{q}^{2}-\alpha^{2 n}[n]_{q}[n+2]_{q} \succ 0
$$

which is the desired result.
Now, we give the $q$-generating function for the sequence $\left\{D_{n}(c)\right\}$ below.

Theorem 3. The $q$-analog generating function for elements of the sequence $\left\{D_{n}(c)\right\}$ is

$$
\begin{equation*}
G(x)=\frac{1+\left(\alpha[2]_{q}-c[1]_{q}\right) x}{x^{2}-c x+1} \tag{22}
\end{equation*}
$$

Proof. To prove the correctness of the claim, the following three equations can be written using the Binet form provided by the elements of the sequence $\left\{D_{n}(c)\right\}$ and the recursive relation:

$$
\begin{gathered}
\sum_{n \geq 0} \alpha^{n}[n+1]_{q} x^{n}=[1]_{q}+\alpha[2]_{q} x+\alpha^{2}[3]_{q} x^{2}+\ldots+\alpha^{n}[n+1]_{q} x^{n}+\ldots, \\
\sum_{n \geq 0} \alpha^{n}[n+1]_{q} x^{n+2}=[1]_{q} x^{2}+\alpha[2]_{q} x^{3}+\alpha^{2}[3]_{q} x^{4}+\ldots+\alpha^{n}[n+1]_{q} x^{n+2}+\ldots,
\end{gathered}
$$

$\sum_{n \geq 0}-c \alpha^{n}[n+1]_{q} x^{n+1}=-c[1]_{q} x-c \alpha[2]_{q} x^{2}-c \alpha^{2}[3]_{q} x^{3}-\ldots-c \alpha^{n}[n+1]_{q} x^{n+1}-\ldots$.
If these equations are rearranged using the recursive relation, then the desired equation is obtained.

The following theorem gives the Catalan identity, which is one of the important identities that the elements of this sequence provide.

Theorem 4. For $n \geq k$, we have

$$
\begin{equation*}
D_{n+k}(c) D_{n-k}(c)-D_{n}^{2}(c)=-q^{n-k+1} \alpha^{2 n}[k]_{q} \tag{23}
\end{equation*}
$$

Proof. From the definition $D_{n}(c)$,

$$
D_{n+k}(c) D_{n-k}(c)-D_{n}^{2}(c)=\alpha^{2 n}\left([n+k+1]_{q}[n-k+1]_{q}-[n+1]_{q}[n+1]_{q}\right)
$$

can be written. If the $q$-analog definition of $n$ numbers is also used, the following equations are obtained.

$$
\begin{gathered}
D_{n+k}(c) D_{n-k}(c)-D_{n}^{2}(c)=\alpha^{2 n} \frac{q^{n+1}\left(2-q^{k}-q^{-k}\right.}{(1-q)^{2}}=\frac{-\alpha^{2 n} q^{n+1}\left(1-q^{k}\right)^{2}}{(1-q)^{2} q^{k}} \\
D_{n+k}(c) D_{n-k}(c)-D_{n}^{2}(c)=-q^{n-k+1} \alpha^{2 n}[k]_{q}
\end{gathered}
$$

Note that this Catalan identity is given when $k=1$ reduces to Cassini's identity. Indeed,

$$
D_{n+1}(c) D_{n-1}(c)-D_{n}^{2}(c)=-q^{n} \alpha^{2 n}
$$

Thus, it is shown that the alleged equality is true.
In the following, we give the elements of the sequence $\left\{D_{n}(c)\right\}$ provide equality as called Vajda's identity.

Theorem 5. For elements of the sequence $\left\{D_{n}(c)\right\}$, we have

$$
\begin{equation*}
D_{n+m}(c) D_{n+k}(c)-D_{n}(c) D_{n+m+k}(c)=\alpha^{2 n+m+k} q^{n+1}[m]_{q}[k]_{q} \tag{24}
\end{equation*}
$$

Proof. From the definition of the formulas $D_{n+m}(c), D_{n+k}(c)$, the first side of the desired equation is written as follows

$$
\begin{aligned}
& \alpha^{2 n+m+k}\left([n+k+1]_{q}[n+m+1]_{q}-[n+1]_{q}[n+m+k+1]_{q}\right) \\
& \quad=\alpha^{2 n+m+k}\left\{\frac{q^{n+k+1}\left(q^{m}-1\right)+q^{n+1}(1-q)^{m}}{(1-q)^{2}}\right\}
\end{aligned}
$$

Then, if necessary simplifications and calculations are taken, then

$$
D_{n+m}(c) D_{n+k}(c)-D_{n}(c) D_{n+m+k}(c)=-\alpha^{2 n+m+k} \frac{\left(1-q^{m}\right) q^{n+1}\left(1-q^{k}\right)}{(1-q)^{2}}
$$

is obtained. Using the definition of $q$-integer in this last equation, the following equality

$$
D_{n+m}(c) D_{n+k}(c)-D_{n}(c) D_{n+m+k}(c)=\alpha^{2 n+m+k} q^{n+1}[m]_{q}[k]_{q}
$$

is obtained so that the proof is finished. For $k=1$, this equality is as follows.

$$
D_{n+m}(c) D_{n+1}(c)-D_{n}(c) D_{n+m+1}(c)=\alpha^{2 n+m+1} q^{n+1}[m]_{q}
$$

In the following, we prove that the elements of the sequence $\left\{D_{n}(c)\right\}$ satisfy the d'Ocagne identity.

Theorem 6. For elements of the sequence $\left\{D_{n}(c)\right\}$, we have

$$
\begin{equation*}
D_{m}(c) D_{n+1}(c)-D_{n}(c) D_{m+1}(c)=\alpha^{n+m+1} q^{n+1}[m-n]_{q} \tag{25}
\end{equation*}
$$

Proof. Using the necessary definitions, the second side of the claimed equation can be written as follows.

$$
\frac{\alpha^{n+m+1}}{(1-q)^{2}}\left\{\left(1-q^{m+1}\right)\left(1-q^{n+2}\right)-\left(1-q^{n+1}\right)\left(1-q^{m+2}\right)\right\}
$$

In here, if the necessary operations are done, then

$$
\begin{gathered}
D_{m}(c) D_{n+1}(c)-D_{n}(c) D_{m+1}(c)=\frac{\alpha^{n+m+1}}{(1-q)^{2}}\left\{q^{n+1}-q^{m+1}\right\} \\
D_{m}(c) D_{n+1}(c)-D_{n}(c) D_{m+1}(c)=\alpha^{n+m+1}\left\{[m+1]_{q}-[n+1]_{q}\right\}
\end{gathered}
$$

is obtained. And then,

$$
\begin{aligned}
D_{m}(c) D_{n+1}(c)-D_{n}(c) D_{m+1}(c) & =\alpha^{n+m+1} q^{n+1}\left(\frac{1-q^{m-n}}{1-q}\right) \\
& =\alpha^{n+m+1} q^{n+1}[m-n]_{q}
\end{aligned}
$$

is achieved. So, the proof is completed.
Theorem 7. The following equality is provided for the elements of sequence $\left\{D_{n}(c)\right\}$;

$$
\begin{equation*}
D_{n+1}(c)+D_{n-1}(c)=c \alpha^{k}[k+1]_{q} \tag{26}
\end{equation*}
$$

Proof. From the definition $D_{n}(c)$,

$$
\begin{aligned}
D_{n+1}(c)+D_{n-1}(c) & =\alpha^{k+1}\left\{\frac{\left(1-q^{k+2}\right)+q\left(1-q^{k}\right)}{1-q}\right\} \\
& =\alpha^{k+1}\left\{\frac{(1+q) q\left(1-q^{k+1}\right.}{1-q}\right\} \\
D_{n+1}(c)+D_{n-1}(c) & =\alpha^{k+1}(1+q)[k+1]_{q}
\end{aligned}
$$

is obtained. Also, considering $\frac{c}{\alpha}$ which is the value of $1+q$ in the last equation,

$$
D_{n+1}(c)+D_{n-1}(c)=c \alpha^{k}[k+1]_{q}
$$

is obtained that the proof is completed.
Corollary 4. The following equality is provided for elements of the sequence $\left\{D_{n}(c)\right\}$;

$$
\begin{equation*}
D_{k+n}(c)+D_{k-n}(c)=\alpha^{k+n}\left(1+q^{n}\right)[k+1]_{q} . \tag{27}
\end{equation*}
$$

Note that if the last equation is used and $n$ is written instead of $k$, then the following equation is obtained, which is a very useful formula.

$$
\begin{equation*}
D_{2 n}(c)=\alpha^{2 n}\left(1+q^{n}\right)[n+1]_{q}-1 \tag{28}
\end{equation*}
$$

## 3 Conclusion

In this paper, we have considered a special sequence related to Delannoy numbers. We have dealt with the properties of the sequence that have been examined using $q$-calculus. Since many properties of sequences can be examined much more easily with the help of $q$-calculus, this study can be used in similar studies.

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