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Codes parameterized by the edges of a bipartite graph with a perfect matching

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Abstract

In this paper we study the main characteristics of some evaluation codes parameterized by the edges of a bipartite graph with a perfect matching.

1 Introduction

This work aims to study certain classes of linear codes, known as parameterized codes (see Definition 2.6). As our main goal is to relate these codes with bipartite graphs with a perfect matching (see Definitions 2.1 and 2.2), the codes are parameterized by the edges of a graph $\mathcal{G}$. The procedure is as follows: given a graph $\mathcal{G}$, we define its toric set $\mathbb{X}$ parameterized by its edges (see Definition 2.4), and then we associate an evaluation code, $C_{\mathbb{X}}(d)$, to this set $\mathbb{X}$. Our primary purpose is the description of the main characteristics of these codes. This article is an interesting generalization of [6], where the authors study the case of an even cycle $\mathcal{G} = C_n$. Furthermore, this work generalizes the case where $\mathcal{G} = \mathcal{K}_{m,m}$, a specific complete bipartite graph. In both instances, $\mathcal{G}$ is a bipartite graph with a perfect matching.

As far as we know, the first approach to this topic is given in [5], where the authors study the codes $C_{\mathbb{X}}(d)$ when $\mathbb{X}$ is the toric set parameterized by the edges of a complete bipartite graph $\mathcal{K}_{m,n}$. The results they obtain come

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from the fact that this code is the tensor product (as linear spaces) of codes associated with the projective torus (see Definition 2.3). The case of the codes $C_{\mathbb{X}}(d)$ when $\mathbb{X}$ is the projective torus $\mathbb{T}_{s-1}$ is studied in [4], although the dimension and the regularity index (see Definition 2.8) are found in [2] because $\mathbb{X}$ is a complete intersection (see Definition 2.9). However, the formula for the minimum distance is given until 2011, in [14]. Actually, in 2018 and 2020, in [1] and [3], the authors found the generalized Hamming weights and the relative generalized Hamming weights, respectively, of the affine cartesian codes, which are introduced in [10]. These weights are a generalization of the minimum distance, and since the codes arising from the projective torus are equal to some particular affine cartesian codes, their value is known when $\mathbb{X} = \mathbb{T}_{s-1}$. Furthermore, the study of the generalized Hamming weights in the case of the cycle C_4 and some complete bipartite graphs of the form $\mathcal{K}_{2,m}$ is given in [9].

The only parameter known for any simple graph, connected or not, is the length of the code. In 2015, [11, Theorem 3.2], the authors found an explicit formula for the cardinality of the set $\mathbb{X}$, which is the length of $C_{\mathbb{X}}(d)$. Also, in the same article, they found the regularity index when $\mathcal{G}$ is an even cycle, [11, Theorem 6.2], and an upper bound for the case of bipartite graphs with subgraphs isomorphic to even cycles that have disjoint edge sets, [11, Theorem 6.3]. This upper bound is attained if the graph is connected, [11, Corollary 6.5]. Moreover, the case when $\mathcal{G}$ is an odd cycle is completely solved because its toric set is the projective torus.

If $\mathcal{G} = \mathcal{K}_n$ is a complete graph, its regularity index is shown in [7, Remark 3]. Moreover, some bounds for the minimum distance of these codes are given in [7, Corollaries 8 and 9]. Also, the regularity index when $\mathcal{G}$ is a complete multipartite graph is computed in [12, Theorem 4.3]. Finally, in [8], there are some general bounds for the main parameters of the code $C_{\mathbb{X}}(d)$ when $\mathbb{X}$ is the toric set parameterized by the edges of any simple graph $\mathcal{G}$.

The contents of this paper are as follows. In Section 2, we introduce the main concepts about graphs and linear codes that will be useful to the development of the article. We define the code $C_{\mathbb{X}}(d)$ when $\mathbb{X}$ is the toric set parameterized by the edges of a bipartite graph with a perfect matching, which is the fundamental structure of this work. In Section 3, we give, in Theorem 3.1, some bounds for the dimension and the minimum distance of $C_{\mathbb{X}}(d)$, and also for the regularity index of the vanishing ideal $I_{\mathbb{X}}$. In Section 4, we define the set $\mathbb{Y}$, which plays a significant role studying the dimension of $C_{\mathbb{X}}(d)$. We prove, in Theorem 4.1, that $\mathbb{Y}$ is a complete intersection, and find a set of generators for its vanishing ideal $I_{\mathbb{Y}}$. Moreover, we give a formula relating $I_{\mathbb{X}}$ and $I_{\mathbb{Y}}$ in Proposition 4.3. In Section 5, we find, in Theorem 5.2, a formula for the dimension of the code $C_{\mathbb{X}}(d)$ in terms of the dimension of the codes associated with the projective torus, and the Hilbert function of $I_{\mathbb{Y}}/I_{\mathbb{X}}$. It allows us to give a tight lower bound for the regularity index of the vanishing ideal $I_{\mathbb{X}}$, which is attained in the cases of even cycles and complete bipartite graphs of the form $\mathcal{K}_{m,m}$. This bound is also attained in the case of graphs such that all of their connected components are even cycles (Corollary 5.5). Finally, in Section 6, we give an example of a graph with two connected components, each of them a square, and describe the main characteristics of the code $C_{\mathbb{X}}(d)$ that were obtained in this work.

2 Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple graph with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and edge set $\mathcal{E} = \{e_1, \ldots, e_s\}$.

Definition 2.1. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called bipartite if there is a partition of $\mathcal{V}$ into two disjoint subsets $\mathcal{V} = \mathcal{U} \cup \mathcal{W}$, such that every edge $e \in \mathcal{E}$ joins a vertex in $\mathcal{U}$ to a vertex in $\mathcal{W}$.

Definition 2.2. A matching $\mathcal{M}$ in $\mathcal{G}$ is a subset of the edge set $\mathcal{E}$ such that for every $e, e' \in \mathcal{M}$ there is no vertex $v \in \mathcal{V}$ such that e and e' are both incidents on v. The matching $\mathcal{M}$ is called perfect if, for every $v \in \mathcal{V}$, there is $e \in \mathcal{M}$ which is incident on v.

It is immediate that if $\mathcal{G}$ has a perfect matching, then $|\mathcal{V}|$ is an even number. If $\mathcal{G}$ is bipartite, then $|\mathcal{U}| = |\mathcal{W}|$. From now on we assume that $\mathcal{G}$ is bipartite, $n = 2k$, and $\mathcal{M} = \{e_1, e_2, \ldots, e_k\}$ is a perfect matching. Without loss of generality we take $e_i = \{v_{2i-1}, v_{2i}\}$ for all $i = 1, \ldots, k$. Therefore we set $\mathcal{U} = \{v_1, v_3, \ldots, v_{2k-1}\}$ and $\mathcal{W} = \{v_2, v_4, \ldots, v_{2k}\}$. Also, from now on, we denote the degree of each vertex v_i as n_i for all $i = 1, \ldots, n$. For this kind of graphs we notice that $n_1 + n_3 + \cdots + n_{2k-1} = s$.

Let $K = \mathbb{F}_q$ be a finite field with q elements. The set of non–zero elements of K is denoted by K^*, and $|\mathbb{X}|$ denotes the cardinality of any set $\mathbb{X}$.

Definition 2.3. The projective torus of dimension $s-1$, which is a group under componentwise multiplication, is given by

$$\mathbb{T}_{s-1} = \{[t_1, \ldots, t_s] \in \mathbb{P}^{s-1} : (t_1, \ldots, t_s) \in (K^*)^s\},$$

where the projective space $\mathbb{P}^{s-1}$ is the quotient space $(K^s \setminus \{\mathbf{0}\})/\sim$, where for any $\vec{x}_1$, $\vec{x}_2 \in K^s \setminus \{\mathbf{0}\}$, $\vec{x}_1 \sim \vec{x}_2$ if and only if there is $\lambda \in K^*$ such that $\vec{x}_1 = \lambda \vec{x}_2$.

Furthermore, we need to introduce some basic facts about linear codes and how we define the linear codes parameterized by the edges of a graph $\mathcal{G}$. We consider that $\mathcal{G}$ has no isolated vertices and it is not necessarily connected. Also we assume that $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the vertex set, and $\mathcal{E} = \{e_1, \ldots, e_s\}$ is the edge set of $\mathcal{G}$. For each edge $e_i = \{v_j, v_k\}$, where $v_j, v_k \in \mathcal{V}$, let $(t_1, \ldots, t_n)^{e_i} = t_j t_k$ for $(t_1, \ldots, t_n) \in (K^*)^n$.

Definition 2.4. The toric set $\mathbb{X}$ parameterized by the edges of the graph $\mathcal{G}$ is the following subset of the projective torus $\mathbb{T}_{s-1}$:

$$\mathbb{X} = \{[(t_1, \ldots, t_n)^{e_1}, \ldots, (t_1, \ldots t_n)^{e_s}] \in \mathbb{P}^{s-1} : t_i \in K^*\}. \tag{1}$$

Equation (1) works for any simple graph. However, when we work with the case where $\mathcal{G}$ is a bipartite graph with a perfect matching and with m connected components, there is no loss of generality if we assume that the toric set parameterized by its edges is given by

$$\mathbb{X} = \{[t_1t_2, \ldots, t_1t_{2i_1}, t_3t_4, \ldots, t_3t_{2i_3}, \ldots, t_{2k-1}t_{2k}, \ldots, t_{2k-1}t_{2i_{2k-1}}] \in \mathbb{P}^{s-1} : t_i \in K^*\},$$

where the first n_1 entries are the edges incident on v_1 (starting with t_1t_2), the second block has n_3 entries which are the edges incident on v_3 (starting with t_3t_4), and so on until the last block of entries which are the n_{2k-1} edges incident with v_{2k-1} (starting with $t_{2k-1}t_{2k}$). Each block of entries starts with the elements of the perfect matching $\mathcal{M}$.

Definition 2.5. A linear code C is a subspace of K^l, where l is a positive integer. This integer l is known as its length. Its dimension as a linear space over K is called its dimension, and it is denoted by $\dim_K C$. Finally, the minimum distance of a code C, δ_C, is defined as follows:

$$\delta_C = \min\{w_H(v) : v \in C,\ v \neq 0\},$$

where $w_H(v)$ is the Hamming weight of v, that is, the number of non–zero entries of v. These three numbers (length, dimension, and minimum distance) are called the main parameters of a code C, and they are related by the Singleton bound:

$$\delta_C \leq l - \dim_K C + 1.$$

Moreover, let $S = K[X_1, \ldots, X_s] = \oplus_{d \geq 0} S_d$ be a polynomial ring and $\mathbb{X} = \{P_1, \ldots, P_{|\mathbb{X}|}\}$ be the toric set parameterized by the edges of the graph $\mathcal{G}$.

Definition 2.6. The code of order d parameterized by the edges of the graph $\mathcal{G}$, which is denoted by $C_{\mathbb{X}}(d)$, is the following subspace of $K^{|\mathbb{X}|}$:

$$C_{\mathbb{X}}(d) = \left\{ \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|\mathbb{X}|})}{X_1^d(P_{|\mathbb{X}|})} \right) : f \in S_d \right\},$$

that is, $C_{\mathbb{X}}(d)$ is the image of the following surjective linear transformation:

$$S_d \longrightarrow K^{|\mathbb{X}|},$$
$$f \mapsto \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|\mathbb{X}|})}{X_1^d(P_{|\mathbb{X}|})}\right).$$

Definition 2.7. The ideal of S that is spanned by the homogeneous polynomials that vanish on $\mathbb{X}$ is called the vanishing ideal of $\mathbb{X}$, and it is denoted by $I_{\mathbb{X}}$. It is a graded ideal, $I_{\mathbb{X}} = \oplus_{d\geq 0} I_{\mathbb{X}}(d)$, and its main characteristics can be seen in [13].

In the case of the code $C_{\mathbb{X}}(d)$, its length is $|\mathbb{X}|$, its dimension is given by the Hilbert function $\dim_K C_{\mathbb{X}}(d) = H_{\mathbb{X}}(d) = \dim_K(S_d/I_{\mathbb{X}}(d))$, and its minimum distance is denoted by $\delta_{\mathbb{X}}(d)$. It is known that the Hilbert function is a strictly increasing function until it stabilizes.

Definition 2.8. The regularity index of $S/I_{\mathbb{X}}$ is the least integer d such that $H_{\mathbb{X}}(d) = |\mathbb{X}|$. It is denoted by $\operatorname{reg}(S/I_{\mathbb{X}})$. Actually, $H_{\mathbb{X}}(d) = |\mathbb{X}|$ for all $d \geq \operatorname{reg}(S/I_{\mathbb{X}})$. For these cases, $C_{\mathbb{X}}(d) = K^{|\mathbb{X}|}$, and then $\delta_{\mathbb{X}}(d) = 1$. Therefore the only interesting codes $C_{\mathbb{X}}(d)$ are those for which $d < \operatorname{reg}(S/I_{\mathbb{X}})$.

Finally, we need to introduce the concept of a complete intersection.

Definition 2.9. A set of points $\mathbb{X} \subseteq \mathbb{P}^{s-1}$ is called a complete intersection if its vanishing ideal $I_{\mathbb{X}}$ is generated by a regular sequence of $s-1$ elements, that is, $I_{\mathbb{X}} = (f_1, \ldots, f_{s-1})$, such that f_1 is not a zero divisor of S, and f_i is not a zero divisor of $S/(f_1, \ldots, f_{i-1})$ for $i = 2, \ldots, s-1$.

3 Some bounds

In the following theorem, we give the length of the code $C_{\mathbb{X}}(d)$ parameterized by the edges of a bipartite graph with m connected components and with a perfect matching, and also we give some bounds for the regularity index of $S/I_{\mathbb{X}}$, the dimension, and the minimum distance. It is worth mentioning that the bounds for the minimum distance of $C_{\mathbb{X}}(d)$ depend on the value of the minimum distance when the toric set is the projective torus, which was computed in [14, Theorem 3.5].

Theorem 3.1. *If $\mathcal{G}$ is a bipartite graph with a perfect matching, with m connected components, and $\mathbb{X}$ is the toric set parameterized by its edges, then:*

1. *The length of the code $C_{\mathbb{X}}(d)$ is given by:*

$$|\mathbb{X}| = (q-1)^{n-m-1}.$$

2. The regularity index of $S/I_{\mathbb{X}}$ is bounded by:

$$\left\lceil \frac{(q-2)(n-1)}{2(q-1)^m} \right\rceil \leq \operatorname{reg}(S/I_{\mathbb{X}}) \leq (q-2)(k-1) + (q-1)^{k-m} - 1,$$

where $n = 2k$.

3. The dimension of the code $C_{\mathbb{X}}(d)$ is bounded by:

$$\dim_K(C_{\mathbb{X}}(d)) \geq \sum_{j=0}^{\lfloor d/(q-1) \rfloor} (-1)^j \binom{k-1}{j} \binom{k-1+d-j(q-1)}{k-1}.$$

4. The minimum distance of the code $C_{\mathbb{X}}(d)$ is bounded by:

$$\left\lceil \frac{\delta_{\mathbb{T}_{n-1}}(2d)}{(q-1)^m} \right\rceil \leq \delta_{\mathbb{X}}(d) \leq (q-1)^{k-m} \delta_{\mathbb{T}_{k-1}}(d).$$

Proof. Assertion (1) follows directly from [11, Theorem 3.2]. Moreover, the lower bounds for the regularity index and the minimum distance given in (2) and (4) follow from [8, Theorem 2], because γ, the number on non–bipartite components of $\mathcal{G}$, is equal to zero.

Furthermore, let $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ be the subgraph of $\mathcal{G}$ with the same vertex set, but with $\mathcal{E}' = \mathcal{M}$. Let $\mathbb{X}$ and $\mathbb{X}'$ be the toric sets associated with the edges of $\mathcal{G}$ and $\mathcal{G}'$, respectively. Thus

$$\mathbb{X}' = \{[t_1t_2, t_3t_4, \cdots, t_{2k-1}t_{2k}] \in \mathbb{P}^{k-1} : t_i \in K^*\}, \tag{2}$$

and therefore $\mathbb{X}' = \mathbb{T}_{k-1}$. Then $|\mathbb{X}'| = (q-1)^{k-1}$, and we notice that

$$|\mathbb{X}| = (q-1)^{n-m-1} = |\mathbb{X}'|(q-1)^{k-m}.$$

Inequality (3) and the upper bounds for the regularity index and the minimum distance given in (2) y (4) follow easily from [8, Theorem 3]. □

4 Vanishing ideals

We continue using the notation of the last sections. Let $n_i = \deg v_i$ for $i = 1, \ldots, n$. Of course, $n_1 + n_3 + \cdots + n_{2k-1} = s$, the number of edges of the graph $\mathcal{G}$. Moreover, let $S = K[X_1, \ldots, X_s]$, as in Section 2, and $R = K[Y_1, Y_3, \ldots, Y_{2k-1}]$ be two polynomial rings. From now on, let $\mathbb{Y}$ be the following subset of the projective space $\mathbb{P}^{s-1}$:

$$\mathbb{Y} = \{[t_1, \ldots, t_1, t_3, \ldots, t_3, \ldots, t_{2k-1}, \ldots, t_{2k-1}] \in \mathbb{P}^{s-1} : t_i \in \mathbb{K}^*\}, \tag{3}$$

where t_i appears n_i times, for all $i = 1, 3, \ldots, 2k-1$. Clearly, $\mathbb{Y} \subseteq \mathbb{X}$. Actually, we get the following result:

Theorem 4.1. *$\mathbb{Y}$ is a complete intersection. In fact, $I_{\mathbb{Y}}$ is spanned by the union of the following sets:*

1. $\mathbb{W}_0 = \{X_{n_1+1}^{q-1} - X_1^{q-1}, X_{n_1+n_3+1}^{q-1} - X_1^{q-1}, \ldots, X_{n_1+n_3+\cdots+n_{2k-3}+1}^{q-1} - X_1^{q-1}\}$.

2. $$\mathbb{W}_1 = \begin{cases} \emptyset & \text{if } \; n_1 = 1, \\ \{X_i - X_1\}_{i=2}^{n_1} & \text{if } \; n_1 \geq 2. \end{cases}$$

3. $$\mathbb{W}_3 = \begin{cases} \emptyset & \text{if } \; n_3 = 1, \\ \{X_i - X_3\}_{i=n_1+2}^{n_1+n_3} & \text{if } \; n_3 \geq 2. \end{cases}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

4. $$\mathbb{W}_{2k-1} = \begin{cases} \emptyset & \text{if } \; n_{2k-1} = 1, \\ \{X_i - X_{2k-1}\}_{i=n_1+n_3+\cdots+n_{2k-3}+2}^{s} & \text{if } \; n_{2k-1} \geq 2. \end{cases}$$

Proof. We want to show that $(\mathbb{W}) = I_{\mathbb{Y}}$, where $\mathbb{W} := \mathbb{W}_0 \cup \mathbb{W}_1 \cup \mathbb{W}_3 \cup \cdots \cup \mathbb{W}_{2k-1}$. At first, we notice that the number of polynomials in $\mathbb{W}$ is given by

$$\begin{aligned} |\mathbb{W}| &= |\mathbb{W}_0| + |\mathbb{W}_1| + |\mathbb{W}_3| + \cdots + |\mathbb{W}_{2k-1}| \\ &= (k-1) + (n_1 - 1) + (n_3 - 1) + \cdots + (n_{2k-1} - 1) \\ &= k - 1 + s - k = s - 1, \end{aligned}$$

because $n_1 + n_3 + \cdots + n_{2k-1} = s$. On the other hand, it is easy to see that $\mathbb{W} \subseteq I_{\mathbb{Y}}$. Furthermore, let $f \in I_{\mathbb{Y}}$. We use the lexicographic ordering $X_s > X_{s-1} > \cdots > X_1$. By the division algorithm, f can be written as:

$$f = \sum_{i=2}^{n_1} f_i(X_i - X_1) + \sum_{i=n_1+2}^{n_1+n_3} g_i(X_i - X_3) + \cdots + \sum_{i=n_1+n_3+\cdots+n_{2k-3}+2}^{s} h_i(X_i - X_{2k-1}) + r,$$

where $f_i, g_i \ldots, h_i, r \in S$, and $r = 0$ (in this case $f \in (\mathbb{W})$, and $I_{\mathbb{Y}} \subseteq (\mathbb{W})$) or r is a K–linear combination of monomials, none of which is divisible by any of

$$X_2, \ldots, X_{n_1}, X_{n_1+2}, \ldots, X_{n_1+n_3}, \ldots, X_{n_1+n_3+\cdots+n_{2k-3}+2}, \ldots, X_s.$$

Therefore,

$$r(X_1,\dots,X_s) = r(X_1, X_{n_1+1}, X_{n_1+n_3+1},\dots, X_{n_1+n_3+\cdots+n_{2k-3}+1}).$$

Also, we observe that the projective torus $\mathbb{T}_{k-1}$ can be written as

$$\mathbb{T}_{k-1} = \{[t_1, t_{n_1+1}, t_{n_1+n_3+1},\dots, t_{n_1+n_3+\cdots+n_{2k-3}+1}] : t_i \in K^*\},$$

and thus ([4, Theorem 1])

$$I_{\mathbb{T}_{k-1}} = (X_{n_1+1}^{q-1} - X_1^{q-1}, X_{n_1+n_3+1}^{q-1} - X_1^{q-1}, \dots, X_{n_1+n_3+\cdots+n_{2k-3}+1}^{q-1} - X_1^{q-1}). \tag{4}$$

Let $P := [t_1,\dots,t_1,t_3,\dots,t_3,\dots,t_{2k-1},\dots,t_{2k-1}] \in \mathbb{Y}$. Then

$$0 = f(P) = r(P) = r(t_1, t_{n_1+1}, t_{n_1+n_3+1},\dots, t_{n_1+n_2+\cdots+n_{2k-3}+1}),$$

for all $t_i \in K^*$. Therefore $r \in I_{\mathbb{T}_{k-1}}$, and by Equation (4), $r \in (\mathbb{W}_0)$, and thus $f \in (\mathbb{W})$. That is, $I_{\mathbb{Y}} = (\mathbb{W})$, and the claim follows □

On the other hand, let θ be the map

$$\theta : S \to R,$$
$$f(X_1,\dots,X_s) \mapsto f(Y_1,\dots,Y_1,Y_3,\dots,Y_3,\dots Y_{2k-1},\dots,Y_{2k-1}),$$

where Y_i appears n_i times, for all $i = 1,3,\dots,2k-1$. We notice that $\theta(X_i) = Y_1$ for all $i = 1,\dots,n_1$, $\theta(X_i) = Y_3$ for all $i = n_1+1,\dots,n_1+n_3$, and so on until $\theta(X_i) = Y_{2k-1}$ for all $i = n_1+n_3+\cdots+n_{2k-3}+1,\dots,s$. Moreover, the following result relates the vanishing ideals of $\mathbb{X}$ and $\mathbb{T}_{k-1}$.

Proposition 4.2. *With the notation introduced above, θ is a ring epimorphism and*

$$\theta(I_{\mathbb{X}}) = I_{\mathbb{T}_{k-1}}.$$

Proof. The fact that θ is a ring epimorphism follows directly from the definitions. Let $f \in I_{\mathbb{X}}$ and $Q = [t_1,t_3,\dots,t_{2k-1}] \in \mathbb{T}_{k-1}$. Also let $P = [t_1,\dots,t_1,t_3,\dots,t_3,\dots,t_{2k-1},\dots,t_{2k-1}] \in \mathbb{Y} \subseteq \mathbb{X}$. Therefore

$$\theta(f)(Q) = f(P) = 0.$$

Thus $\theta(f) \in I_{\mathbb{T}_{k-1}}$ and then $\theta(I_{\mathbb{X}}) \subseteq I_{\mathbb{T}_{k-1}}$. On the other hand, as $\mathbb{X} \subseteq \mathbb{T}_{s-1}$, we get that $I_{\mathbb{T}_{s-1}} \subseteq I_{\mathbb{X}}$. Furthermore, $I_{\mathbb{T}_{k-1}}$ can be written as

$$I_{\mathbb{T}_{k-1}} = (Y_3^{q-1} - Y_1^{q-1},\dots,Y_{2k-1}^{q-1} - Y_1^{q-1}).$$

But

$$\begin{aligned} Y_3^{q-1} - Y_1^{q-1} &= \theta(X_{n_1+1}^{q-1} - X_1^{q-1}), \ldots, Y_{2k-1}^{q-1} - Y_1^{q-1} \\ &= \theta(X_{n_1+n_3+\cdots+n_{2k-3}+1}^{q-1} - X_1^{q-1}). \end{aligned}$$

Moreover,

$$\{X_{n_1+1}^{q-1} - X_1^{q-1}, \ldots, X_{n_1+n_3+\cdots+n_{2k-3}+1}^{q-1} - X_1^{q-1}\} \subseteq I_{\mathbb{T}_s-1} \subseteq I_{\mathbb{X}}.$$

Therefore $I_{\mathbb{T}_{k-1}} \subseteq \theta(I_{\mathbb{X}})$, and the claim follows. □

Since $\mathbb{Y} \subseteq \mathbb{X}$ we obtain that $I_{\mathbb{X}} \subseteq I_{\mathbb{Y}}$. The following result relates the vanishing ideals of $\mathbb{X}$, $\mathbb{Y}$, and the map θ.

Proposition 4.3. *The vanishing ideal of* $\mathbb{Y}$ *is given by*

$$I_{\mathbb{Y}} = I_{\mathbb{X}} + \ker \theta.$$

Proof. We notice that $\mathbb{W}_0 \subseteq I_{\mathbb{T}_{s-1}} \subseteq I_{\mathbb{X}}$ and $\mathbb{W}_{2i-1} \subseteq \ker \theta$ for all $i = 1, \ldots, k$. Therefore, by using Theorem 3.1, we conclude that

$$I_{\mathbb{Y}} \subseteq I_{\mathbb{X}} + \ker \theta. \quad (5)$$

Furthermore, let $f \in I_{\mathbb{X}}$, $g \in \ker \theta$, and

$$P := [t_1, \ldots, t_1, t_3, \ldots, t_3, \ldots, t_{2k-1}, \ldots, t_{2k-1}] \in \mathbb{Y}.$$

Thus, because $P \in \mathbb{Y} \subseteq \mathbb{X}$, $f(P) = 0$. Moreover, because $g \in \ker \theta$,

$$0 = \theta(g)(t_1, t_3, \ldots, t_{2k-1}) = g(P),$$

and then $(f + g)(P) = 0$. That is,

$$I_{\mathbb{X}} + \ker \theta \subseteq I_{\mathbb{Y}}. \quad (6)$$

The claim follows from (5) and (6). □

5 Dimension and the regularity index

Now, our goal is to compute the dimension of the code $C_{\mathbb{X}}(d)$ parameterized by the edges of a bipartite graph $\mathcal{G}$ with a perfect matching. To do this, we need the following lemma.

Lemma 5.1. *Let ψ the following map:*

$$\psi : S/I_{\mathbb{X}} \to R/I_{\mathbb{T}_{k-1}},$$
$$f + I_{\mathbb{X}} \to \theta(f) + I_{\mathbb{T}_{k-1}}.$$

Therefore, ψ is a ring epimorphism, and $\ker\psi = I_{\mathbb{Y}}/I_{\mathbb{X}}$.

Proof. At first, we notice that the map ψ is well–defined because if $f + I_{\mathbb{X}} = g + I_{\mathbb{X}}$ then $f - g \in I_{\mathbb{X}}$. Thus, because of Proposition 4.2, $\theta(f) - \theta(g) = \theta(f - g) \in \theta(I_{\mathbb{X}}) = I_{\mathbb{T}_{k-1}}$, and therefore $\theta(f) + I_{\mathbb{T}_{k-1}} = \theta(g) + I_{\mathbb{T}_{k-1}}$. Also, that ψ is a ring epimorphism follows immediately from the fact that θ is a ring epimorphism.

Let $f + I_{\mathbb{X}} \in \ker\psi$. Then $\theta(f) \in I_{\mathbb{T}_{k-1}} = \theta(I_{\mathbb{X}})$. Thus, there exists $g \in I_{\mathbb{X}}$ such that $\theta(f) = \theta(g)$. That is, $\theta(f-g) = \theta(f)-\theta(g) = 0$, and it implies that $f - g \in \ker\theta$. Therefore, there exists $h \in \ker\theta$ such that $f - g = h$, that is, $f = g + h$. Then $f \in I_{\mathbb{X}} + \ker\theta = I_{\mathbb{Y}}$, and we conclude that

$$\ker\psi \subseteq I_{\mathbb{Y}}/I_{\mathbb{X}}. \tag{7}$$

On the other hand, let $f + I_{\mathbb{X}} \in I_{\mathbb{Y}}/I_{\mathbb{X}}$. As $f \in I_{\mathbb{Y}} = I_{\mathbb{X}} + \ker\theta$, we get that $f = f_1 + f_2$ for some $f_1 \in I_{\mathbb{X}}$, $f_2 \in \ker\theta$. Thus $\theta(f) = \theta(f_1) + \theta(f_2) = \theta(f_1) \in \theta(I_{\mathbb{X}}) = I_{\mathbb{T}_{k-1}}$. Therefore $\psi(f + I_{\mathbb{X}}) = \theta(f) + I_{\mathbb{T}_{k-1}} = I_{\mathbb{T}_{k-1}}$, and then $f + I_{\mathbb{X}} \in \ker\psi$. It proves that

$$I_{\mathbb{Y}}/I_{\mathbb{X}} \subseteq \ker\psi, \tag{8}$$

and the result follows from (7) and (8). □

From now on we set $H_\psi(d) := \dim_K(I_{\mathbb{Y}}(d)/I_{\mathbb{X}}(d))$ for all $d \geq 0$. The main result of this section gives the dimension of $C_{\mathbb{X}}(d)$ in terms of $H_\psi(d)$ and $H_{\mathbb{T}_{k-1}}(d)$, where we know that ([4, Lemma 1] and [14, Corollary 2.2])

$$H_{\mathbb{T}_{k-1}}(d) = \sum_{j=0}^{\lfloor \frac{d}{q-1} \rfloor} (-1)^j \binom{k-1}{j} \binom{k-1+d-j(q-1)}{k-1}.$$

Theorem 5.2. *The dimension of the code $C_{\mathbb{X}}(d)$ parameterized by the edges of a bipartite graph $\mathcal{G}$ with a perfect matching is given by*

$$\dim_K C_{\mathbb{X}}(d) = H_{\mathbb{X}}(d) = H_\psi(d) + H_{\mathbb{T}_{k-1}}(d),$$

for all $d \in \mathbb{Z}, d \geq 0$.

Proof. Let ψ_d be the following linear map

$$\begin{aligned}\psi_d : S_d/I_{\mathbb{X}}(d) &\to R_d/I_{\mathbb{T}_{k-1}}(d),\\ f + I_{\mathbb{X}}(d) &\to \theta(f) + I_{\mathbb{T}_{k-1}}(d).\end{aligned}$$

ψ_d is a surjective linear transformation and then $(S_d/I_{\mathbb{X}}(d))/\ker\psi_d$ is isomorphic, as a linear space, to $R_d/I_{\mathbb{T}_{k-1}}(d)$. But $\ker\psi_d = I_{\mathbb{Y}}(d)/I_{\mathbb{X}}(d)$, and thus

$$H_{\mathbb{X}}(d) - \dim_K I_{\mathbb{Y}}(d)/I_{\mathbb{X}}(d) = H_\psi(d),$$

and the result follows. □

Although in Section 3 we gave some bounds for the regularity index of $S/I_{\mathbb{X}}$, where $\mathbb{X}$ is the toric set parameterized by the edges of a bipartite graph $\mathcal{G}$ with a perfect matching, in the following result we give a formula of this number in terms of the corresponding regularity index associated with the projective torus, which is given by ([4, Lemma 1])

$$\operatorname{reg}(R/I_{\mathbb{T}_{k-1}}) = (q-2)(k-1), \tag{9}$$

and the regularity of $I_{\mathbb{Y}}/I_{\mathbb{X}}$.

Corollary 5.3. *The regularity index of the quotient ring $S/I_{\mathbb{X}}$ is given by*

$$\operatorname{reg}(S/I_{\mathbb{X}}) = \max\{\operatorname{reg}(R/I_{\mathbb{T}_{k-1}}), \operatorname{reg}(I_{\mathbb{Y}}/I_{\mathbb{X}})\}.$$

Proof. Let φ the linear map

$$\begin{aligned}\varphi : I_{\mathbb{Y}}(d)/I_{\mathbb{X}}(d) &\to I_{\mathbb{Y}}(d+1)/I_{\mathbb{X}}(d+1),\\ f + I_{\mathbb{X}}(d) &\to X_1 f + I_{\mathbb{X}}(d+1).\end{aligned}$$

If $f + I_{\mathbb{X}}(d) = g + I_{\mathbb{X}}(d)$, with $f, g \in I_{\mathbb{Y}}(d)$, then

$$X_1 f - X_1 g = X_1(f-g) \in I_{\mathbb{X}}(d+1),$$

and thus $X_1 f + I_{\mathbb{X}}(d+1) = X_1 g + I_{\mathbb{X}}(d+1)$. It implies that φ is a well–defined map. It suffices to show that this is an injective map because, in this case, $H_\psi(d) \leq H_\psi(d+1)$ and H_ψ is a non–decreasing function. Let $f + I_{\mathbb{X}}(d) \in \ker\varphi$. Then $X_1 f \in I_{\mathbb{X}}(d+1)$. Let

$$P = [t_1t_2, \ldots, t_1t_{2i_1}, t_3t_4, \ldots, t_3t_{2i_3}, \ldots, t_{2k-1}t_{2k}, \ldots, t_{2k-1}t_{2i_{2k-1}}] \in \mathbb{X}.$$

Therefore, $t_1t_2 f(P) = 0$ for all $t_1, t_2 \in K^*$. That is, $f(P) = 0$ for all $P \in \mathbb{X}$. Then $f \in I_{\mathbb{X}}(d)$ and it implies that

$$\ker\varphi = I_{\mathbb{X}}(d).$$

Thus φ is an injective map, and the claim follows.

□

Remark 5.4. From Corollary 5.3 and Equation (9), we get that

$$\operatorname{reg}(S/I_{\mathbb{X}}) \geq \operatorname{reg}(R/I_{\mathbb{T}_{k-1}}) = (q-2)(k-1),$$

and this is a tight lower bound because it is attained in the case of even cycles ([11, Theorem 6.2]), and when the graph $\mathcal{G}$ is a complete bipartite graph of the form $\mathcal{K}_{m,m}$ ([5, Corollary 5.4]). Both graphs are bipartite graphs with a perfect matching.

Corollary 5.5. *Let $\mathcal{G}$ be a graph such that each of its m connected components is an even cycle C_{2l_i} and let $\mathbb{X}$ be the toric set parameterized by its edges. Then*

$$\operatorname{reg}(S/I_{\mathbb{X}}) = (q-2)(n-1).$$

Proof. We notice that in this case $\sum_{i=1}^{m} l_i = k$, where $n = s = 2k$. Furthermore, by [11, Theorem 6.3], we get that

$$\operatorname{reg}(S/I_{\mathbb{X}}) \leq (q-2)(s - \sum_{i=1}^{m} l_i - 1) = (q-2)(k-1).$$

As $\mathcal{G}$ is a bipartite graph with a perfect matching, the claim follows by Remark 5.4. $\square$

6 Example

Let $\mathcal{G}$ be the graph with two connected components, each of them a square (a cycle with four vertices), and let $\mathbb{X}$ be the toric set parameterized by its edges.

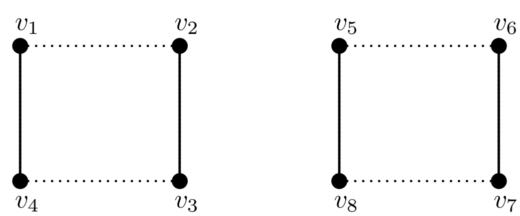

Figure 1: The graph $\mathcal{G}$.

A bipartition of the vertex set is given by

$$\mathcal{U} = \{v_1, v_3, v_5, v_7\}, \text{ and } \mathcal{W} = \{v_2, v_4, v_6, v_8\}.$$

Also, a perfect matching is $\mathcal{M} = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}\}$, and its edges appear with dotted lines in Figure 1. The toric set parameterized by the edges of $\mathcal{G}$ is given by

$$\mathbb{X} = \{[t_1t_2, t_1t_4, t_3t_4, t_3t_2, t_5t_6, t_5t_8, t_7t_8, t_7t_6] \in \mathbb{P}^7 : t_i \in K^*\},$$

and the length of the code $C_{\mathbb{X}}(d)$ is $|\mathbb{X}| = (q-1)^5$. Moreover, by using Corollary 5.5,

$$\operatorname{reg}(S/I_{\mathbb{X}}) = 3q-6,\ q > 2.$$

On the other hand, the set $\mathbb{Y}$, defined in Equation (3), is given by

$$\mathbb{Y} = \{[t_1, t_1, t_3, t_3, t_5, t_5, t_7, t_7] \in \mathbb{P}^7 : t_i \in K^*\},$$

and its vanishing ideal, according with Theorem 3.1, is

$$I_{\mathbb{Y}} = (X_3^{q-1} - X_1^{q-1}, X_5^{q-1} - X_1^{q-1}, X_7^{q-1} - X_1^{q-1}, X_2 - X_1, X_4 - X_3, X_6 - X_5, X_8 - X_7).$$

Furthermore, if we take $q = 5$, then the Hilbert functions involved in Theorem 5.2 are described in Table 1. Of course, we are interested in the cases $1 \leq d < \operatorname{reg}(S/I_{\mathbb{X}}) = 9$.

Table 1: The different Hilbert functions involved in Theorem 5.2 with $q = 5$, and $\mathbb{X}$ being the toric set parameterized by the edges of the graph $\mathcal{G}$ of Figure 1.

d	1	2	3	4	5	6	7	8
$H_{\mathbb{T}_{k-1}}(d)$	4	10	20	32	44	54	60	63
$H_{\psi}(d)$	4	24	84	208	396	616	796	912
$H_{\mathbb{X}}(d)$	8	34	104	240	440	670	856	975

Moreover, if we set $l_d = \left\lceil \frac{\delta_{\mathbb{T}_{n-1}}(2d)}{(q-1)^m} \right\rceil = \left\lceil \frac{\delta_{\mathbb{T}_7}(2d)}{4^2} \right\rceil$, $u_d = (q-1)^{k-m} \cdot \delta_{\mathbb{T}_{k-1}}(d) = 4^2 \cdot \delta_{\mathbb{T}_3}(d)$ (both bounds appear in Theorem 3.1), and $B_d = (q-1)^{n-m-1} - H_{\mathbb{X}}(d) + 1 = 4^5 - H_{\mathbb{X}}(d) + 1$ is the Singleton bound, then we get Table 2.

Table 2: Some bounds for the minimum distance of the code $C_{\mathbb{X}}(d)$ parameterized by the edges of the graph $\mathcal{G}$ of Figure 1.

d	1	2	3	4	5	6	7	8
l_d	512	192	64	32	12	4	2	1
u_d	768	512	256	192	128	64	48	32
B_d	1017	991	921	785	585	355	169	50

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