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Fibonacci and Lucas Polynomials in n-gon

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Abstract

In this paper, we bring into light, study the polygonal structure of Fibonacci polynomials that are placed clockwise on these by a number corresponding to each vertex. Also, we find the relation between the numbers with such vertices. We present a relation for obtained sequence in an n-gon yielding the m-th term formed at k vertices. Also, we apply these situations to Lucas polynomials and find new recurrence relations. Then, the numbers obtained by writing the coefficients of these polynomials in step form are shown in OEIS.

1 Introduction

Knowing about sequences of numbers, especially the Fibonacci numbers, is an interesting tool for many researchers [1, 6, 7, 8, 9, 10, 11, 19]. Starting from the Fibonacci sequences, many new sequences were defined, and their properties were examined, often by changing the initial conditions [5, 20].
At the same time, many authors work on polynomials defined with the help of number sequences [3, 4, 12, 13, 14, 16, 17, 18].
One of the latest studies in this subject is [2] where the authors placed Pell numbers clockwise on the vertices of the polygons and some properties about obtained new sequences. Then in [21], the work in [2] was moved to the Pell and Pell-Lucas polynomials. They obtained interesting properties about the coefficients of their polynomials for the new polynomial sequences formed at the vertices.

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Different concrete mixed initial-boundary values problems are addressed in the papers works [22, 23, 24]. Here, techniques are used for the existence of the solutions of the considered problems, for their uniqueness or for continuous dependence in relation to the initial data, or boundary conditions or supply terms.

In this study, we made a new study for Fibonacci and Lucas polynomials based on [2]. The general term for the polynomial sequences formed at each vertex of the regular n-gon is given. Moreover, when the coefficients of both new polynomial sequences and term derivatives formed at the vertices of a regular n-gone were examined, it was seen that they were known number sequences in OEIS [15].

Now, let us remind the following definitions that are very well known.

The Fibonacci numbers are given by

$$F_{n+2} = F_n + F_{n+1}, n \geq 0$$

with $F_0 = 0$ and $F_1 = 1$.

$x^2 - x - 1 = 0$, the characteristic equation, and the roots, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, Binet formula, $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

The Lucas numbers are given by

$$L_n = L_{n-1} + L_{n-2} , n \geq 2 \ (n \in \mathbb{N})$$

with $L_0 = 2$ and $L_1 = 1$.

Its Binet formula is $L_n = \alpha^n + \beta^n$.

For $n \geq 2$, Fibonacci polynomial is given by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

such that $F_0(x) = 0$, $F_1(x) = 1$.

For $n \geq 2$, Lucas polynomials is given by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

such that $L_0(x) = 2$, $L_1(x) = x$.

And we know that

$$F_{-n}(x) = (-1)^{n-1} F_n(x)$$

and

$$L_{-n}(x) = (-1)^n L_n(x).$$

We know that the limit of consecutive terms of the Fibonacci sequence gives the golden ratio. In fact, if the golden ratio is interpreted geometrically, it depends on the regular pentagon. Equal sides of a regular pentagon and diagonals passing through equal 108° angles between equal sides form a star-shaped pentagram. The ratio of the side lengths of a regular pentagon to the diagonal lengths gives the golden ratio. From this point of view, in this article, new recurrences on n-gon are defined by focusing on how the proportional relationship can be generalized.

2 Main Results

2.1 New Relations for Fibonacci Polynomials

Let k, m show the vertex number and the order of term of the sequence occurring at any vertex in any n-gon such that $0 \leq k < n$ and $m \geq 1$, respectively.
For $n = 1$, let's write the Fibonacci polynomials over a dot as shown in Figure 1.

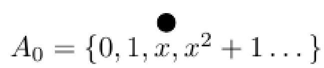

Figure 1: Fibonacci polynomials placed over a dot.

For $n = 1$, we get $k = 0$. So, we obtain

$$F_{(m-1)n+k}(x) = F_{m-1}(x) = L_1(x) F_{(m-2)}(x) + F_{(m-3)}(x)$$

which is the Fibonacci polynomials. Let's write the Fibonacci polynomials for $n = 2$ consecutively at start and end points of a line segment as shown in Figure 2.

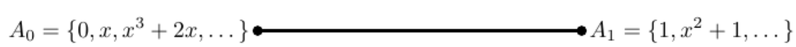

Figure 2: Fibonacci polynomials at the endpoints of a segment.

We have

$$F_{(m-1)n+k}(x) = F_{(m-1)2+k}(x) = L_2(x) F_{(m-2)2+k}(x) + F_{(m-3)2+k}(x)$$

For $n = 3$, let's place the Fibonacci polynomials as shown in Figure 3.
So, we get

$$F_{(m-1)n+k}(x) = F_{(m-1)3+k}(x) = L_3(x) F_{(m-2)3+k}(x) + F_{(m-3)3+k}(x)$$

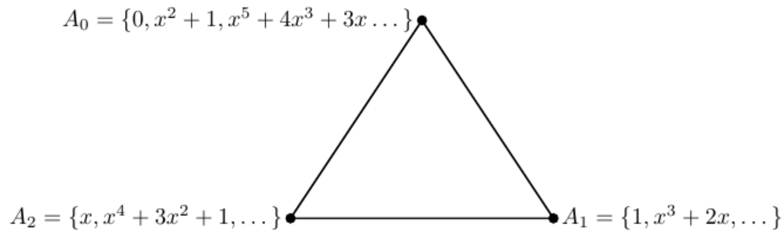

Figure 3: Fibonacci polynomials placed at the vertices of a triangle

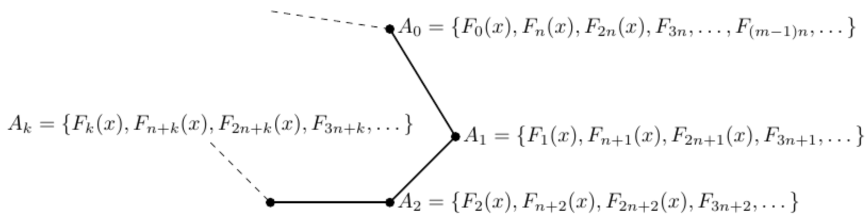

Figure 4: Fibonacci polynomials in the vertices of an n-gon

Let's place the Fibonacci polynomials on an n-gon as shown in Figure 4.
Let's exemplify this situation. In a 3-gon, let's try to find the 2nd term of the sequence at point A_1 In this case, $n = 3$, $m = 2$ and $k = 1$, $F_4(x)$. The value we will find corresponds to the second term of the sequence $A_1 = \left\{1, x^3+2x, x^6+5x^4+6x^2+1, \ldots\right\}$ in Figure 3. That is, x^3+2x. Here, the numbering of terms for the sequence in the corner is started with 1, $A_k = \{x_1, x_2, \ldots\}$.
So, we have

$$F_{(m-1)n+k}(x) = F_{(m-1)3+k}(x) = (x^3+3x)F_{(m-2)3+k}(x) + F_{(m-3)3+k}(x)$$

$$F_4(x) = (x^3+3x)F_1(x) + F_{-2}(x)$$
$$= (x^3+3x)1 - F_2(x) = (x^3+3x)1 - x = x^3+2x.$$

Notice that the coefficient of $F_{(m-2)n+k}(x)$ and $F_{(m-3)n+k}(x)$ is the $L_n(x)$ and $(-1)^{n+1}$, respectively.
After all this preparation, we can now give the first main theorem.

Theorem 2.1.1. *The mth term of the sequence in the vertex $A_k$is as follows*

$$F_{(m-1)n+k}(x) = L_n(x)F_{(m-2)n+k}(x) - (-1)^n F_{(m-3)n+k}(x), 0 \le k < n \quad (2.1)$$

Proof. From Binet formulas,

$$\begin{aligned}
& F_{(m-2)n+k}(x)\, L_n(x) - (-1)^n F_{(m-3)n+k}(x) \\
= \quad & \frac{\alpha^{(m-2)n+k}(x) - \beta^{(m-2)n+k}(x)}{\alpha(x)-\beta(x)} . \left(\alpha^n(x) + \beta^n(x)\right) \\
- \quad & (-1)^n . \frac{\alpha^{(m-3)n+k}(x) - \beta^{(m-3)n+k}(x)}{\alpha(x)-\beta(x)} \\
= \quad & \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x)-\beta(x)} \\
+ \quad & \frac{\beta^n(x)\,\alpha^{mn-2n+k}(x) - \alpha^n(x)\,\beta^{mn-2n+k}(x)}{\alpha(x)-\beta(x)} \\
- \quad & \frac{(-1)^n.\alpha^{mn-3n+k}(x) + (-1)^n.\beta^{mn-3n+k}(x)}{\alpha(x)-\beta(x)} \\
= \quad & \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1n+k}(x)}{\alpha(x)-\beta(x)} \\
= \quad & F_{(m-1)n+k}(x).
\end{aligned}$$

□

Theorem 2.1.2. *The relation between the polynomials corresponding to the vertex $A_k$is follows*

$$\begin{aligned}
& F_{mn+k}(x) \\
= \quad & F_{n+k}(x) \left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}} (-1)^{(n+1)t} \binom{m-t-1}{t} L_n^{m-2t-1}(x) \right) \\
+ \quad & F_k(x) \left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}} (-1)^{(n+1)(t+1)} \binom{m-t-2}{t} L_n^{m-2t-2}(x) \right)
\end{aligned}$$

where $0 \le k < n$.

Proof. From the induction method on m, for $m=1$, we obtain

$$\begin{aligned} F_{n+k}(x) &= F_{n+k}(x)\left(\sum_{t=0}^{0}(-1)^{(n+1)t}\binom{-t}{t}L_n^{-2t}(x)\right) \\ &+ F_k(x)\left(\sum_{t=0}^{-1}(-1)^{(n+1)(t+1)}\binom{-t-1}{t}L_n^{-2t-2}(x)\right) \\ &= F_{n+k}(x)\left((-1)^0\binom{0}{0}(L_n(x))^0\right)+F_n(x)\left((-1)^{n+1}\binom{-1}{0}(L_n(x))^{-2}\right) \\ &= F_{n+k}(x) \end{aligned}$$

The result is true for $m=s$. So,

$$\begin{aligned} F_{sn+k}(x) = F_{n+k}(x)&\left(\sum_{t=0}^{\frac{2s-3-(-1)^s}{4}}(-1)^{(n+1)t}\binom{s-t-1}{t}L_n^{s-2t-1}(x)\right) \\ &+ F_k(x)\left(\sum_{t=0}^{\frac{2s-5-(-1)^{s-1}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-2}{t}L_n^{s-2t-2}(x)\right) \end{aligned} \tag{2.2}$$

For $m=s+1$, if we use equation (2.1) then we obtain

$$F_{(s+1)n+k}(x)=F_{sn+k}(x)L_n(x)-(-1)^nF_{(s-1)n+k}(x)$$

By using equation equation (2.2), we have

$$\begin{aligned} &F_{(s+1)n+k}(x) \\ &=F_{n+k}(x)\left(\sum_{t=0}^{\frac{2s-3-(-1)^s}{4}}(-1)^{(n+1)t}\binom{s-t-1}{t}L_n^{s-2t-1}(x)\right).L_n(x) \\ &+F_k(x)\left(\sum_{t=0}^{\frac{2s-5-(-1)^{s-1}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-2}{t}L_n^{s-2t-2}(x)\right).L_n(x) \\ &-(-1)^nF_{n+k}(x)\left(\sum_{t=0}^{\frac{2s-5-(-1)^{s-1}}{4}}(-1)^{(n+1)t}\binom{s-t-2}{t}L_n^{s-2t-2}(x)\right) \\ &-(-1)^nF_k(x)\left(\sum_{t=0}^{\frac{2s-7-(-1)^{s-2}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-3}{t}L_n^{s-2t-3}(x)\right). \end{aligned}$$

When $s=2r$, we have

$$
\begin{aligned}
F_{(s+1)n+k}(x) = F_{n+k}(x) & \left(\sum_{t=0}^{r-1}(-1)^{(n+1)t}\binom{2r-t-1}{t}L_n^{2r-2t}(x)\right) \\
& +F_{n+k}(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-2}{t}L_n^{2r-2t-2}(x)\right) \\
& +F_k(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-2}{t}L_n^{2r-2t-1}(x)\right) \\
& +F_k(x)\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+2)}\binom{2r-t-3}{t}L_n^{2r-2t-3}(x)\right) \\
& =F_{n+k}(x)\left[\binom{2r-1}{0}L_n^{2r}(x)+(-1)^{n+1}\binom{2r-2}{1}L_n^{2r-2}(x)\right. \\
& \quad +\binom{2r-3}{2}L_n^{2r-4}(x)+\ldots+(-1)^{(n+1)(r-1)}\binom{r}{r-1}L_n^2(x) \\
& \quad +(-1)^{n+1}\binom{2r-2}{0}L_n^{2r-2}(x)+\binom{2r-3}{1}L_n^{2r-4}(x) \\
& \left.+\cdots+(-1)^{(n+1)(r-1)}L_n^2(x)+(-1)^{(n+1)r}\binom{r-1}{r-1}\right] \\
& +F_k(x)\left[(-1)^{n+1}\binom{2r-2}{0}L_n^{2r-1}(x)+\binom{2r-3}{1}L_n^{2r-3}(x)\right. \\
& \quad +(-1)^{n+1}\binom{2r-4}{2}L_n^{2r-5}(x) \\
& +\cdots+(-1)^{(n+1)r}\binom{r-1}{r-1}L_n(x)+\binom{2r-3}{0}L_n^{2r-3}(x) \\
& \quad +(-1)^{n+1}\binom{2r-4}{1}L_n^{2r-5}(x) \\
& +\binom{2r-5}{2}L_n^{2r-7}(x)+\cdots+(-1)^{(n+1)r}\binom{r-1}{r-2}L_n(x) \\
& =F_{n+k}(x)\left(\sum_{t=0}^{r}(-1)^{(n+1)t}\binom{2r-t}{t}L_n^{2r-2t}(x)\right) \\
& +F_k(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-1}{t}L_n^{2r-2t-1}(x)\right) \\
& =F_{n+k}(x)\left(\sum_{t=0}^{\frac{2s-1-(-1)^{s+1}}{4}}(-1)^{(n+1)t}\binom{s-t}{t}L_n^{s-2t}(x)\right) \\
& +F_k(x)\left(\sum_{t=0}^{\frac{2s-3-(-1)^{s}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-1}{t}L_n^{s-2t-1}(x)\right)
\end{aligned}
$$

when $s = 2r - 1$, we have

$$\begin{aligned}
&= F_{n+k}(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)t}\binom{2r-t-2}{t}L_n^{2r-2t-1}(x)\right)\\
&+F_{n+k}(x)\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+1)}\binom{2r-t-3}{t}L_n^{2r-2t-3}(x)\right)\\
&+F_k(x)\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+1)}\binom{2r-t-3}{t}L_n^{2r-2t-2}(x)\right)\\
&+F_k(x)\left(\sum_{t=0}^{r-2}(-1)^{(n+1)(t+2)}\binom{2r-t-4}{t}L_n^{2r-2t-4}(x)\right)\\
&=F_{n+k}(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)t}\binom{2r-t-1}{t}L_n^{2r-2t-1}(x)\right)\\
&+F_k(x)\left(\sum_{t=0}^{r-1}(-1)^{(n+1)(t+1)}\binom{2r-t-2}{t}L_n^{2r-2t-2}(x)\right)\\
&=F_{n+k}(x)\left(\sum_{t=0}^{\frac{2s-1-(-1)^{s+1}}{4}}(-1)^{(n+1)t}\binom{s-t}{t}L_n^{s-2t}(x)\right)\\
&+F_k(x)\left(\sum_{t=0}^{\frac{2s-3-(-1)^{s}}{4}}(-1)^{(n+1)(t+1)}\binom{s-t-1}{t}L_n^{s-2t-1}(x)\right).
\end{aligned}$$

□

2.2 New Relations for Lucas Polynomials

Here, we write the Lucas polynomials on the vertices of an n-gon. Note that the mth term of the polynomials sequence at the point A_k is $L_{(m-1)n+k}$. Let us give this situation with the second main theorem as follows.

Theorem 2.2.1. *Let the Lucas polynomials be written on the vertices of an n-gon clockwise. The mth term of the polynomial sequence corresponding to the vertex A_k is follows*

$$L_{(m-1)n+k}(x) = L_n(x)L_{(m-2)n+k}(x) - (-1)^n L_{(m-3)n+k}(x), 0 \leq k < n.$$

Proof. From Binet's formula for Lucas polynomials, we find

$$\begin{aligned}
&L_n(x)\,L_{(m-2)n+k}\,(x) - (-1)^n L_{(m-3)n+k}\,(x)\\
&= \left(\alpha^{(m-2)n+k}(x) + \beta^{(m-2)n+k}(x)\right)\left(\alpha^n(x) + \beta^n(x)\right)\\
&-(-1)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)\\
&=\alpha^{mn-n+k}(x) + \beta^{mn-n+k}(x) + \alpha^{(m-2)n+k}(x)\beta^n(x) + \alpha^n(x)\beta^{(m-2)n+k}(x)\\
&-(-1)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)\\
&=\alpha^{mn-n+k}(x) + \beta^{mn-n+k}(x) + (\alpha\beta)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)\\
&-(-1)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)
\end{aligned}$$

Because of $\alpha\beta = -1$, we have

$$\begin{aligned}
L_n(x)\,L_{(m-2)n+k}\,(x) - (-1)^n L_{(m-3)n+k}\,(x) &= \alpha^{mn-n+k}(x) + \beta^{mn-n+k}(x)\\
&+(-1)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)\\
&-(-1)^n\left(\alpha^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)\right)\\
&=\alpha^{mn-n+k}(x) + \beta^{mn-n+k}(x) = L_{(m-1)n+k}(x).
\end{aligned}$$

□

Theorem 2.2.2. *For $0 \le k < n$, there is a following relation between the polynomials corresponding to the vertex A_k .*

$$\begin{aligned}
L_{mn+k}(x) = L_{n+k}(x)&\left(\sum_{t=0}^{\frac{2m-3-(-1)^m}{4}} (-1)^{(n+1)t} \binom{m-t-1}{t} L_n^{m-2t-1}(x)\right)\\
+L(x)&\left(\sum_{t=0}^{\frac{2m-5-(-1)^{m-1}}{4}} (-1)^{(n+1)(t+1)} \binom{m-t-2}{t} L_n^{m-2t-2}(x)\right)
\end{aligned}$$

Proof. The proof is omitted here as it is similar to the proof of Theorem 2.2.2. □

2.3 Some Properties of coefficients in new recurrences of Fibonacci polynomials

In this section, let's examine some value of any Fibonacci polynomials corresponding $F_{mn+k}(x)$ for some m,n and k. For $x = 1$ in Table 2.3, we give

some terms numbered in OEIS of the Fibonacci polynomials $F_{mn+k}(x)$. For $x=1, n=3$ and $k=0,1,2$, we find the the following sequences.
A few terms of them is follows.

$$\{A_2\}_{m\in\mathbb{N}} = \{1,5,21,89,377,1597,\dots\}$$

$$\{A_1\}_{m\in\mathbb{N}} = \{1,3,13,55,233,987,\dots\}$$

$$\{A_0\}_{m\in\mathbb{N}} = \{0,2,8,34,144,610,\dots\}.$$

For $x=1, n=4$ and $k=0,1,2,3$, we get

$$\{A_3\}_{m\in\mathbb{N}} = \{2,13,89,610,\dots\}$$

$$\{A_2\}_{m\in\mathbb{N}} = \{1,8,55,377,\dots\}$$

$$\{A_1\}_{m\in\mathbb{N}} = \{1,5,34,233,\dots\}$$

$$\{A_0\}_{m\in\mathbb{N}} = \{0,3,21,144,\dots\}.$$

It is worthy to be noted that numbers sequence which above are referenced in OEIS [15].

Fibonacci polynomial sequences as numbered in OEIS.

F_{3m+2}	A015448
F_{3m+1}	A033887
F_{3m}	A014445
F_{4m+3}	A033891
F_{4m+2}	A033890
F_{4m+1}	A033889
F_{4m}	A033888

For $n=3$ and $k=2$, if the coefficients of the polynomial sequences are written in step form, then we have following table.

$$\{A_2\}_{m\in\mathbb{N}} = \{x, x^4+3x^2+1, x^7+6x^5+10x^3+4x, x^{10}+9x^8+28x^6+ \\ 35x^4+15x^2+1, x^{13}+12x^{11}+55x^9+120x^7+126x^5+56x^3+7x,\dots\}$$

The coefficients for $n=3$ and $k=2$

1	0									
1	0	3	0	1						
1	0	6	0	10	0	4	0			
1	0	9	0	28	0	35	0	15	0	1
1	0	12	0	55	0	120	0	126	0	56

Note that the number of terms in the expansion of $(a_1 + a_2 + \cdots + a_{n+3-j})^n$ is each entrance in the diagonal beginning with $\{1, j, \dots\}$ for $n \geq j-2$. For example, for $n \geq 1$, sequence $\{1, 3, 10, 35, 126, \dots\}$ is the number of terms in the expansion of $(a_1 + a_2 + \cdots + a_{n-1})^n$.
Similarly, let's take some derivatives

$$\{A_2\}_{m\in\mathbb{N}} = \{x, x^4 + 3x^2 + 1, x^7 + 6x^5 + 10x^3 + 4x, x^{10} + 9x^8 + 28x^6 + 35x^4 + 15x^2 + 1, x^{13} + 12x^{11} + 55x^9 + 120x^7 + 126x^5 + 56x^3 + 7x, \dots\}$$

and rewrite them in digit form.

$$\{A_2\}'_{m\in\mathbb{N}} = \{1, 4x^3 + 6x, 7x^6 + 30x^4 + 30x^2 + 4, 10x^9 + 72x^7 + 168x^5 + 140x^3 + 30x, 13x^{12} + 132x^{10} + 495x^8 + 840x^6 + 630x^4 + 168x^2 + 7, \dots\}$$

$$\{A_2\}''_{m\in\mathbb{N}} = \{0, 12x^2 + 6, 42x^5 + 120x^3 + 60x, 90x^8 + 504x^6 + 840x^4 + 420x^2 + 30, \dots\}$$

Fibonacci first derivative polynomial sequences for n= 3 and k= 2.

1						
4	6					
7	30	30	4			
10	72	168	140	30		
13	132	495	840	630	168	7

In the above Table, it is seen that the numbers in the diagonal give the sequence A002457 and the numbers in the vertical column give the sequence A152743 (6 times pentagonal numbers) These numbers are referenced in the OEIS [15].

Fibonacci second derivative polynomial sequences for n= 3 and k= 2.

0					
12	6				
42	120	60			
90	504	840	420	30	
156	1320	3960	5040	2520	336

In the above Table, it is seen that the numbers in the diagonal give sequence A089431 and the numbers in the vertical column give sequence A054776 in OEIS [15].

3 Conclusion

In this work, we examine the sequence of polynomials corresponding to each vertex by placing Fibonacci polynomials clockwise into the regular n-gon. We have obtained the formula that gives the general term for the sequence of polynomials at each vertex.
Then we were interested in the coefficients of the polynomials corresponding to the vertices. We have seen that these coefficients are known special number sequences.
We did all this for Lucas polynomials as well.

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