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Semi r-ideals of commutative rings

Hani A. Khashan and Ece Yetkin Celikel

Abstract

For commutative rings with identity, we introduce and study the concept of semi r-ideals which is a kind of generalization of both r-ideals and semiprime ideals. A proper ideal I of a commutative ring R is called semi r-ideal if whenever $a^2 \in I$ and $Ann_R(a) = 0$, then $a \in I$. Several properties and characterizations of this class of ideals are determined. In particular, we investigate semi r-ideal under various contexts of constructions such as direct products, localizations, homomorphic images, idealizations and amalagamations rings. We extend semi r-ideals of rings to semi r-submodules of modules and clarify some of their properties. Moreover, we define submodules satisfying the D-annihilator condition and justify when they are semi r-submodules.

1 Introduction

Throughout, all rings are supposed to be commutative with identity and all modules are unital. Let R be a ring and M an R-module. We recall that a proper ideal I of a R is called semiprime if whenever $a \in R$ such that $a^2 \in I$, then $a \in I$. It is well-known that I is semiprime in R if and only if I is a radical ideal, that is $I = \sqrt{I}$ where $\sqrt{I} = \{x \in R : x^m \in I \text{ for some } m \in \mathbb{Z}\}$. In 2015, R. Mohamadian [15] introduced the concept of r-ideals of commutative rings. A proper ideal I of a ring R is called an r-ideal (resp. pr -ideal) if whenever $a, b \in R$ such that $ab \in I$ and $Ann_R(a) = 0$, then $b \in I$ (resp. $b \in \sqrt{I}$) where $Ann_R(a) = \{b \in R : ab = 0\}$. Prime and r-ideals are not comparable in general; but it is verified that every maximal r-ideal in a ring is a prime

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ideal, while every minimal prime ideal is an r -ideal. In 2017, Tekir, Koc and Oral [18] introduced the concept of n-ideals as a special kind of r-ideals by considering the set of nilpotent elements instead of zero divisors. Recently, in [20], Yetkin Celikel and Khashan generalized n-ideals by defining and studying the class of semi n-ideals. A proper ideal I of R is called a semi n-ideal if for $a \in R$, $a^2 \in I$ and $a \notin \sqrt{0}$ imply $a \in I$. Later, some other generalizations of semiprime, n-ideals and r-ideals have been introduced, see for example,[4], [10]-[12] and [19].

Motivated by semiprime ideals and semi n-ideals, we define a proper ideal I of a ring R to be a semi r-ideal if whenever $a \in R$ such that $a^2 \in I$ and $Ann_R(a) = 0$, then $a \in I$. It is clear that the class of semi r-ideals is a generalization of that of semiprime and r-ideals. We start section 2 by giving some examples (see Example 1) to show that this generalization is proper. Next, we determine several equivalent characterizations of semi r-ideals (see Theorem 1). Among many other results in this paper, we characterize rings in which every ideal is a semi r-ideal (see Theorem 3). We investigate semi r-ideals under various contexts of constructions such as homomorphic images, quotient rings, localizations and polynomial rings (see Propositions 1 and 3, Corollary 3, Theorem 4). Moreover, we discuss and characterize semi r-ideals of cartesian product of rings (see Proposition 5, Theorems 5 and 6, Corollaries 4 and 5). Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. We study some forms of semi r-ideals of the amalgamation ring $R \bowtie^f J$ of R with S along J with respect to f (see Theorems 7 and 8).

Let M be an R-module, N be a submodule of M and I be an ideal of R. As usual, we will use the notations $(N :_R M)$ and $(N :_M I)$ for the sets $\{r \in R : rm \in N \text{ for all } m \in M\}$ and $\{m \in M : Im \subseteq N\}$, respectively. In particular, the annihilator of an element $m \in M$ (resp. $r \in R$) denoted by $Ann_R(m)$ (resp. $Ann_M(r)$), is $(0 :_R m)$ (resp. $(0 :_M r)$. We recall that the torsion subgroup $T(M)$ of an R-module M is defined as $T(M) = \{m \in M :$ there exists $0 \neq r \in R$ such that $rm = 0\}$. It is easy to see that $T(M)$ is a submodule of M, called the torsion submodule. A module is torsion (resp. torsion-free) if $T(M) = M$ (resp. $T(M) = \{0\}$).

In 2009, the concept of semiprime submodules is presented. A proper submodule is said to be semiprime if whenever $r \in R$, $m \in M$ and $r^2m \in N$, then $rm \in N$, [16]. Afterwards, the notions of r-submodule and sr-submodules are introduced and studied in [13]. A proper submodule N is called an r-submodule (resp. sr-submodule) of M if whenever $rm \in N$ and $Ann_M(r) = 0_M$ (resp. $Ann_R(m) = 0$), then $m \in N$ (resp. $r \in (N :_R M)$). As a new generalization of above structures, in Section 3, we define a proper submodule N of M to be a semi r-submodule if whenever $r \in R$, $m \in M$ with $r^2m \in N$, $Ann_M(r) = 0_M$ and $Ann_R(m) = 0$, then $rm \in N$. We illustrate (see Example

4) that this generalization of r-submodules is proper. However, it is observed that semi r-submodules coincides with semiprime submodules in any torsion-free module. Then, we introduce a new condition for submodules, namely, D-annihilator condition as follows: A proper submodule N of an R-module M is said to satisfy the D-annihilator condition if whenever K is a submodule of M and $r \in R$ such that $rK \subseteq N$ and $Ann_M(r) = 0_M$, then either $K \subseteq N$ or $K \cap T(M) = \{0_M\}$. By using this condition, we totally characterize semi r-submodules of finitely generated faithful multiplication R-modules (see Proposition 8, Theorems 9 and 10, Corollary 6).

We recall that the idealization of an R-module M denoted by $R(+)M$, is the commutative ring $R \times M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. It is well known from [2] that

$$zd(R(+)M) = \{(r, m)|\ r \in zd(R) \cup Z(M),\ m \in M\}$$

In Proposition 11, we clarify the relation between semi r-ideals of the idealization ring $R(+)M$ and those of R which enables us to build some interesting examples of semi r-ideals.

Let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1,\ j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [8], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module. The last section is devoted to clarify semi r-submodules of the amalgamation of modules.

2 Properties of semi r-ideals

This section deals with many properties of semi r-ideals. We justify the relations among the concepts of semiprime ideals, semi n-ideals and our new class of ideals. Moreover, several characterizations and examples are presented. In particular, we characterize rings in which every ideal is a semi r-ideal.

Definition 1. *Let I be a proper ideal of a ring R. I is called a semi r-ideal of R if whenever $a \in R$ such that $a^2 \in I$ and $Ann_R(a) = 0$, then $a \in I$.*

For any non-zero subset A of a ring R, we note that $Ann_R(A)$ is a semi r-ideal of R. It is clear that the classes of semiprime ideals, r-ideals and semi n-ideals are contained in the class of semi r-ideals. However, in general these containments are proper as we illustrate in the following examples.

Example 1. *Let p and q be prime integers.*

1. *Any non-zero semiprime ideal in an integral domain is a semi r-ideal that is not an r-ideal.*
2. *In the ring $\mathbb{Z}_{p^2q}$, the ideal $\left\langle \overline{p^2} \right\rangle$ is a semi r-ideal that is not a semi n-ideal.*
3. *The zero ideal of a ring R is always a semi r-ideal but it is not a semiprime ideal unless R is a semiprime ring.*
4. *Every ideal of a Boolean ring (a ring of which every element is idempotent) is semi r-ideal. Consider the ideal $I = 0 \times 0 \times \mathbb{Z}_2$ of the Boolean ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then I is a semi r-ideal that is not prime.*
5. *In general pr-ideals and semi r-ideals are not comparable. Let T be a reduced ring with subring $\mathbb{Z}$ and P be a nonzero minimal prime ideal in T with $P \cap \mathbb{Z} = (0)$. From [15, Example 2.17], $J = x^2P[x]$ is a pr -ideal of the ring $R = \mathbb{Z} + xT[x]$. Choose an element $0 \neq p \in P$. Then $(xp)^2 \in J$ and $Ann_R(xa) = 0$ but $xa \notin J$. Thus, J is not a semi r-ideal. Moreover, any non-zero prime ideal in an integral domain is clearly a semi r-ideal that is not a pr-ideal.*

If I and J are semi r-ideals of a ring R, then IJ and $I+J$ need not be so as we can see in the following example.

Example 2. *Consider the ideals $I = \langle x \rangle$ and $J = \langle x-4 \rangle$ of the ring $R = \mathbb{Z}[x]$. Then I and J are (semi) prime ideals and so are semi r-ideals of R. On the other hand, $I + J = \langle x, x-4 \rangle = \langle x, 4 \rangle$ is not a semi r-ideal of R. Indeed, $(2+x)^2 \in I + J$ and $Ann_R(2+x) = 0$, but $2 + x \notin I + J$. Also, $I^2 = \left\langle x^2 \right\rangle$ is not a semi r-ideal of R as $x^2 \in I^2$ and $Ann_R(x) = 0$, but $x \notin I^2$.*

Next, we give the following characterization of semi r-ideals. By $zd(R)$ we denote the set of all zero divisor elements of a ring R. Moreover, $reg(R)$ denotes the set $R \backslash zd(R)$.

Theorem 1. *Let I be a proper ideal of a ring R and k be a positive integer. The following statements are equivalent.*

1. I is a semi r-ideal of R.

2. Whenever $a \in R$ with $0 \neq a^2 \in I$ and $Ann_R(a) = 0$, then $a \in I$.

3. Whenever $a \in R$ with $a^k \in I$ and $Ann_R(a) = 0$, then $a \in I$.

4. $\sqrt{I} \subseteq zd(R) \cup I$.

Proof. (1)$\Leftrightarrow$(2). Suppose (2) holds and let $a \in R$ such that $a^2 \in I$ and $Ann_R(a) = 0$. If $a^2 = 0$, then $a = 0$ and the result follows obviously. If $a^2 \neq 0$, then we are also done by (2). The converse part is obvious.

(1)$\Rightarrow$(3). Suppose $a^k \in I$ and $Ann_R(a) = 0$ for $a \in R$. We use the mathematical induction on k. If $k \leq 2$, then the claim is clear. We now assume that (3) holds for all $2 < t < k$ and show that it is also true for k. Suppose k is even, say, $k = 2m$ for some positive integer m. Since $a^k = (a^m)^2 \in I$ and clearly $Ann_R(a^m) = 0$, then $a^m \in I$ as I is a semi r-ideal. By the induction hypothesis, we conclude that $a \in I$ as needed. Suppose k is odd, so that $k + 1 = 2s$ for some $s < k$. Then similarly, we have $(a^s)^2 \in I$ and $Ann_R(a^s) = 0$ which imply that $a^s \in I$ and again by the induction hypothesis, we conclude $a \in I$.

(3)$\Rightarrow$(4). Let $a \in \sqrt{I}$. Then $a^k \in I$ for some $k \geq 1$ and so by (3) $a \in zd(R)$ or $a \in I$. Thus, $\sqrt{I} \subseteq zd(R) \cup I$.

(4)$\Rightarrow$(1). Straightforward. □

Corollary 1. *Let I be a semi r-ideal of a ring R and k be a positive integer. If J is an ideal of R with $J^k \subseteq I$ and $J \cap zd(R) = \{0\}$, then $J \subseteq I$.*

Proof. Suppose that $J^k \subseteq I$ and $J \cap zd(R) = \{0\}$ for some ideal J of R. Let $0 \neq a \in J$. From the assumption $J \cap zd(R) = \{0\}$, we have $Ann_R(a) = 0$. Thus, $a^k \in I$ implies that $a \in I$ by Theorem 1 (3). □

Corollary 2. *Let I and J be proper ideals of a ring R such that $I \cap zd(R) = J \cap zd(R) = \{0\}$.*

1. If I and J are semi r-ideals of a ring R with $I^2 = J^2$, then $I = J$.

2. If I^2 is a semi r-ideal, then $I^2 = I$.

Proof. (1) Since $I^2 \subseteq J$ and $J \cap zd(R) = \{0\}$, then we have $I \subseteq J$ by Corollary 1. On the other hand, since $J^2 \subseteq I$ and $J \cap zd(R) = \{0\}$, we have $J \subseteq I$ again by Corollary 1, so we are done.

(2) A direct consequence of (1). □

We note by example 1 that unlike r-ideals, if I is a semi r-ideal of a ring R, then I need not be contained in $zd(R)$. Also, clearly, semi r-ideals which contain the zero divisors of a ring R are semiprime.

Next, we present a condition for a semi r-ideal to be an r-ideal. First, we need the following lemma.

Lemma 1. *Let S be a non-empty subset of R where $S \cap zd(R) = \emptyset$. If I is a semi r-ideal of R with $S \nsubseteq I$, then $(I : S)$ is a semi r-ideal of R.*

Proof. Let $a \in R$ such that $a^2 \in (I : S)$ and $Ann_R(a) = 0$. Then $(as)^2 \in I$ for all $s \in S$. As I is a semi r-ideal of R, we have either $as \in zd(R)$ or $as \in I$ for all $s \in S$. If $as \in zd(R)$, then $S \cap zd(R) = \emptyset$ implies $a \in zd(R)$, a contradiction. Thus, $as \in I$ for all $s \in S$ and so $a \in (I : S)$ as required. □

Theorem 2. *If I is maximal among all semi r-ideals of a ring R contained in $zd(R)$, then I is an r-ideal.*

Proof. Let I be maximal among all semi r-ideals of a ring R contained in $zd(R)$. Suppose that $ab \in I$ and $Ann_R(a) = 0$. Then $a \notin I \cup zd(R)$ and so $(I :_R a)$ is a semi r-ideal of R by Lemma 1. Since clearly, $(I :_R a) \subseteq zd(R)$ and $I \subseteq (I :_R a)$, then the maximality of I implies, $I = (I :_R a)$. Thus, $b \in I$ and I is an r-ideal. □

Following [15], we call a ring R a uz-ring if $R = U(R) \cup zd(R)$. It is proved in [15] that R is a uz-ring if and only if every ideal in R is an r-ideal. In particular, a direct product of fields is an example of a uz-ring. Next, we generalize this result to semi r-ideals.

Theorem 3. *The following statements are equivalent for a ring R.*

1. R is a uz-ring.
2. Every proper ideal of R is an r-ideal.
3. Every proper ideal of R is a semi r-ideal.
4. Every proper principal ideal of R is a semi r-ideal.
5. Every semi r-ideal is an r-ideal.

Proof. (1)⇒(2). Follows by [15, Proposition 3.4].

(2)⇒(3)⇒(4). Clear.

(4)⇒(1). Let $x \in R \backslash zd(R)$. If $\langle x^2 \rangle = R$, then $x \in U(R)$. Suppose $\langle x^2 \rangle$ is proper in R. Since $x^2 \in \langle x^2 \rangle$ and $Ann_R(x) = 0$, then by assumption, $x \in \langle x^2 \rangle$. Thus, $x = rx^2$ for some $r \in R$ and so $rx = 1$ as $Ann_R(x) = 0$. Thus, again $x \in U(R)$ and $R = U(R) \cup zd(R)$ as needed.

(1)⇒(5). Clear by (1)⇔(2).

(5)⇒(1). Since a maximal ideal of R is clearly a semi r-ideal, then by (5), every maximal ideal in R is an r-ideal. Let $r \in R$. If $r \notin U(R)$, then $r \in M$ for some maximal ideal M of R and so $r \in zd(R)$ by [15, Remark 2.3(d)]. Therefore, $R = U(R) \cup zd(R)$ and R is a uz-ring. □

Next, we discuss the behavior of semi r-ideals under homomorphisms.

Proposition 1. *Let $f : R_1 \to R_2$ be a ring homomorphism. The following statements hold.*

1. If f is an epimorphism, $I_1 \subseteq Ker(f)$ and I_1 is a semi r-ideal of R_1 such that $I_1 \cap zd(R_1) = \{0\}$, then $f(I_1)$ is a semi r-ideal of R_2.
2. If f is an isomorphism and I_2 is a semi r-ideal of R_2, then $f^{-1}(I_2)$ is a semi r-ideal of R_1.

Proof. (1) Let $a \in R_2$ such that $a^2 \in f(I_1)$ and $a \notin f(I_1)$. Then there exists $x \in R_1 \backslash I_1$ such that $a = f(x)$. Since $f(x^2) = a^2 \in f(I_1)$, then $x^2 \in I_1$ as $Ker(f) \subseteq I_1$. Now, I_1 is a semi r-ideal of R_1 implies $x \in zd(R_1)$. If $x = 0$, then $a = f(x) \in zd(R_2)$. Suppose $x \neq 0$ and choose $0 \neq y \in R$ such that $xy = 0$. Then $f(y) \neq 0$ since otherwise $y \in I_1 \cap zd(R_1)$, a contradiction. Thus, again $a = f(x) \in zd(R_2)$ and $f(I_1)$ is a semi r-ideal of R_2.

(2) Suppose I_2 is a semi r-ideal of R_2. Let $x \in R_1$ such that $x^2 \in f^{-1}(I_2)$ and $x \notin f^{-1}(I_2)$. Then $f(x^2) = f(x)^2 \in I_2$ and $f(x) \notin I_2$ which imply $f(x) \in zd(R_2)$. Since f is an isomorphism, then clearly $x \in zd(R_1)$ and $f^{-1}(I_2)$ is a semi r-ideal of R_1. □

In view of Proposition 1, we have the following result for quotient rings.

Corollary 3. *Let I and J be ideals of a ring R with $J \subseteq I$.*

1. If I is a semi r-ideal of R and $I \cap zd(R) = \{0\}$, then I/J is a semi r-ideal of R/J.
2. If I/J is a semi r-ideal of R/J and J is an r-ideal of R, then I is a semi r-ideal of R.

Proof. (1). Consider the natural epimorphism $\pi : R \to R/J$ with $Ker(\pi) = J$ and apply Proposition 1.

(2). Let $a \in R$ such that $a^2 \in I$ and $a \notin zd(R)$. Then $(a + J)^2 = a^2 + J \in I/J$. If $a + J \in zd(R/I)$, then there is $b \notin J$ such that $ab \in J$. Since J is a semi r-ideal of R, we get $a \in zd(R)$, a contradiction. Thus, $a + J \notin zd(R/I)$ which yields $a + J \in I/J$ as I/J is a semi n-ideal of R/J and so $a \in I$. □

If $I \cap zd(R) \neq \{0\}$ in Corollary 3(1), then the result need not be true. For example, $4\mathbb{Z}(+)\mathbb{Z}_4$ is a semi r-ideal of $\mathbb{Z}(+)\mathbb{Z}_4$, see Remark 11. But $4\mathbb{Z}(+)\mathbb{Z}_4/0(+)\mathbb{Z}_4 \cong 4\mathbb{Z}$ is not a semi r-ideal of $\mathbb{Z}(+)\mathbb{Z}_4/0(+)\mathbb{Z}_4 \cong \mathbb{Z}$. We also note that the condition " J is an r-ideal" in Corollary 3(2) is crucial. For example $8\mathbb{Z}/16\mathbb{Z}$ is a semi r-ideal of $\mathbb{Z}/16\mathbb{Z}$ but $8\mathbb{Z}$ is not a semi r-ideal of $\mathbb{Z}$.

In particular, Corollary 3 holds if $J \subseteq zd(R)$.

Proposition 2. *The intersection of any family of semi r-ideals is a semi r-ideal.*

Proof. Let $\{I_\alpha : \alpha \in \Lambda\}$ is a family of semi r-ideals. Suppose $a^2 \in \bigcap_{\alpha\in\Lambda} I_\alpha$ and $a \notin \bigcap_{\alpha\in\Lambda} I_\alpha$. Then $a \notin I_\gamma$ for some $\gamma \in \Lambda$. Since I_γ is a semi r-ideal, we have $a \in zd(R)$ and so $\bigcap_{\alpha\in\Lambda} I_\alpha$ is a semi r-ideal. $\square$

Let I be a proper ideal of R. In the following we give the relationship between semi r-ideals of a ring and those of its localization ring by using the notation $Z_I(R)$ which denotes the set $\{r \in R \mid rs \in I \text{ for some } s \in R\backslash I\}$.

Proposition 3. *Let S be a multiplicatively closed subset of a ring R such that $S \cap zd(R) = \emptyset$. Then the following hold.*

1. If I is a semi r-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a semi r-ideal of $S^{-1}R$.
2. If $S^{-1}I$ is a semi r-ideal of $S^{-1}R$ and $S \cap Z_I(R) = \emptyset$, then I is a semi r-ideal of R.

Proof. (1) Suppose for $\frac{a}{s} \in S^{-1}R$ that $\left(\frac{a}{s}\right)^2 \in S^{-1}I$ and $\left(\frac{a}{s}\right) \notin S^{-1}I$. Then there exits $u \in S$ such that $ua^2 \in I$ and so $(ua)^2 \in I$. Since clearly $ua \notin I$ and I is a semi r-ideal, we have $ua \in zd(R)$, say, $(ua)b = 0$ for some $0 \neq b \in R$. Thus, $\frac{a}{s} \cdot \frac{b}{1} = \frac{uab}{us} = 0_{S^{-1}R}$ and $\frac{b}{1} \neq 0_{S^{-1}R}$ as $S \cap zd(R) = \emptyset$. Thus, $\frac{a}{s} \in zd(S^{-1}R)$ and $S^{-1}I$ is a semi r-ideal of $S^{-1}R$.

(2) Suppose $a^2 \in I$ for $a \in R$. Since $S^{-1}I$ is a semi n-ideal of $S^{-1}R$ and $\left(\frac{a}{1}\right)^2 \in S^{-1}I$, we have either $\frac{a}{1} \in S^{-1}I$ or $\frac{a}{1} \in zd(S^{-1}R)$. If $\frac{a}{1} \in S^{-1}I$, then there exists $u \in S$ such that $ua \in I$. Since $S \cap zd(R) = \emptyset$, we conclude that $a \in I$. If $\frac{a}{1} \in zd(S^{-1}R)$, then there is $\frac{b}{t} \neq 0_{S^{-1}R}$ such that $\frac{ab}{t} = \frac{a}{1} \cdot \frac{b}{t} = 0_{S^{-1}R}$. Hence, $vab = 0$ for some $v \in S$ and so $ab = 0$ as $S \cap zd(R) = \emptyset$. Thus, $a \in zd(R)$ as $b \neq 0$ and I is a semi r-ideal of R. $\square$

We recall that if $f = \sum_{i=1}^{m} a_i x^i \in R[x]$, then the ideal $\langle a_1, a_2, \cdots, a_m\rangle$ of R generated by the coefficients of f is called the content of f and is denoted by $c(f)$. It is well known that if f and g are two polynomials in $R[x]$, then the content formula $c(g)^{m+1}c(f) = c(g)^m c(fg)$ holds where m is the degree of f , [9, Theorem 28.1]. For an ideal I of R, it can be easily seen that $I[x] = \{f(x) \in R[x] : c(f) \subseteq I\}$.

Definition 2. *A ring R is said to satisfy the property $(*)$ if whenever $f \in reg(R[x])$, then $c(f)\backslash\{0\} \subseteq reg(R)$.*

Theorem 4. *Let I be an ideal of a ring R.*

1. If $I[x]$ is a semi r-ideal of $R[x]$, then I is a semi r-ideal of R.
2. If R satisfies the property $(*)$ and I is a semi r-ideal of R, then $I[x]$ is a semi r-ideal of $R[x]$

Proof. (1) Suppose $I[x]$ is a semi r-ideal of $R[x]$. Let $a \in R$ such that $a^2 \in I$ and $Ann_R(a) = 0$. Then Clearly, $a^2 \in I[x]$ and $Ann_{R[x]}(a) = 0$. By assumption, $a \in I[x]$ and so $a \in I$ as required.

(2) Suppose R satisfies the property $(*)$ and I is a semi r-ideal of R. Let $f(x) \in R[x]$ such that $(f(x))^2 \in I[x]$ and $Ann_{R[x]}(f(x)) = 0$. Then $c(f^2) \subseteq I$ and so by the content formula, $(c(f))^2 = c(f^2) \subseteq I$. Moreover, $c(f) \cap zd(R) = \{0\}$ as R satisfies the property $(*)$ and so $c(f) \subseteq I$ by Corollary 1. It follows that $f(x) \in I[x]$ and we are done. $\square$

In general, if S is an overring of a ring R, then we may find a semi r-ideal J of S where $J \cap R$ is not a semi r-ideal in R.

Example 3. *Let $S = \mathbb{Z} \times \mathbb{Z}$ and consider the ring homomorphism $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by $\varphi(x) = (x, 0)$. Then φ is a monomorphism and so $R = \varphi(\mathbb{Z})$ is a domain. Now, $J = Ann_S((0, 1))$ is a nonzero (semi) r-ideal in S. However, clearly, $R \subseteq J$ and so $J \cap R = R$ is not a semi r-ideal in R.*

Let S be an overring of a ring R . Following [15], R is said to be essential in S if $J \cap R \neq \{0\}$ for every nonzero ideal J of S .

Proposition 4. *Let $R \subseteq S$ be rings such that R is essential in S. If J is a semi r -ideal of S, then $J \cap R$ is a semi r-ideal in R.*

Proof. Let $a \in R$ such that $a^2 \in J \cap R$ and $Ann_R(a) = 0$. Then $a \in S$ with $a^2 \in J$ and $Ann_S(a) = 0$. Indeed, if $Ann_S(a) \neq 0$, then R being essential implies $Ann_S(a) \cap R \neq \{0\}$. Thus, there exists $0 \neq r \in R$ such that $r \in Ann_S(a)$ and so $r \in Ann_R(a)$, a contradiction. Since J is a semi r -ideal of S, then $a \in J \cap R$ and the result follows., $\square$

The rest of this section is devoted to discuss semi r-ideals of cartesian products of rings and their particular subrings: the amalgamation rings.

Proposition 5. *Let $R = R_1 \times R_2$ where R_1 and R_2 are two rings and I_1, I_2 be proper ideals of R_1 and R_2, respectively. Then $I_1 \times R_2$ (resp. $R_1 \times I_2$) is a semi r-ideal of R if and only if I_1 is a semi r-ideal of R_1 (resp. I_2 is a semi r-ideal of R_2).*

Proof. Let $I_1 \times R_2$ be a semi r-ideal of R and $a \in R_1$ with $a^2 \in I_1$ and $Ann_{R_1}(a) = 0$. Then $(a, 1)^2 \in I_1 \times R_2$ and $Ann_R(a, 1) = (0, 0)$ imply that $(a, 1) \in I_1 \times R_2$ and so $a \in I_1$. Thus I_1 is a semi r-ideal of R_1. Conversely, suppose that $(a, b)^2 \in I_1 \times R_2$ and $Ann_R(a, b) = (0, 0)$. Then $a^2 \in I_1$ and clearly $Ann_{R_1}(a) = 0$ which implies $a \in I_1$. Hence, $(a, b) \in I_1 \times R_2$, so we are done. The proof of the case $R_1 \times I_2$ is similar. □

The following corollary generalizes Proposition 5.

Corollary 4. *Let $R_1, R_2, \cdots, R_n$ be rings, $R = R_1 \times R_2 \times \cdots \times R_n$ and I_i be a proper ideal of R_i for each $i = 1, 2, \cdots n$. Then for all $j = 1, 2, \cdots n$, $I = R_1 \times \cdots \times R_{j-1} \times I_j \times R_{j+1} \times \cdots \times R_n$ is a semi r-ideal of R if and only if I_j is a semi r-ideal of R_j.*

Theorem 5. *Let R_1 and R_2 be two rings, $R = R_1 \times R_2$ and I_1, I_2 be proper ideals in R_1 and R_2, respectively.*

1. If I_1 and I_2 are semi r-ideals of R_1 and R_2, respectively, then $I = I_1 \times I_2$ is a semi r-ideal of R.
2. If $I = I_1 \times I_2$ is a semi r-ideal of R, then either I_1 is a semi r-ideal of R_1 or I_2 is a semi r-ideal of R_2.
3. If $I = I_1 \times I_2$ is a semi r-ideal of R and $I_2 \nsubseteq zd(R_2)$, then I_1 is a semi r-ideal of R_1.
4. If $I = I_1 \times I_2$ is a semi r-ideal of R and $I_1 \nsubseteq zd(R_1)$, then I_2 is a semi r-ideal of R_2.

Proof. (1) Let $(a, b) \in R$ such that $(a^2, b^2) = (a, b)^2 \in I$ and $Ann_R(a, b) = (0, 0)$. Then $a^2 \in I_1$, $b^2 \in I_2$ and clearly $Ann_{R_1}(a) = Ann_{R_2}(b) = 0$. Therefore, $a \in I_1$, $b \in I_2$ and so $(a, b) \in I$ as needed.

(2).Suppose $I = I_1 \times I_2$ is a semi r-ideal of R but I_1 and I_2 are not semi r-ideals of R_1 and R_2, respectively. Choose $a \in R_1$ and $b \in R_2$ such that $a^2 \in I_1$, $b^2 \in I_2$, $Ann_{R1}(a) = 0$ and $Ann_{R_2}(b) = 0$ but $a \notin I_1$ and $b \notin I_2$. Then $(a, b)^2 \in I$ and clearly, $Ann_R(a, b) = (0, 0)$. By assumption, we have $(a, b) \in I$ which is a contradiction. Therefore, either I_1 is a semi r-ideal of R_1 or I_2 is a semi r-ideal of R_2.

(3) Suppose $a^2 \in I_1$ for some $a \in R_1$ with $Ann_{R_1}(a) = 0$. Since $I_2 \nsubseteq Z(R_2)$, we can choose $b \in I_2 \cap reg(R_2)$. Then $(a, b)^2 \in I$ and $Ann_R(a, b) = (0, 0)$. It follows that $(a, b) \in I$; and hence $a \in I_1$.

(4) is similar to (3). □

The converse of Theorem 5(1) is not true in general. For example, $4\mathbb{Z} \times 0$ is a semi r-ideal in $\mathbb{Z} \times \mathbb{Z}$ by Proposition 2. On the other hand, the ideal $4\mathbb{Z}$ is not a semi r-ideals of $\mathbb{Z}$.

The following corollary generalizes Theorem 5 to any finite direct product of rings. The proof is similar to that of Theorem 5.

Corollary 5. *Let $R_1, R_2, \cdots, R_n$ be rings, $R = R_1 \times R_2 \times \cdots \times R_n$ and I_i be a proper ideal of R_i for each $i = 1, 2, \cdots n$.*

1. If I_i is a semi r-ideals of R_i for each $i = 1, 2, \cdots n$, then $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R.
2. If $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R, then I_j is a semi r-ideal of R_j for at least one $j \in \{1, 2, \cdots, n\}$.
3. If $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R and $I_j \nsubseteq Z(R_j)$ for all $j \neq i$, then I_i is a semi r-ideal of R_i.

Lemma 2. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i's are rings and R_j is reduced ring for some $j = 1, ..., n$. If I_i is an ideal of R_i for all $i \neq j$, then $I = I_1 \times \cdots \times I_{j-1} \times 0 \times I_{j+1} \times \cdots \times I_n$ is a semi r-ideal of R.*

Proof. Let $a = (a_1, a_2, ..., a_n) \in R$ with $a^2 \in I$. Then $a_j^2 = 0$ which implies $a_j = 0$ as R_j is reduced. Since $Ann_R(a) = Ann_R(a_1, ..., a_{j-1}, 0, a_{j+1}, ..., a_n) \neq 0$, I is a semi r-ideal of R. □

Next, we present a characterization for semi r-ideals of cartesian products of domains.

Theorem 6. *Let $R_1, R_2, \cdots, R_n$ $(n \geq 2)$ be domains, $R = R_1 \times R_2 \times \cdots \times R_n$ and I_i be an ideal of R_i for each $i = 1, 2, \cdots n$. Then $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R if and only if one of the following statements holds*

1. $I_j = \{0\}$ for at least one $j \in \{1, 2, \cdots, n\}$.
2. There exists $j \in \{1, 2, \cdots n\}$ such that I_i is a semi r-ideal of R_i for all $i = 1, \cdots, j$ and $I_i = R_i$ for all $i = j + 1, \cdots, n$.
3. I_i is a semi r-ideals of R_i for each $i = 1, 2, \cdots n$.

Proof. Suppose $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R. Suppose that all I_i's are nonzero. If for all $i \in \{1, 2, \cdots n\}$, I_i is proper in R_i, then I_i is a semi r-ideals of R_i by Corollary 5(3). Without loss of generality assume that $I_1, ..., I_j$ are proper in $R_1, \cdots, R_j$, respectively and $I_i = R_i$ for all $i \in \{j + 1, ..., n\}$. For each $i \in \{2, ..., j\}$, choose a nonzero element $b_i \in I_i$.

Let $a \in R_1$ such that $a^2 \in I_1$. Since $(a, b_2, b_3, ...b_j, 1_{R_{j+1}}, ..., 1_{R_n})^2 \in I$ and $Ann_R(a, b_2, b_3, ...b_j, 1_{R_{j+1}}, ..., 1_{R_n}) = 0$, we have $(a, b_2, b_3, ...b_j, 1_{R_{j+1}}, ..., 1_{R_n}) \in I$ and so $a \in I_1$. Therefore, I_1 is a semi r-ideal of R_1. Similarly, I_i is a semi r-ideals of R_i for all $i \in \{1, ..., j\}$.

Conversely, if (1) holds, then I is clearly a semi r-ideal of R. Suppose that $I_1, ..., I_j$ are semi r-ideals and $I_k = R_k$ for all $k \in \{j+1, ..., n\}$. Let $a = (a_1, a_2, ..., a_n) \in R$ with $a^2 \in I$ and $Ann_R(a) = 0$. Then for each $i \in \{1, ..., j\}$, $a_i^2 \in I$ and $Ann_{R_i}(a_i) = 0$ as R_i's are domain. Thus, $a_i \in I_i$ and so $a \in I$. Finally, if (3) holds, then $I = I_1 \times I_2 \times \cdots \times I_n$ is a semi r-ideal of R by Corollary 5(1). □

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. As a subring of $R \times S$, the amalgamation of R and S along J with respect to f is defined by $R \bowtie^f J = (a, f(a) + j) : a \in R$, $j \in J\}$. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a+j) : a \in R,\ j \in J\}$. For more related definitions and several properties of this kind of rings, one can see [6]. If I is an ideal of R and K is an ideal of $f(R) + J$, then $I \bowtie^f J = \{(i, f(i) + j) : i \in I,\ j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R,\ j \in J,\ f(a) + j \in K\}$ are ideals of $R \bowtie^f J$, [7].

Lemma 3. *[3] Let R, S, J and f be as above. Let $A = \{(r, f(r) + j) | r \in zd(R)\}$ and $B = \{(r, f(r) + j) | j'(f(r) + j) = 0$ for some $j' \in J \backslash \{0\}\}$. Then $zd(R \bowtie^f J) \subseteq A \cup B$.*

Next, we determine conditions under which $I \bowtie^f J$ and $\bar{K}^f$ are semi r-ideals of $R \bowtie^f J$.

Theorem 7. *Let R, S, J and f be as above. If I is a semi r-ideal of R, then $I \bowtie^f J$ is a semi r-ideal of $R \bowtie^f J$. The converse is true if $f(reg(R)) \cap Z(J) = \emptyset$*

Proof. Suppose I is a semi r-ideal of R. Let $(a, f(a) + j) \in R \bowtie^f J$ such that $(a, f(a) + j)^2 = (a^2, f(a^2) + 2jf(a) + j^2) \in I \bowtie^f J$ and $(a, f(a) + j) \notin zd(R \bowtie^f J)$. Then $a^2 \in I$ and $a \notin zd(R)$ by Lemma 3. Therefore, $a \in I$ and so $(a, f(a) + j) \in I \bowtie^f J$ as needed. Now, suppose $f(reg(R)) \cap Z(J) = \emptyset$ and $I \bowtie^f J$ is a semi r-ideal of $R \bowtie^f J$. Let $a^2 \in I$ for $a \in R$ and $a \notin zd(R)$. Then $(a, f(a)) \in R \bowtie^f J$ with $(a, f(a))^2 = (a^2, f(a^2)) \in I \bowtie^f J$. If $(a, f(a)) \in zd(R \bowtie^f J)$, then Lemma 3 implies $f(a) \in Z(J)$ which is a contradiction. Therefore, $(a, f(a)) \notin zd(R \bowtie^f J)$ and so $(a, f(a)) \in I \bowtie^f J$ as $I \bowtie^f J$ is a semi r-ideal of $R \bowtie^f J$. Thus, $a \in I$ as required. □

Theorem 8. *Let $f : R \to S$ be a ring homomorphism and J, K be ideals of S. If K is a semi r-ideal of $f(R) + J$, then $\bar{K}^f$ is a semi r-ideal of $R \bowtie^f J$.*

1. If K is a semi r-ideal of $f(R)+J$ and $zd(f(R)+J) = Z(J)$, then $\bar{K}^f$ is a semi r-ideal of $R \bowtie^f J$.

2. If $\bar{K}^f$ is a semi r-ideal of $R \bowtie^f J$, $f(zd(R)) \subseteq zd(f(R)+J)$ and $f(zd(R))J = 0$, then K is a semi r-ideal of $f(R)+J$.

Proof. (1) Suppose K is a semi r-ideal of $f(R)+J$. Let $(a, f(a)+j) \in R \bowtie^f J$ such that $(a, f(a)+j)^2 = (a^2, (f(a)+j)^2) \in \bar{K}^f$ and $(a, f(a)+j) \notin zd(R \bowtie^f J)$. Then $(f(a)+j)^2 \in K$ and by Lemma 3, $f(a)+j \notin Z(J) = zd(f(R)+J)$. Therefore, $f(a)+j \in K$ and $(a, f(a)+j) \in \bar{K}^f$ as needed.

(2) Suppose $\bar{K}^f$ is a semi r-ideal of $R \bowtie^f J$ and $f(zd(R))J = 0$. Let $f(a)+j \in f(R)+J$ such that $(f(a)+j)^2 \in K$ and $f(a)+j \notin zd(f(R)+J)$. Then $(a, f(a)+j) \in R \bowtie^f J$ with $(a, f(a)+j)^2 \in \bar{K}^f$. Suppose $(a, f(a)+j) \in zd(R \bowtie^f J)$. Then as $Z(J) \subseteq zd(f(R)+J)$ and by Lemma 3, we conclude that $a \in zd(R)$. Since $f(a) \in zd(f(R)+J)$, then $f(a)f(b) = 0$ for some $0 \neq f(b) \in f(R)$. Thus, $(f(a)+j)f(b) = 0$ as $f(zd(R))J = 0$ which contradicts that $f(a)+j \notin zd(f(R)+J)$. Therefore, $(a, f(a)+j) \notin zd(R \bowtie^f J)$ and so $(a, f(a)+j) \in \bar{K}^f$. It follows that $f(a)+j \in K$ and K is a semi r-ideal of $f(R)+J$. $\square$

3 Semi r-submodules of modules over commutative rings

The aim of this section is to extend semi r-ideals of commutative rings to semi r-submodules of modules over commutative rings. Recall that a module M is said to be faithful if $Ann_R(M) = (0 :_R M) = 0_R$.

Definition 3. *Let M be an R-module and N a proper submodule of M.*

1. N is called a semiprime submodule if whenever $r^2m \in N$, then $rm \in N$. [16]

2. N is called a r-submodule if whenever $rm \in N$ and $Ann_M(r) = 0_M$, then $m \in N$. [13]

3. N is called a sr-submodule if whenever $rm \in N$ and $Ann_R(m) = 0$, then $m \in N$. [13]

Definition 4. *Let M be an R-module and N a proper submodule of M. We call N a semi r-submodule if whenever $r \in R$, $m \in M$ with $r^2m \in N$, $Ann_M(r) = 0_M$ and $Ann_R(m) = 0$, then $rm \in N$.*

The reader clearly observe that any semi r-submodule of an R-module R is a semi r-ideal of R. The zero submodule is always a semi r-submodule of M. Also, see the implications:

$$\begin{array}{ccc} r\text{-submodule} & & \\ & \searrow & \\ sr\text{-submodule} & \rightarrow & \text{semi } r\text{-submodule} \\ & \nearrow & \\ \text{semiprime submodule} & & \end{array}$$

However, the next examples show that these arrows are irreversible.

Example 4.

1. Consider the submodule $N = 6\mathbb{Z} \times \langle 0 \rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$. Let $r \in \mathbb{Z}$ and $m = (m_1, m_2) \in M$ such that $r^2 \cdot (m_1, m_2) \in N$. Then $r^2 m_1 \in 6\mathbb{Z}$, $r^2 m_2 = 0$ and $Ann_{\mathbb{Z}}(r) = Ann_{\mathbb{Z}}(m_1) = Ann_{\mathbb{Z}}(m_2) = 0$ as $\mathbb{Z}$ is a domain. Since $6\mathbb{Z}$ and $\langle 0 \rangle$ are semi r-ideals of $\mathbb{Z}$, then $r \cdot (m_1, m_2) \in N$ and so N is a semi r-submodule of M. On the other hand, we have $2 \cdot (3, 0) \in N$ with $Ann_M(2) = 0_M$ and $Ann_{\mathbb{Z}}((3, 0)) = 0$ but $(3, 0) \notin N$ and so N is neither r-submodule nor sr-submodule of M.

2. Consider the submodule $N = \langle \bar{4} \rangle \times \langle 0 \rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_8 \times \mathbb{Z}$. Let $r \in \mathbb{Z}$ and $m = (m_1, m_2) \in M$ such that $r^2 \cdot (m_1, m_2) \in N$. Then it is clear to observe that $Ann_{\mathbb{Z}}(r) = Ann_{\mathbb{Z}}(m_1) = Ann_{\mathbb{Z}}(m_2) = 0$. Since again N is a semi r-submodule of M as $\langle \bar{4} \rangle$ is a semi r-ideal of $\mathbb{Z}_8$ and $\langle 0 \rangle$ is a semi r-ideals of $\mathbb{Z}$. However, $2^2 \cdot (\bar{1}, 0) \in N$ but $2 \cdot (\bar{1}, 0) \notin N$ and so N is not a semiprime submodule of M.

Proposition 6. *Let M be an R-module, N a proper submodule of M and k any positive integer. Then N is a semi r-submodule of M if and only if whenever $r \in R$, $m \in M$ with $r^k m \in N$, $Ann_M(r) = 0_M$ and $Ann_R(m) = 0$, then $rm \in N$.*

Proof. The proof follows by mathematical induction on k in a similar way to that of Theorem 1 (3). $\square$

We recall that a module M is torsion (resp. torsion-free) if $T(M) = M$ (resp. $T(M) = \{0\}$) where $T(M) = \{m \in M : \text{there exists } 0 \neq r \in R \text{ such that } rm = 0\}$. It is clear that any torsion-free module is faithful.

Proposition 7. *Semi r-submodules and semiprime submodules are coincide in any torsion-free module.*

Proof. Since every semiprime submodule is semi r-submodule, we need to show the converse. Let N be a semi r-submodule of an R-module M, $r \in R$, $m \in M$ with $r^2 m \in N$. Keeping in mind that M is torsion-free, we have

$Ann_R(m) = 0$. Now, suppose that $m' \in Ann_M(r)$. Then $rm' = 0$ and if $r = 0$, then clearly $rm \in N$. If $r \neq 0$, then $m' = 0$ again as M is torsion-free. Since N is a semi r-submodule, we conclude $rm \in N$, as required. □

Definition 5. *A proper submodule N of an R-module M is said to satisfy the D-annihilator condition if whenever K is a submodule of M and $r \in R$ such that $rK \subseteq N$ and $Ann_M(r) = 0_M$, then either $K \subseteq N$ or $K \cap T(M) = \{0_M\}$.*

Obviously, any r-submodule satisfies the D-annihilator condition. The converse is not true in general. For example the submodule $N = 6\mathbb{Z} \times \langle 0 \rangle$ of the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ clearly satisfies the D-annihilator condition. On the other hand, N is not an r-submodule of M, (see Example 4(1)). It is clear that any proper submodule of a torsion-free module satisfies the D-annihilator condition. However, we may find a submodule satisfying the D-annihilator condition in a torsion module. For example, for any positive integer n, every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_n$ satisfies the D-annihilator condition. Indeed, suppose that $rm \in \langle \bar{d} \rangle$ for some integer d dividing n. Put $n = cd$ then $cr\bar{m} = 0$. Since $Ann_M(r) = 0_M$, we get $c\bar{m} = 0$ and so $\bar{m} \in \langle \bar{d} \rangle$.

Proposition 8. *Let N be a proper submodule of an R-module M satisfying the D-annihilator condition. Then the following are equivalent.*

1. N is a semi r-submodule of M.
2. For $r \in R$ and a submodule K of M with $r^2K \subseteq N$ and $Ann_M(r) = 0_M$, then $rK \subseteq N$.

Proof. (1)⇒(2). Suppose that $r^2K \subseteq N$ and $Ann_M(r) = 0_M = Ann_M(r^2)$. If $K \subseteq N$, then we are done. If $K \nsubseteq N$, then $Ann_R(k) = 0_R$ for each $k \in K$ since by assumption $K \cap T(M) = \{0_M\}$. Since N is a semi r-submodule, we conclude that $rk \in N$. Therefore, $rk \in N$ for all $k \in K$ and the result follows.

(2)⇒(1). is straightforward. □

Recall that an R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Moreover, we have $N = (N :_R M)M$. Next, we conclude a useful characterization for semi r-submodules. First, recall the following lemmas.

Lemma 4. *[17] Let N be a submodule of a finitely generated faithful multiplication R-module M. For an ideal I of R, $(IN :_R M) = I(N :_R M)$, and in particular, $(IM :_R M) = I$.*

Lemma 5. *[1] Let N is a submodule of faithful multiplication R-module M. If I is a finitely generated faithful multiplication ideal of R, then*

1. *$N = (IN :_M I)$.*

2. *If $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.*

Theorem 9. *Let M be a finitely generated faithful multiplication R-module. Then a submodule $N = IM$ satisfying the D-annihilator condition is a semi r-submodule of M if and only if I is a semi r-ideal of R.*

Proof. Suppose $N = IM$ is a semi r-submodule of M and let $r \in R$ such that $r^2 \in I$ with $Ann_R(r) = 0$. We claim that $Ann_M(r) = 0_M$. Indeed, if there is $0_M \neq m \in M$ such that $rm = 0_M$, then $\langle r\rangle (\langle m\rangle :_R M) = (\langle rm\rangle :_R M) = (0_M :_R M) = 0$ by Lemma 4. Thus, $(\langle m\rangle :_R M) = 0$ as $Ann_R(r) = 0$ and then $\langle m\rangle = (\langle m\rangle :_R M)M = 0_M$, a contradiction. Since N satisfies the D-annihilator condition and $r^2M \subseteq IM$, then $rM \subseteq IM$ by Proposition 8. Thus, $r \in (rM :_R M) \subseteq (IM :_R M) = I$, as needed.

Conversely, suppose that I is a semi r-ideal of R. Let $r \in R$ and $K = JM$ be a submodule of M such that $r^2JM = r^2K \subseteq IM$ and $Ann_M(r) = 0_M$. Take $A = rJ$ and note that $A^2 \subseteq r^2JM : M \subseteq (IM :_R M) = I$ by Lemma 4. Now, we claim that $A \cap zd(R) = \{0\}$. Suppose on contrary that there exists $0 \neq a = rj \in A$ such that $Ann_R(a) \neq 0$. Choose $0 \neq b \in R$ with $ab = rjb = 0$. Then $rjbM = 0_M$ and so $jbM = 0_M$ as $Ann_M(r) = 0_M$. Since $b \neq 0$, $jM \subseteq K$ and N satisfies the D-annihilator condition, then $jM = 0$ and we conclude $j = 0$ as M is faithful, which is a contradiction. Therefore, $A \cap zd(R) = \{0\}$ and $A \subseteq I$ by Corollary 1. Thus, $rK = rJM = AM \subseteq IM = N$ as needed. $\square$

In view of Theorem 9 we give the following characterization.

Corollary 6. *Let R be a ring and M be a finitely generated faithful multiplication R-module. For a submodule N of M satisfying the D-annihilator condition, the following statements are equivalent.*

1. *N is a semi r-submodule of M.*

2. *$(N :_R M)$ is semi r-ideal of R.*

3. *$N = IM$ for some semi r-ideal I of R.*

Let N be a submodule of an R-module M and I be an ideal of R. The residual of N by I is the set $(N :_M I) = \{m \in M : Im \subseteq N\}$. It is clear that $(N :_M I)$ is a submodule of M containing N. More generally, for any subset $S \subseteq R$, $(N :_M S)$ is a submodule of M containing N. We recall that M-$rad(N)$ denotes the intersection of all prime submodules of M containing N. Moreover, if M is finitely generated faithful multiplication, then M-$rad(N) = \sqrt{(N :_R M)}M$, [17].

Proposition 9. *Let M be a finitely generated multiplication R-module and N be a semi r-submodule of M satisfying the D-annihilator condition.*

1. For any ideal I of R with $(N :_M I) \neq M$, $(N :_M I)$ is a semi r-submodule of M.
2. If M is faithful, then $(M\text{-}rad(N) :_R M) \subseteq zd(R) \cup \sqrt{(N :_R M)}$.

Proof. (1) First, we show that $(N :_M I)$ satisfies the D-annihilator condition. Let K be a submodule of M and $r \in R$ such that $rK \subseteq (N :_M I)$, $K \nsubseteq (N :_M I)$ and $Ann_M(r) = 0_M$. Then $rIK \subseteq N$ and so $IK \cap T(M) = \{0_M\}$. It follows clearly that $K \cap T(M) = \{0_M\}$ as needed. Suppose N is a semi r-submodule of M. Let K be a submodule of M such that $r^2K \subseteq (N :_M I)$ and $Ann_M(r) = 0_M$. Then $r^2IK \subseteq N$ which implies that $rIK \subseteq N$ by Proposition 8 and thus, $rK \subseteq (N :_M I)$. Therefore, $(N :_M I)$ is a semi r-submodule of M again by Proposition 8.

(2) Since N be a semi r-submodule, $(N :_R M)$ is a semi r-ideal of R by Corollary 6. Then the claim follows as $M\text{-}rad(N) = \sqrt{(N :_R M)}M$ and by using Theorem 1(4). $\square$

Next, we discuss when IN is a semi r-submodule of a finitely generated multiplication module M where I is an ideal of R and N is a submodule of M. Recall that a submodule N of an R-module M is said to be pure if $JN = JM \cap N$ for every ideal J of R.

Theorem 10. *Let I be an ideal of a ring R, M be a finitely generated faithful multiplication R-module and N be a submodule of M such that IN satisfies the D-annihilator condition.*

1. If I is a semi r-ideal of R and N is a pure semi r-submodule of M, then IN is a semi r-submodule of M.
2. Let I be a finitely generated faithful multiplication ideal of R. If IN is semi r-submodule of M, then either I is a semi r-ideal of R or N is a semi r-submodule of M.

Proof. (1) Suppose that $r^2K \subseteq IN$ and $Ann_M(r) = 0_M$ for some $r \in R$ and a submodule $K = JM$ of M. If we take $A = rJ$, then $A^2 \subseteq r^2JM : M \subseteq (IN : M) = I(N : M) \subseteq I \cap (N : M)$. By Theorem 9, $(N :_R M)$ is a semi r-ideal. We show that $A \cap zd(R) = \{0\}$. Let $x \in A \cap zd(R)$, say, $x = ry$ for some $y \in J$. Choose a nonzero $z \in R$ such that $xz = ryz = 0$. Then $ryzM = 0_M$ and since $Ann_M(r) = 0_M$, we have $yzM = 0_M$. Since M is faithful and $z \neq 0$, we conclude that $yM = 0_M$ and so $y = 0$. Thus $x = 0$, as required. Since $(N :_R M)$ is a semi r-ideal, then $A \subseteq (N :_R M)$ by Corollary 1. Therefore,

$rK = AM \subseteq (N :_R M)M = N$. On the other hand, since I is also a semi r-ideal, we have $A \subseteq I$ and so $rK = AM \subseteq IM$. Since N is pure, we conclude that $rK \subseteq IM \cap N = IN$ and we are done.

(2) First, by using Lemma 5, we note clearly that N satisfies the D-annihilator condition. We have two cases.

Case I. Let $N = M$. Then $I = I(N :_R M) = (IN :_R M)$ is a semi r-ideal of R by Corollary 6.

Case II. Let N be proper. Observe that by Lemma 5, we have the equality $(N :_R M) = ((IN :_M I) :_R M) = (I(N :_R M) :_M I)$. Suppose that $r^2 \in (N :_R M)$ and $r \notin zd(R)$. Then $(rI)^2 \subseteq r^2 I \subseteq I(N :_R M) = (IN :_R M)$ by Lemma 4. Here, similar to the proof of Theorem 9, it can be easily verify that $rI \cap zd(R) = \{0\}$. Since $(IN :_R M)$ is a semi r-ideal, $rI \subseteq (IN :_R M) = I(N :_R M)$ which means $r \in (I(N :_R M) :_M I) = (N :_R M)$ by Lemma 5. Thus, $(N :_R M)$ is a semi r-ideal of R and Corollary 6 implies that N is a semi r-submodule of M. □

Next, we study the behavior of the semi r-submodule property under module homomorphisms.

Proposition 10. *Let M and M' be R-modules and $f : M \to M'$ be an R-module homomorphism.*

1. If f is an epimorphism and N is a semi r-submodule of M such that $Ker(f) \subseteq N$ and $N \cap T(M) = \{0_M\}$, then $f(N)$ is a semi r-submodule of M'.
2. If f is an isomorphism and N' is a semi r-submodule of M', then $f^{-1}(N')$ is a semi r-submodule of M.

Proof. (1). Let N be a semi r-submodule of M and $r \in R$, $m' := f(m) \in M'$ $(m \in M)$ such that $r^2 m' \in f(N)$, $Ann_{M''}(r) = 0_{M\ '}$ and $Ann_R(f(m)) = 0_{M\ '}$. Then $r^2 m \in N$ as $Ker(f) \subseteq N$. We show that $Ann_M(r) = 0_M$. If $r = 0$, then the claim is obvious. Suppose $r \neq 0$ and there is $m_1 \in M$ such that $rm_1 = 0_M$. Then $rf(m_1) = 0_{M'}$ and so $f(m_1) = 0_{M'}$ as $Ann_{M''}(r) = 0_{M\ '}$. Thus, $m_1 \in Ker(f) \cap T(M) \subseteq N \cap T(M) = \{0_M\}$ as needed. Also, it is clear that $Ann_R(m) = 0_M$. Therefore, $rm \in N$ and so $rm' \in f(N)$ as required.

(2). Let N' is a semi r-submodule of M'. Suppose that $r^2 m \in f^{-1}(N')$, $Ann_M(r) = 0_M$ and $Ann_R(m) = 0$ for some $r \in R$ and $m \in M$. Then $r^2 f(m) = f(r^2 m) \in N'$, $Ann_{M'}(r) = 0_{M'}$ and $Ann_R(f(m)) = 0$. Indeed, if $rm' = 0$ for some $0 \neq m' = f(m_1) \in M'$, then $rm_1 \in K\text{erf} = \{0_M\}$ and clearly $0 \neq m_1 \in M$, a contradiction. Similarly, if there exists $0 \neq c \in R$ such that $cf(m) = 0_{M'}$, then $cm = 0_M$ which is also a contradiction. Since N' is a semi R-submodule, then $rf(m) \in N'$ and so $rm \in f^{-1}(N')$. Thus, $f^{-1}(N')$ is a semi r-submodule of M. □

In the following, we discuss semi r-submodules of localizations of modules. Here, the notation $Z_N(R)$ denotes the set $\{r \in R\colon\ rm \in N$ for some $m \in M\backslash N\}$.

Theorem 11. *Let S be a multiplicatively closed subset of a ring R and M be an R-module such that $S \cap Z(M) = \emptyset$.*

1. If N is a semi r-submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a semi r-submodule of $S^{-1}M$.
2. If $S^{-1}N$ is a semi r-submodule of $S^{-1}R$ and $S \cap Z_N(R) = \emptyset$, then N is a semi r-submodule of M.

Proof. (1) Let $\frac{r}{s} \in S^{-1}R$, $\frac{m}{t} \in S^{-1}M$ with $\left(\frac{r}{s}\right)^2\left(\frac{m}{t}\right) \in S^{-1}N$, $Ann_{S^{-1}M}(\frac{r}{s}) = 0_{S^{-1}M}$ and $Ann_{S^{-1}R}(\frac{m}{t}) = 0_{S^{-1}R}$. Choose $u \in S$ such that $r^2(um) \in N$. We show that $Ann_M(r) = 0_M$ and $Ann_R(um) = 0$. First, assume that $rm' = 0_M$ for some $m' \in M$. Then $\left(\frac{r}{s}\right)\left(\frac{m'}{1}\right) = 0_{S^{-1}M}$ and so $\frac{m'}{1} = 0_{S^{-1}M}$ as $Ann_{S^{-1}M}(\frac{r}{s}) = 0_{S^{-1}M}$. Hence, there exists $v \in S$ such that $vm' = 0_M$. Since $S \cap Z(M) = \emptyset$, then $m' = 0_M$ and so $Ann_M(r) = 0_M$. Secondly, assume that $r'um = 0$ for some $r' \in R$. Then $\frac{r'u}{1}\frac{m}{t} = 0_{S^{-1}M}$ and $Ann_{S^{-1}R}(\frac{m}{t}) = 0_{S^{-1}R}$ imply that $r'us = 0$ for some $s \in S$. But, clearly, $um \neq 0_M$ and so $us \in S \cap Z(M) = \emptyset$, a contradiction. Hence, $Ann_R(um) = 0$. Therefore, $r^2(um) \in N$ implies that $rum \in N$ and so $\frac{r}{s}\frac{m}{t} = \frac{rum}{sut} \in S^{-1}N$.

(2) Suppose that $r^2m \in N$ with $Ann_M(r) = 0_M$ and $Ann_R(m) = 0$ for some $r \in R$ and $m \in M$. Now, $\left(\frac{r}{1}\right)^2\frac{m}{1} \in S^{-1}N$. If $Ann_{S^{-1}M}(\frac{r}{1}) \neq 0_{S^{-1}M}$, then there exists $0_{S^{-1}M} \neq \frac{m'}{t} \in S^{-1}M$ such that $\frac{r}{1}\frac{m'}{t} = 0_{S^{-1}M}$ which implies $urm' = 0_M$ for some $u \in S$. Since $Ann_M(r) = 0_M$, we have $um' = 0_M$ and $\frac{m'}{t} = \frac{um'}{ut} = 0_{S^{-1}M}$, a contradiction. Now, assume that $Ann_{S^{-1}R}(\frac{m}{1}) \neq 0_{S^{-1}R}$. Then $\frac{r'}{s'}\frac{m}{1} = 0_{S^{-1}M}$ for some $0_{S^{-1}R} \neq \frac{r'}{s'} \in S^{-1}R$. Thus, $r'vm = 0$ for some $v \in S$ and clearly $r'm \neq 0_M$. Hence, again $v \in S \cap Z(M) = \emptyset$, a contradiction. Thus, $Ann_{S^{-1}M}(\frac{r}{1}) = 0_{S^{-1}M}$ and $Ann_{S^{-1}R}(\frac{m}{1}) = 0_{S^{-1}R}$ imply that $\frac{r}{1}\frac{m}{1} \in S^{-1}N$ and so $wrm \in N$ for some $w \in S$. Since $S \cap Z_N(M) = \emptyset$, we conclude that $rm \in N$, as desired. □

We recall from [2] that for an R-module M, we have

$$zd(R(+)M) = \{(r,m)|\ r \in zd(R) \cup Z(M),\ m \in M\}$$

where $Z(M) = \{r \in R : rm = 0 \text{ for some } 0_M \neq m \in M\}$. In the following proposition, we justify the relation between semi r-ideals of R and those of the idealization ring $R(+)M$.

Proposition 11. *Let M be an R-module and I be a proper ideal of R.*

1. *If I is a semi r-ideal of R, then $I(+)M$ is a semi r-ideal of $R(+)M$. Moreover, the converse is true if $Z(M) \subseteq zd(R)$.*

2. *If I is a semi r-ideal of R and N is an r-submodule of M, then $I(+)N$ is a semi r-ideal of $R(+)M$. Moreover, the converse is true if $Z(M) \subseteq zd(R)$.*

Proof. (1). Suppose that $(a,m)^2 \in I(+)M$ and $(a,m) \notin zd(R(+)M)$. Then $a^2 \in I$ and $a \notin zd(R)$. Since I is a semi r-ideal, we conclude that $a \in I$ and so $(a,m) \in I(+)M$. Now, assume that $Z(M) \subseteq zd(R)$ and $I(+)M$ is a semi r-ideal of $R(+)M$. Let $a \in R$ such that $a^2 \in I$ but $a \notin I$. Then $(a,0)^2 \in I(+)M$ and $(a,0) \notin I(+)M$ which imply that $(a,0) \in zd(R(+)M)$. Since $Z(M) \subseteq zd(R)$, we conclude that $a \in zd(R)$ and we are done.

(2). Suppose that $(a,m)^2 \in I(+)N$ and $(a,m) \notin zd(R(+)M)$. Then $a \in I$ as in (1). Moreover, $a.m \in N$ as $IM \subseteq N$. Since also, $a \notin Z(M)$, then $Ann_M(a) = 0$. Therefore, $m \in N$ as N is an r-submodule of M and $(a,m) \in I(+)N$ as needed. If $Z(M) \subseteq zd(R)$, then similar to the proof of (1), the converse holds. □

Remark 1. *In general, if $Z(M) \nsubseteq zd(R)$, then the converse of Proposition 11 need not be true. For example, consider the idealization ring $R = \mathbb{Z}(+)\mathbb{Z}_4$ and the ideal $4\mathbb{Z}(+)\mathbb{Z}_4$ of R. Let $(a,m)^2 \in 4\mathbb{Z}(+)\mathbb{Z}_4$ for $(a,m) \in R$. Then $a^2 \in 4\mathbb{Z}$ and so $(a,m) \in 2\mathbb{Z} \times \mathbb{Z}_4 = zd(R)$. Thus, $4\mathbb{Z}(+)\mathbb{Z}_4$ is a (semi) r-ideal of R. On the other hand, $4\mathbb{Z}$ is not a semi r-ideal of $\mathbb{Z}$.*

4 Semi r-submodules of amalgamated modules

Let R be a ring, J an ideal of R and M an R-module. Recently, in [5], the duplication of the R-module M along the ideal J (denoted by $M \bowtie J$) is defined as

$$M \bowtie J = \{(m,m') \in M \times M : m - m' \in JM\}$$

which is an $(R \bowtie J)$-module with scaler multiplication defined by $(r, r+j) \cdot (m,m') = (rm, (r+j)m')$ for $r \in R$, $j \in J$ and $(m,m') \in M \bowtie J$. For various properties and results concerning this kind of modules, one may see [5].

Let J be an ideal of a ring R and N be a submodule of an R-module M. Then

$$N \bowtie J = \{(n,m) \in N \times M : n - m \in JM\}$$

and

$$\bar{N} = \{(m,n) \in M \times N : m - n \in JM\}$$

are clearly submodules of $M \bowtie J$. Moreover,

$$Ann_{R\bowtie J}(M \bowtie J) = (r, r+j) \in R \bowtie I | r \in Ann_R(M) \text{ and } j \in Ann_R(M) \cap J\}$$

and so $M \bowtie J$ is a faithful $R \bowtie J$ -module if and only if M is a faithful R-module, [5, Lemma 3.6].

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1, j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [8], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scaler product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2)$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, one can easily justify that the sets

$$N_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : m_1 \in N_1\}$$

and

$$\overline{N_2}^\varphi = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : \ \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^\varphi JM_2$.

Note that if $R = R_1 = R_2$, $M = M_1 = M_2$, $f = Id_R$ and $\varphi = Id_M$, then the amalgamation of M_1 and M_2 along J with respect to φ is exactly the duplication of the R-module M along the ideal J. Moreover, in this case, we have $N_1 \bowtie^\varphi JM_2 = N \bowtie J$ and $\overline{N_2}^\varphi = \bar{N}$.

Theorem 12. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as above. Assume $JM_2 = \{0_{M_2}\}$ and let N_1 be submodule of M_1. Then*

1. N_1 is an r-submodule of M_1 if and only if $N_1 \bowtie^\varphi JM_2$ is an r-submodule of $M_1 \bowtie^\varphi JM_2$.
2. If N_1 is a semi r-submodule of M_1, then $N_1 \bowtie^\varphi JM_2$ is a semi r-submodule of $M_1 \bowtie^\varphi JM_2$.

3. If M_2 is faithful and $N_1 \bowtie^{\varphi} JM_2$ is a semi r-submodule of $M_1 \bowtie^{\varphi} JM_2$, then N_1 is a semi r-submodule of M_1.

Proof. (1) Let N_1 be an r-submodule of M_1 and let $(r_1, f(r_1)+j) \in R_1 \bowtie^f J$, $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$ such that $(r_1, f(r_1)+j)(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$ and $Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, f(r_1)+j)) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Then $r_1 m_1 \in N_1$ and we prove that $Ann_{M_1}(r_1) = 0_{M_1}$. Suppose $r_1 m_1' = 0_{M_1}$ for some $m_1' \in M_1$. Then $(r_1, f(r_1)+j)(m_1', \varphi(m_1')) = (0_{M_1}, j\varphi(m_1')) = (0_{M_1}, 0_{M_2})$ as $JM_2 = \{0_{M_2}\}$. Thus, $(m_1', \varphi(m_1')) \in Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, f(r_1)+j)) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Hence, $m_1' = 0_{M_1}$ and $Ann_{M_1}(r_1) = 0_{M_1}$. By assumption, $m_1 \in N_1$ and then $(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$, as needed.

Conversely, let $r_1 \in R_1$ and $m_1 \in M_1$ such that $r_1 m_1 \in N_1$ and $Ann_{M_1}(r_1) = 0_{M_1}$. Then $(r_1, f(r_1)) \in R_1 \bowtie^f J$, $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$ and $(r_1, f(r_1))$ $(m_1, \varphi(m_1)) = (r_1 m_1, \varphi(r_1 m_1)) \in N_1 \bowtie^{\varphi} JM_2$.
Moreover, $Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, f(r_1))) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Indeed, suppose that there $(m_1', \varphi(m_1')) \in M_1 \bowtie^{\varphi} JM_2$ such that $(r_1, f(r_1))(m_1', \varphi(m_1')) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Then $(m_1', \varphi(m_1')) = (0_{M_1}, 0_{M_2})$ as $Ann_{M_1}(r_1) = 0_{M_1}$. Since $N_1 \bowtie^{\varphi} JM_2$ is an r-submodule of $M_1 \bowtie^{\varphi} JM_2$, then $(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$ so that $m_1 \in N_1$ and we are done.

(2) Let $(r_1, f(r_1)+j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$ such that $(r_1, f(r_1)+j)^2(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$, $Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, f(r_1)+j)) = 0_{M_1 \bowtie^{\varphi} JM_2}$ and $Ann_{R_1 \bowtie^f J}((m_1, \varphi(m_1))) = 0_{R_1 \bowtie^f J}$. Then $r_1^2 m_1 \in N_1$ and similar to the proof of (1), we have $Ann_{M_1}(r_1) = 0_{M_1}$. We show that $Ann_{R_1}(m_1) = 0_{R_1}$. Assume on the contrary that there is nonzero element $r_1 \in R_1$ such that $r_1 m_1 = 0_{R_1}$. Then, $(r_1, f(r_1))(m_1, \varphi(m_1)) = 0_{M_1 \bowtie^{\varphi} JM_2}$, but our assumption $Ann_{R_1 \bowtie^f J}((m_1, \varphi(m_1))) = 0_{R_1 \bowtie^f J}$ implies that $(r_1, f(r_1)) = 0_{R_1 \bowtie^f J}$; i.e. $r_1 = 0_{R_1}$, a contradiction. Thus $Ann_{R_1}(m_1) = 0_{R_1}$, and it follows that $r_1 m_1 \in N_1$ and so $(r_1, f(r_1)+j)(m_1, \varphi(m_1)+m_2) \in N_1 \bowtie^{\varphi} JM_2$.

(3) Since M_2 is faithful, then clearly $J = \{0_{R_2}\}$. Let $r_1 \in R_1$ and $m_1 \in M_1$ such that $r_1^2 m_1 \in N_1$, $Ann_{M_1}(r_1) = 0_{M_1}$ and $Ann_{R_1}(m_1) = 0_{R_1}$. Then $(r_1, f(r_1))^2(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$ where $(r_1, f(r_1)) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$. Again, similar to the proof of (1), we have $Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, f(r_1))) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Moreover, suppose there is $(r_1', f(r_1')) \in R_1 \bowtie^f J$ such that $(r_1' m_1, \varphi(r_1' m_1)) = (r_1', f(r_1')+j)(m_1, \varphi(m_1)) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Then $(r_1', f(r_1')) = (0_{R_1}, 0_{R_2})$ as $Ann_{R_1}(m_1) = 0_{R_1}$. Therefore, $Ann_{R_1 \bowtie^f J}((m_1, \varphi(m_1))) = 0_{M_1 \bowtie^{\varphi} JM_2}$. By assumption, $(r_1, f(r_1))(m_1, \varphi(m_1))$ $\in N_1 \bowtie^{\varphi} JM_2$. It follows that $r_1 m_1 \in N_1$ and N_1 is a semi r-submodule of M_1. □

Corollary 7. *Let N be a submodule of an R-module M and J be an ideal of R. Then*

1. If $N \bowtie J$ is an r-submodule of $M \bowtie J$, then N is an r-submodule of M. The converse is true if $JM = 0_M$.
2. If $N \bowtie J$ is a semi r-submodule of $M \bowtie J$, then N is a semi r-submodule of M. The converse is true if $JM = 0_M$.

Proof. (1) Let $r \in R$ and $m \in M$ such that $rm \in N$ and $Ann_M(r) = 0_M$. Then $(r,r)(m,m) \in N \bowtie J$ and clearly, $Ann_{M \bowtie J}((r,r)) = 0_{M \bowtie J}$. Thus, $(m,m) \in N \bowtie J$ and so $m \in N$ as needed. Conversely, suppose $JM = 0_M$ and let $(r, r+j) \in R \bowtie J$, $(m, m+m') \in M \bowtie J$ such that $(r, r+j)(m, m+m') \in N \bowtie J$ and $Ann_{M \bowtie J}((r, r+j)) = 0_{M \bowtie J}$. If $rm'' = 0_M$ for some $m'' \in M$, then $(r, r+j)(m'', m'') = (0, jm'') = (0_M, 0_M)$ as $JM = 0_M$. Thus, $m'' = 0_M$ and $Ann_M(r) = 0_M$. Since $rm \in N$, then $m \in N$ and so $(m, m+m') \in N \bowtie J$.

(2) Let $r \in R$ and $m \in M$ such that $r^2 m \in N$, $Ann_M(r) = 0_M$ and $Ann_R(m) = 0_R$. Then $(r,r)^2(m,m) \in N \bowtie J$. If there exists an element (m', m'') of $M \bowtie J$, $(r,r)(m', m'') = (0_M, 0_M)$, then clearly $(m', m'') = (0_M, 0_M)$ as $Ann_M(r) = 0_M$; and so $Ann_{M \bowtie J}((r,r)) = 0_{M \bowtie J}$. Also, if for $(r', r'+j) \in R \bowtie J$, $(r', r'+j)(m,m) = (0_M, 0_M)$, then $(r', r'+j) = (0_R, 0_R)$ and $Ann_{R \bowtie J}((m,m)) = 0_{R \bowtie J}$. By assumption, $(r,r)(m,m) \in N \bowtie J$ and so $rm \in N$. The proof of the converse part is similar to that of the converse of (1). □

Theorem 13. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 12 and let N_2 be a submodule of M_2.*

1. *If N_2 is an r-submodule of M_2, $JM_2 \neq \{0_{M_2}\}$ and $T(M_2) \subseteq JM_2$, then $\overline{N_2}^\varphi$ is an r-submodule of $M_1 \bowtie^\varphi JM_2$. Moreover, if f is an epimorphism and φ is an isomorphism, then the converse holds.*
2. *If f and φ are isomorphisms and $\overline{N_2}^\varphi$ is a semi r-submodule of $M_1 \bowtie^\varphi JM_2$, then N_2 is a semi r-submodule of M_2.*

Proof. (1). Suppose N_2 is an r-submodule of M_2. Let $(r_1, f(r_1)+j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)+m_2) \in M_1 \bowtie JM_2$ such that $(r_1, f(r_1)+j)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi$ and $Ann_{M_1 \bowtie^\varphi JM_2}((r_1, f(r_1)+j)) = 0_{M_1 \bowtie^\varphi JM_2}$. Then $(f(r_1)+j)(\varphi(m_1)+m_2) \in N_2$ and $Ann_{M_2}((f(r_1)+j)) = 0_{M_2}$. Indeed, suppose $(f(r_1)+j)m_2' = 0_{M_2}$ for some $0_{M_2} \neq m_2' \in M_2$. If $m_2' \in JM_2$, then $(r_1, f(r_1)+j)(0_{M_1}, 0_{M_2} + m_2') = 0_{M_1 \bowtie JM_2}$ where $(0_{M_1}, 0_{M_2} + m_2') \neq 0_{M_1 \bowtie JM_2}$, a contradiction. If $m_2' \notin JM_2$, then $m_2' \notin T(M_2)$ and so $(f(r_1)+j) = 0_{R_2}$. If we choose $0 \neq m_2'' \in JM_2$, then $(r_1, f(r_1)+j)(0_{M_1}, m_2'') = 0_{M_1 \bowtie JM_2}$ which is also a contradiction. By assumption, $\varphi(m_1) + m_2 \in N_2$ and so $(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^\varphi$.

Conversely, suppose φ is an isomorphism and $\overline{N_2}^\varphi$ is an r-submodule of $M_1 \bowtie^\varphi JM_2$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = \varphi(m_1) \in M_2$ such that $r_2 m_2 \in$

N_2 and $Ann_{M_2}(r_2) = 0_{M_2}$. Then $(r_1, r_2) \in R_1 \bowtie^f J$, $(m_1, m_2) \in M_1 \bowtie^{\varphi} JM_2$ and $(r_1, r_2)(m_1, m_2) \in \overline{N_2}^{\varphi}$. Suppose on contrary that there is $(m_1', \varphi(m_1') + m_2') \neq 0_{M_1 \bowtie^{\varphi} JM_2}$ such that $(r_1, r_2)(m_1', \varphi(m_1') + m_2') = 0_{M_1 \bowtie^{\varphi} JM_2}$. If $\varphi(m_1') + m_2' \neq 0_{M_2}$, we get a contradiction. If $\varphi(m_1') + m_2' = 0_{M_2}$ (and so $m_1' \neq 0_{M_1}$), then clearly $r_2 m_2' = 0_{M_2}$ and then $m_2' = 0_{M_2}$. It follows that $\varphi(m_1') = 0_{M_2}$ and so $m_1' = 0_{M_1}$, a contradiction. Since $\overline{N_2}^{\varphi}$ is an r-submodule of $M_1 \bowtie^{\varphi} JM_2$, then $(m_1, m_2) \in \overline{N_2}^{\varphi}$ and so $m_2 \in N_2$ as required.

(3) Let $r_2 = f(r_1) \in R_2$ and $m_2 = \varphi(m_1) \in M_2$ such that $r_2^2 m_2 \in N_2$, $Ann_{M_2}(r_2) = 0_{M_2}$ and $Ann_{R_2}(m_2) = 0_{R_2}$. Then $(r_1, r_2))^2(m_1, m_2) \in \overline{N_2}^{\varphi}$ where $(r_1, f(r_1)) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$. Similar to the proof of the converse part of (1), we have $Ann_{M_1 \bowtie^{\varphi} JM_2}((r_1, r_2)) = 0_{M_1 \bowtie^{\varphi} JM_2}$. We prove that $Ann_{R_1 \bowtie^f J}((m_1, m_2)) = 0_{R_1 \bowtie^f J}$. Let $(r_1', f(r_1') + j') \in R_1 \bowtie^f J$ such that $(r_1', f(r_1') + j')(m_1, m_2) = 0_{M_1 \bowtie^{\varphi} JM_2}$. Then $f(r_1') + j' = 0_{R_2}$ and $r_1' m_1 = 0_{M_1}$. Thus, $f(r_1') m_2 = 0$ and so $f(r_1') = 0_{R_2}$. Since f is one to one, then $r_1' = 0_{R_1}$ and so $(r_1', f(r_1') + j') = 0_{R_1 \bowtie^f J}$ as needed. By assumption, $(r_1, r_2))(m_1, m_2) \in \overline{N_2}^{\varphi}$ and so $r_2 m_2 \in N_2$. □

Corollary 8. *Let N be a submodule of an R-module M and J be an ideal of R. Then*

1. If $\bar{N}$ is an r-submodule of $M \bowtie J$, then N is an r-submodule of M. The converse is true if $JM = 0_M$.

2. If $\bar{N}$ is a semi r-submodule of $M \bowtie J$, then N is a semi r-submodule of M. The converse is true if $JM = 0_M$.

Proof. The proof is similar to that of Corollary 7 and left to the reader. □

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Hani A. KHASHAN,
Department of Mathematics,
Faculty of Science,
Al al-Bayt University,
Al Mafraq, Jordan.
Email: hakhashan@aabu.edu.jo

Ece YETKIN CELIKEL,
Department of Basic Sciences,
Faculty of Engineering,
Hasan Kalyoncu University
Gaziantep, Turkey.
Email:ece.celikel@hku.edu.tr, yetkinece@gmail.com