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Regular and Boolean elements in hoops and constructing Boolean algebras using regular filters

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Abstract

We study hoops in order to give some new characterizations for regular and Boolean elements in hoops and we study the relationship between them. Specially, we prove that any bounded $\veebar$-hoop is a Stone algebra if and only if MV-center set and Boolean elements set are equal. Then we define the concept of regular filter in hoops and $\veebar$-hoops with RF-property and peruse some properties of them. In addition, we show that each $\veebar$-hoop with RF-property, is a Boolean algebra and any hoop A with RF-property such that $\mathcal{B}(A) = \{0, 1\}$, is a local hoop. Finally, we prove that any hoop A has RF-property if and only if $Spec(A) = Max(A)$ and if and only if A is a hyperarchimedean.

1 Introduction

The residuated is a basic concept of arranged structures and categories and have been studied by many mathematicians. In Idziak (1984) showed that the family of residuated lattices is equational which are called BCK-lattices and full BCK-algebras, FLew-algebras in Ono and Komori (1985), and integral, residuated, commutative ℓ-monoids in Höhle (1995). Ward (1940), Ward and Dilworth (1939) were the first who introduced the concept of a residuated lattice as an extension of ideal lattices of a ring. In their original definition, a

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residuated lattice is what we would call it an integral commutative one. Over the last ten years, with the computers and information in science developing rapidly, the residuated lattice theory made great progress. Many experts and scholars had carried out thorough systematical research into it, they studied it from different points of view. For example, Blount and Tsinakis (2003) took it as an expansion of ℓ-groups, they discussed it from the view of ℓ-groups; Kühr (2008), van Altene et al. investigated it from the view of variety; Galatos et al. (2007) investigated it from the view of semiring. Residual lattice theory was used to develop the algebraic counterparts of fuzzy logics in Turunen (1999) and infrastructure logic, in Ono (2003). Hájek (1998) defined the concept of BL-algebras, filters and prime filters in it for proving the completeness theorem of Basic Logic, Turunen (1999) studied on BL-algebras and their filters. MTL-logic is a weaker logic that BL-logic, and it was defined by Esteva and Godo (2001) and they showed by Jenei and Montagna (2002) that MTL-logic is the logic of left continuous t-norms and their residua. Algebra corresponding to MTL-logic is an MTL-algebra. Also, a residuated lattice such as $\mathcal{L}$ is said to be an MTL-algebra if the prelinearity condition holds in $\mathcal{L}$. Cignoli (2008) investigated the structure of free algebras in the subvarieties of Stonean residuated lattices and proved that each algebra in a variety $\mathbf{V}$ of bounded residuated lattices can be represented as a weak Boolean product of directly indecomposable algebras in $\mathbf{V}$ over the Stone space of it's Boolean skeleton (Theorem 1.3). In fact, free algebras are weak Boolean products of directly indecomposable algebras. In order to obtain his characterization the steps were to consider the sets of regular and dense elements ($Reg(\mathcal{L}) = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k}'' = \mathfrak{k}\}, D(\mathcal{L}) = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k}' = 0\}$) and characterize the Boolean skeleton of residuated lattices. Based on the importance of regular, dense and Boolean elements in constructing free algebras, we realize that it will be interesting to investigate different notions of regular substructures of hoop ($B(\mathcal{L})$ in Kowalski and Ono (2002), $Reg(\mathcal{L})$ in Cignoli (2008), $R(\mathcal{L})$ and $MV(\mathcal{L})$ in D. Busneag et al.(2013)).

Hoops are introduced by Bosbach in [7, 8]. In recent years, many mathematicians have studied this algebraic structure from different perspectives such as ideals, filters, relationships with other algebraic structures, etc., and good results have been achieved in this regard which can be found in [3, 4, 5, 6, 20, 21, 22, 25]. Given the importance of the above notions, such as Boolean element and regular, and the results obtained in this field on other algebraic structures, we decided to examine this concept on hoop.

Now, we study hoops in order to give new characterizations for regular and Boolean elements in hoops. Also, we introduce regular filters of hoop and $\veebar$-hoop with RF-property and study some properties of them.

2 Preliminaries

Next, we will mention some of the features that we use in this article.

Definition 2.1. [2] An algebraic structure $(\mathcal{L}, \rtimes, \looparrowright, 1)$ is a ***hoop*** if for every $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$ we have:
$(\mathcal{HP}1)$ $(\mathcal{L}, \rtimes, 1)$ is a commutative monoid,
$(\mathcal{HP}2)$ $\mathfrak{k} \looparrowright \mathfrak{k} = 1$,
$(\mathcal{HP}3)$ $(\mathfrak{k} \rtimes \mathfrak{s}) \looparrowright \mathfrak{p} = \mathfrak{k} \looparrowright (\mathfrak{s} \looparrowright \mathfrak{p})$,
$(\mathcal{HP}4)$ $\mathfrak{k} \rtimes (\mathfrak{k} \looparrowright \mathfrak{s}) = \mathfrak{s} \rtimes (\mathfrak{s} \looparrowright \mathfrak{k})$.

From now on, $\mathcal{L}$ is a hoop.

We define a binary order $\leqq$ on $\mathcal{L}$ where $\mathfrak{k} \leqq \mathfrak{s}$ iff $\mathfrak{k} \looparrowright \mathfrak{s} = 1$. Easily $\langle \mathcal{L}, \leqq \rangle$ is a Poset. $\mathcal{L}$ is ***bounded*** if it has a least element $0 \in \mathcal{L}$ where $0 \leqq \mathfrak{k}$, for each $\mathfrak{k} \in \mathcal{L}$. Consider $\mathfrak{k}^0 = 1$, $\mathfrak{k}^n = \mathfrak{k}^{n-1} \rtimes \mathfrak{k}$, for every $n \in \mathbb{N}$. Assume $\mathcal{L}$ is bounded. Define an operation " $'$ " on $\mathcal{L}$ as, $\mathfrak{k}' = \mathfrak{k} \looparrowright 0$, for each $\mathfrak{k} \in \mathcal{L}$. If $(\mathfrak{k}')' = \mathfrak{k}$, for all $\mathfrak{k} \in \mathcal{L}$, then $\mathcal{L}$ has ***(DNP) property***.

Proposition 2.2. [7, 8] *For all* $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$, *we have:*
(i) $(\mathcal{L}, \leqq)$ *is a* $\overline{\overline{\wedge}}$*-semilattice with* $\mathfrak{k} \,\overline{\overline{\wedge}}\, \mathfrak{s} = \mathfrak{k} \rtimes (\mathfrak{k} \looparrowright \mathfrak{s})$,
(ii) $\mathfrak{k} \rtimes \mathfrak{s} \leqq \mathfrak{p}$ *iff* $\mathfrak{k} \leqq \mathfrak{s} \looparrowright \mathfrak{p}$,
(iii) $\mathfrak{k} \rtimes \mathfrak{s} \leqq \mathfrak{k}, \mathfrak{s}$,
(iv) $\mathfrak{k} \leqq \mathfrak{s} \looparrowright \mathfrak{k}$,
(v) $\mathfrak{k} \looparrowright 1 = 1$,
(vi) $1 \looparrowright \mathfrak{k} = \mathfrak{k}$,
(vii) $\mathfrak{k} \rtimes (\mathfrak{k} \looparrowright \mathfrak{s}) \leqq \mathfrak{s}$,
(viii) $\mathfrak{k} \leqq \mathfrak{s}$ *implies* $\mathfrak{k} \rtimes \mathfrak{p} \leqq \mathfrak{s} \rtimes \mathfrak{p}$, $\mathfrak{p} \looparrowright \mathfrak{k} \leqq \mathfrak{p} \looparrowright \mathfrak{s}$ *and* $\mathfrak{s} \looparrowright \mathfrak{p} \leqq \mathfrak{k} \looparrowright \mathfrak{p}$,
(ix) $\mathfrak{k} \looparrowright (\mathfrak{s} \,\overline{\overline{\wedge}}\, \mathfrak{p}) = (\mathfrak{k} \looparrowright \mathfrak{s}) \,\overline{\overline{\wedge}}\, (\mathfrak{k} \looparrowright \mathfrak{p})$,
(x) $((\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{s}) \looparrowright \mathfrak{s} = \mathfrak{k} \looparrowright \mathfrak{s}$,
(xi) $(\mathfrak{k} \looparrowright \mathfrak{s}) \rtimes (\mathfrak{s} \looparrowright \mathfrak{p}) \leqq \mathfrak{k} \looparrowright \mathfrak{p}$.

Proposition 2.3. [12] *Suppose* $\mathcal{L}$ *is bounded. Then for all* $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$, *we get:*
(i) $\mathfrak{k} \leqq \mathfrak{k}''$,
(ii) $\mathfrak{k} \rtimes \mathfrak{k}' = 0$ *and* $\mathfrak{k}''' = \mathfrak{k}'$,
(iii) $\mathfrak{k} \leqq \mathfrak{k}' \looparrowright \mathfrak{s}$,
(iv) *if* $\mathfrak{k} = \mathfrak{k}''$, *then* $\mathfrak{k} \looparrowright \mathfrak{s} = \mathfrak{s}' \looparrowright \mathfrak{k}'$.

Proposition 2.4. [12] *For every* $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$ *define,* $\mathfrak{k} \veebar \mathfrak{s} = ((\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{s}) \,\overline{\overline{\wedge}}\, ((\mathfrak{s} \looparrowright \mathfrak{k}) \looparrowright \mathfrak{k})$. *Then the next conditions are equivalent:*
(i) $\veebar$ *is an associative operation on* $\mathcal{L}$,
(ii) $\mathfrak{k} \leqq \mathfrak{s}$ *implies* $\mathfrak{k} \veebar \mathfrak{p} \leqq \mathfrak{s} \veebar \mathfrak{p}$,
(iii) $\mathfrak{k} \veebar (\mathfrak{s} \,\overline{\overline{\wedge}}\, \mathfrak{p}) \leqq (\mathfrak{k} \veebar \mathfrak{s}) \,\overline{\overline{\wedge}}\, (\mathfrak{k} \veebar \mathfrak{p})$,
(iv) $\veebar$ *is the join operation on* $\mathcal{L}$.

Definition 2.5. $\mathcal{L}$ is a $\veebar$***-hoop***, if $\veebar$ satisfies in one of the conditions in Proposition 2.4.

Remark 2.6. [12, Remark 2.4] $\veebar$-hoop $(\mathcal{L}, \veebar, \doublebarwedge)$ is a distributive lattice.

Proposition 2.7. [12] *Assume $\mathcal{L}$ is a $\veebar$-hoop. Then, for all $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$, we have:*
(i) $(\mathfrak{k} \veebar \mathfrak{s}) \looparrowright \mathfrak{p} = (\mathfrak{k} \looparrowright \mathfrak{p}) \doublebarwedge (\mathfrak{s} \looparrowright \mathfrak{p})$,
(ii) $\mathfrak{k} \rtimes (\mathfrak{s} \veebar \mathfrak{p}) = (\mathfrak{k} \rtimes \mathfrak{s}) \veebar (\mathfrak{k} \rtimes \mathfrak{p})$.

Definition 2.8. [12] Consider $\emptyset \neq \mathcal{G} \subseteq \mathcal{L}$. Then $\mathcal{G}$ is said to be a ***filter*** of $\mathcal{L}$ if, for any $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$,
$(F1)$ $\mathfrak{k}, \mathfrak{s} \in \mathcal{G}$ implies $\mathfrak{k} \rtimes \mathfrak{s} \in \mathcal{G}$,
$(F2)$ $\mathfrak{k} \leqq \mathfrak{s}$ and $\mathfrak{k} \in \mathcal{G}$ imply $\mathfrak{s} \in \mathcal{G}$.
The set $\mathcal{F}(\mathcal{L})$ contains all filters of $\mathcal{L}$. Clearly, $1 \in \mathcal{G}$, for each $\mathcal{G} \in \mathcal{F}(\mathcal{L})$. $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ is ***proper*** if $\mathcal{G} \neq \mathcal{L}$. Obviously, $0 \notin \mathcal{G}$ if $\mathcal{L}$ is bounded.

Assume $\emptyset \neq \mathcal{U} \subseteq \mathcal{L}$. The symbol $\langle \mathcal{U} \rangle$ is the filter ***generated*** by $\mathcal{U}$ which is the smallest filter of $\mathcal{L}$ that containing $\mathcal{U}$. If $\mathcal{U} = \{\mathfrak{k}\}$, then $\langle \mathfrak{k} \rangle$ where $\langle \mathfrak{k} \rangle = \{\mathfrak{f} \in \mathcal{L} \mid \mathfrak{k}^n \leqq \mathfrak{f},\ \text{for some } n \in \mathbb{N}\}$. Furthermore, if $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ and $\mathfrak{k} \in \mathcal{L}$, then

$$\langle \mathcal{G} \cup \{\mathfrak{k}\} \rangle = \{\mathfrak{f} \in \mathcal{L} \mid \mathfrak{s} \rtimes \mathfrak{k}^n \leqq \mathfrak{f},\ \text{for some } n \in \mathbb{N} \text{ and } \mathfrak{s} \in \mathcal{G}\}.$$

Moreover, a proper filter $\mathcal{G}$ of $\mathcal{L}$ is ***prime***, if for any $\mathcal{W}, \mathcal{V} \in \mathcal{F}(\mathcal{L})$ such that $\mathcal{W} \cap \mathcal{V} \subseteq \mathcal{G}$, then $\mathcal{W} \subseteq \mathcal{G}$ or $\mathcal{V} \subseteq \mathcal{G}$, or equivalently if $\mathcal{L}$ is a $\veebar$-hoop, then $\mathfrak{k} \veebar \mathfrak{s} \in \mathcal{G}$, for some $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$, then $\mathfrak{k} \in \mathcal{G}$ or $\mathfrak{s} \in \mathcal{G}$. Also, a proper filter $\mathcal{G}$ of $\mathcal{L}$ is ***maximal*** if it is not properly contained in the any other proper filters of $\mathcal{L}$. Moreover, all prime filters of $\mathcal{L}$ and maximal filters of $\mathcal{L}$ is shown by $Spec(\mathcal{L})$ and $Max(\mathcal{L})$, respectively. $\mathcal{L}$ is called ***local*** iff it has just a unique maximal filter such as $\mathcal{Q}$, and easily prove $\mathcal{Q} = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k}^n \neq 0,\ \text{for any } n \in \mathbb{N}\}$. (See [4]).

Definition 2.9. [1] Suppose $\mathcal{L}$ is a $\veebar$-hoop and $\emptyset \neq \mathcal{U} \subseteq \mathcal{L}$. The set $\mathcal{U}^{\ddagger} = \{\mathfrak{f} \in \mathcal{L} \mid \mathfrak{f} \veebar \mathfrak{k} = 1,\ \text{for all } \mathfrak{k} \in \mathcal{U}\}$ is said to be a ***co-annihilator*** of X.

Proposition 2.10. [1] *Consider $\emptyset \neq \mathcal{U} \subseteq \mathcal{L}$. Then $\mathcal{U}^{\ddagger} \in \mathcal{F}(\mathcal{L})$ and $\mathcal{U} \cap \mathcal{U}^{\ddagger} = \{1\}$.*

Proposition 2.11. [4] *Assume $\mathcal{L}$ is bounded, $\mathcal{Q} \in Max(\mathcal{L})$ iff $\mathfrak{k} \notin \mathcal{Q}$ implies $n \in \mathbb{N}$ where $(\mathfrak{k}^n)' \in \mathcal{Q}$.*

Theorem 2.12. [4] *Suppose $\mathcal{L}$ is a $\veebar$-hoop. Then every maximal filter of $\mathcal{L}$ is prime.*

Remark 2.13. [4] Clearly by using Zorns Lemma, we get that for each proper filter $\mathcal{G}$, there exists $\mathcal{V} \in Spec(\mathcal{L})$ such that $\mathcal{G} \subseteq \mathcal{V}$.

Theorem 2.14. [1] *Consider $\mathcal{L}$ is a bounded $\veebar$-hoop. Then $(\mathcal{F}(\mathcal{L}), \subseteq, \doublebarwedge, \veebar, \{1\}, \mathcal{L})$ is a bounded distributive lattice, where $\mathcal{G} \doublebarwedge \mathcal{V} = \mathcal{G} \cap \mathcal{V}$ and $\mathcal{G} \veebar \mathcal{V} = \langle \mathcal{G} \cup \mathcal{V} \rangle$, for any $\mathcal{G}, \mathcal{V} \in \mathcal{F}(\mathcal{L})$.*

Theorem 2.15. [1] *Assume $\mathcal{L}$ is a bounded $\veebar$-hoop. Then structure $(\mathcal{F}(\mathcal{L}), \subseteq, \doublebarwedge, \veebar, \{1\}, \mathcal{L})$ is a pseudo-complement and for each $\mathcal{G} \in \mathcal{F}(\mathcal{L})$, it is pseudo-complement of $\mathcal{G}^{\ddagger}$.*

Notation. From now on, $(\mathcal{L}, \rtimes, \looparrowright, 1)$ or $\mathcal{L}$, for short, is a bounded hoop.

3 Regular and Boolean elements in hoops

We define the notions of regular and Boolean elements in hoops and investigate the equivalence definitions and some properties of them.

Note. Define an operator $\boxplus$ by $\mathfrak{k} \boxplus \mathfrak{s} = \mathfrak{k}' \looparrowright \mathfrak{s}$, for every $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$.

Proposition 3.1. *For any $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$, we have:*
(i) $\mathfrak{k} \boxplus 0 = \mathfrak{k}''$, $\mathfrak{k} \boxplus 1 = 1$ *and* $\mathfrak{k} \boxplus \mathfrak{k}' = 1$,
(ii) *if $\mathcal{L}$ has (DNP), then* $\mathfrak{k} \boxplus \mathfrak{s} = \mathfrak{s} \boxplus \mathfrak{k}$,
(iii) *if $\mathcal{L}$ has (DNP), then* $(\mathfrak{k} \boxplus \mathfrak{s}) \boxplus \mathfrak{p} = \mathfrak{k} \boxplus (\mathfrak{s} \boxplus \mathfrak{p})$,
(iv) *if* $\mathfrak{k} \leqq \mathfrak{s}$, *then* $\mathfrak{k} \boxplus \mathfrak{p} \leqq \mathfrak{s} \boxplus \mathfrak{p}$.

Proof. The proof is straightforward. □

Proposition 3.2. *Consider $\mathcal{L}$ is a bounded $\veebar$-hoop. Then for any $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$, we get:*
(i) *if* $\mathfrak{k} \veebar \mathfrak{s} = 1$, *then* $\mathfrak{k} \boxplus \mathfrak{s} = 1$,
(ii) $\mathfrak{k} \boxplus (\mathfrak{s} \veebar \mathfrak{p})' = (\mathfrak{k} \boxplus \mathfrak{s}') \doublebarwedge (\mathfrak{k} \boxplus \mathfrak{p}')$,
(iii) $\mathfrak{k} \boxplus (\mathfrak{s} \doublebarwedge \mathfrak{p}) = (\mathfrak{k} \boxplus \mathfrak{s}) \doublebarwedge (\mathfrak{k} \boxplus \mathfrak{p})$,
(iv) *if $\mathcal{L}$ has (DNP), then* $\mathfrak{k} \boxplus (\mathfrak{s}' \veebar \mathfrak{p}')' = (\mathfrak{k} \boxplus \mathfrak{s}) \doublebarwedge (\mathfrak{k} \boxplus \mathfrak{p})$.

Proof. *(i)* By Proposition 2.2(iv), $\mathfrak{s} \leqq \mathfrak{k}' \looparrowright \mathfrak{s}$ and by Proposition 2.3(iii), $\mathfrak{k} \leqq \mathfrak{k}' \looparrowright \mathfrak{s}$. Then

$$1 = \mathfrak{k} \veebar \mathfrak{s} \leqq \mathfrak{k}' \looparrowright \mathfrak{s} = \mathfrak{k} \boxplus \mathfrak{s}.$$

Hence, $\mathfrak{k} \boxplus \mathfrak{s} = 1$.
(ii) Let $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$. Then by definition of $\boxplus$ and Proposition 2.7(i), we have

$$\begin{aligned} \mathfrak{k} \boxplus (\mathfrak{s} \veebar \mathfrak{p})' &= \mathfrak{k}' \looparrowright (\mathfrak{s} \veebar \mathfrak{p})' = (\mathfrak{s} \veebar \mathfrak{p}) \looparrowright \mathfrak{k}'' = (\mathfrak{s} \looparrowright \mathfrak{k}'') \doublebarwedge (\mathfrak{p} \looparrowright \mathfrak{k}'') \\ &= (\mathfrak{k}' \looparrowright \mathfrak{s}') \doublebarwedge (\mathfrak{k}' \looparrowright \mathfrak{p}') = (\mathfrak{k} \boxplus \mathfrak{s}') \doublebarwedge (\mathfrak{k} \boxplus \mathfrak{p}'). \end{aligned}$$

(iii) Let $\mathfrak{k}, \mathfrak{s}, \mathfrak{p} \in \mathcal{L}$. Then by definition of $\boxplus$ and Proposition 2.2(ix), we get

$$\mathfrak{k} \boxplus (\mathfrak{s} \doublebarwedge \mathfrak{p}) = \mathfrak{k}' \looparrowright (\mathfrak{s} \doublebarwedge \mathfrak{p}) = (\mathfrak{k}' \looparrowright \mathfrak{s}) \doublebarwedge (\mathfrak{k}' \looparrowright \mathfrak{p}) = (\mathfrak{k} \boxplus \mathfrak{s}) \doublebarwedge (\mathfrak{k} \boxplus \mathfrak{p}).$$

(iv) According to (iii), the proof is straightforward. □

Definition 3.3. Assume $\mathcal{L}$ is a bounded $\veebar$-hoop. Then $\mathfrak{e} \in \mathcal{L}$ is called a **Boolean element** if $\mathfrak{e} \veebar \mathfrak{e}' = 1$. The set $\mathcal{B}(\mathcal{L})$ contains all Boolean elements of $\mathcal{L}$.

Example 3.4. Suppose $\mathcal{L} = \{0, \mathfrak{f}, \mathfrak{v}, \mathfrak{i}, \mathfrak{o}, 1\}$ such that $0 \leqq \mathfrak{f} \leqq \mathfrak{v} \leqq 1$, $0 \leqq \mathfrak{f} \leqq \mathfrak{o} \leqq 1$ and $0 \leqq \mathfrak{i} \leqq \mathfrak{o} \leqq 1$. Define the operations $\looparrowright$ and $\rtimes$ on $\mathcal{L}$ by:

$\looparrowright$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	1	1	1	1	1	1
$\mathfrak{f}$	$\mathfrak{o}$	1	1	$\mathfrak{o}$	1	1
$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1	$\mathfrak{i}$	$\mathfrak{o}$	1
$\mathfrak{i}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	1	1	1
$\mathfrak{o}$	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{o}$	1	1
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

$\rtimes$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	0	0	0	0	0	0
$\mathfrak{f}$	0	0	$\mathfrak{f}$	0	0	$\mathfrak{f}$
$\mathfrak{v}$	0	$\mathfrak{f}$	$\mathfrak{v}$	0	$\mathfrak{f}$	$\mathfrak{v}$
$\mathfrak{i}$	0	0	0	$\mathfrak{i}$	$\mathfrak{i}$	$\mathfrak{i}$
$\mathfrak{o}$	0	0	$\mathfrak{f}$	$\mathfrak{i}$	$\mathfrak{i}$	$\mathfrak{o}$
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

Then $(\mathcal{L}, \looparrowright, \rtimes, 0, 1)$ is a bounded $\veebar$-hoop. Obviously, $\mathcal{B}(\mathcal{L}) = \{0, \mathfrak{v}, \mathfrak{i}, 1\}$.

Proposition 3.5. *If $\mathcal{L}$ is a bounded $\veebar$-hoop and $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, then $\mathfrak{e} = \mathfrak{e}^2$, $\mathfrak{e} = \mathfrak{e}''$ and $\mathfrak{e}' \looparrowright \mathfrak{e} = \mathfrak{e}$.*

Proof. Let $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$. Then by Proposition 2.2(ii), $\mathfrak{e}^2 \leqq \mathfrak{e}$. Since $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, we have $\mathfrak{e} \veebar \mathfrak{e}' = 1$. Then

$$\mathfrak{e} \looparrowright \mathfrak{e}^2 = (1 \rtimes \mathfrak{e}) \looparrowright \mathfrak{e}^2 = ((\mathfrak{e} \veebar \mathfrak{e}') \rtimes \mathfrak{e}) \looparrowright \mathfrak{e}^2.$$

By Propositions 2.7(ii) and 2.3(ii),

$$((\mathfrak{e} \veebar \mathfrak{e}') \rtimes \mathfrak{e}) \looparrowright \mathfrak{e}^2 = ((\mathfrak{e} \rtimes \mathfrak{e}) \veebar (\mathfrak{e}' \rtimes \mathfrak{e})) \looparrowright \mathfrak{e}^2 = \mathfrak{e}^2 \looparrowright \mathfrak{e}^2 = 1.$$

Hence, $\mathfrak{e} \leqq \mathfrak{e}^2$, and so $\mathfrak{e} = \mathfrak{e}^2$. Now, we prove that $\mathfrak{e} = \mathfrak{e}''$. For this, by Proposition 2.3(i), $\mathfrak{e} \leqq \mathfrak{e}''$. Since $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, we have $\mathfrak{e} \veebar \mathfrak{e}' = 1$, then by Propositions 2.7(ii) and 2.3(ii),

$$\mathfrak{e}'' \looparrowright \mathfrak{e} = ((\mathfrak{e} \veebar \mathfrak{e}') \rtimes \mathfrak{e}'') \looparrowright \mathfrak{e} = ((\mathfrak{e} \rtimes \mathfrak{e}'') \veebar (\mathfrak{e}' \rtimes \mathfrak{e}'')) \looparrowright \mathfrak{e} = (\mathfrak{e} \rtimes \mathfrak{e}'') \looparrowright \mathfrak{e}.$$

Thus, by Proposition 2.2(iii), $(\mathfrak{e} \rtimes \mathfrak{e}'') \looparrowright \mathfrak{e} = 1$. Hence, $\mathfrak{e}'' \looparrowright \mathfrak{e} = 1$, and so $\mathfrak{e} = \mathfrak{e}''$. Finally, for proving $\mathfrak{e}' \looparrowright \mathfrak{e} = \mathfrak{e}$, by Proposition 2.2(iv), we have $\mathfrak{e} \leqq \mathfrak{e}' \looparrowright \mathfrak{e}$. Sufficiently, we have to show $\mathfrak{e}' \looparrowright \mathfrak{e} \leqq \mathfrak{e}$. For this, since $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, we get $\mathfrak{e}' \in \mathcal{B}(\mathcal{L})$, moreover, $\mathfrak{e} = \mathfrak{e}''$ and $\mathfrak{e}^2 = \mathfrak{e}$, so we consequence that,

$$(\mathfrak{e}' \looparrowright \mathfrak{e}) \looparrowright \mathfrak{e} = (\mathfrak{e}' \looparrowright \mathfrak{e}'') \looparrowright \mathfrak{e}'' = (\mathfrak{e}' \rtimes \mathfrak{e}')' \looparrowright \mathfrak{e}'' = \mathfrak{e}'' \looparrowright \mathfrak{e}'' = 1.$$

Thus, $\mathfrak{e}' \looparrowright \mathfrak{e} = \mathfrak{e}$. $\square$

Proposition 3.6. *If $\mathcal{L}$ is a bounded $\veebar$-hoop and for each $\mathfrak{k},\mathfrak{s}\in\mathcal{L}$, $\mathfrak{k}\veebar\mathfrak{s}=1$, then $\mathfrak{k},\mathfrak{s}\in\mathcal{B}(\mathcal{L})$.*

Proof. By Proposition 2.2(vii) and (iv),

$$\mathfrak{k}\leqq(\mathfrak{k}\looparrowright\mathfrak{s})\looparrowright\mathfrak{s}\ \text{ and }\ \mathfrak{s}\leqq(\mathfrak{k}\looparrowright\mathfrak{s})\looparrowright\mathfrak{s}.$$

Then $1=\mathfrak{k}\veebar\mathfrak{s}\leqq(\mathfrak{k}\looparrowright\mathfrak{s})\looparrowright\mathfrak{s}$. Thus, $(\mathfrak{k}\looparrowright\mathfrak{s})\looparrowright\mathfrak{s}=1$, and so $\mathfrak{k}\looparrowright\mathfrak{s}\leqq\mathfrak{s}$. Also, by Proposition 2.2(iv), $\mathfrak{s}\leqq\mathfrak{k}\looparrowright\mathfrak{s}$. Hence, $\mathfrak{k}\looparrowright\mathfrak{s}=\mathfrak{s}$. By the similar way, $\mathfrak{s}\looparrowright\mathfrak{k}=\mathfrak{k}$. According to definition of $\veebar$, we get

$$\mathfrak{k}\veebar\mathfrak{k}'=((\mathfrak{k}\looparrowright\mathfrak{k}')\looparrowright\mathfrak{k}')\doublebarwedge((\mathfrak{k}'\looparrowright\mathfrak{k})\looparrowright\mathfrak{k})=(\mathfrak{k}'\looparrowright\mathfrak{k}')\doublebarwedge(\mathfrak{k}\looparrowright\mathfrak{k})=1.$$

Similarly, $\mathfrak{s}\veebar\mathfrak{s}'=1$. Therefore, $\mathfrak{k},\mathfrak{s}\in\mathcal{B}(\mathcal{L})$. □

Proposition 3.7. *Suppose $\mathcal{L}$ is a bounded $\veebar$-hoop. Then $\mathfrak{k}\in\mathcal{B}(\mathcal{L})$ iff $\mathfrak{k}\boxplus\mathfrak{k}=\mathfrak{k}$, $\mathfrak{k}''=\mathfrak{k}$, $\mathfrak{k}^2=\mathfrak{k}$ and $(\mathfrak{k}\looparrowright\mathfrak{k}')'\veebar(\mathfrak{k}'\looparrowright\mathfrak{k})'=1$.*

Proof. ($\Rightarrow$) Let $\mathfrak{k}\in\mathcal{B}(\mathcal{L})$. Then by Proposition 3.5, $\mathfrak{k}''=\mathfrak{k}$, $\mathfrak{k}^2=\mathfrak{k}$ and since $\mathfrak{k}'\looparrowright\mathfrak{k}=\mathfrak{k}$, we get that $\mathfrak{k}\boxplus\mathfrak{k}=\mathfrak{k}$. Also, from $\mathfrak{k}\in\mathcal{B}(\mathcal{L})$,

$$(\mathfrak{k}\looparrowright\mathfrak{k}')'\veebar(\mathfrak{k}'\looparrowright\mathfrak{k})'=(\mathfrak{k}\rtimes\mathfrak{k})''\veebar(\mathfrak{k}\boxplus\mathfrak{k})'=\mathfrak{k}''\veebar\mathfrak{k}'=\mathfrak{k}\veebar\mathfrak{k}'=1.$$

($\Leftarrow$) Let $\mathfrak{k}\in\mathcal{L}$. Then by assumptions, we obtain that

$$1=(\mathfrak{k}\looparrowright\mathfrak{k}')'\veebar(\mathfrak{k}'\looparrowright\mathfrak{k})'=(\mathfrak{k}\rtimes\mathfrak{k})''\veebar(\mathfrak{k}\boxplus\mathfrak{k})'=\mathfrak{k}''\veebar\mathfrak{k}'=\mathfrak{k}\veebar\mathfrak{k}'.$$

Hence, $\mathfrak{k}\in\mathcal{B}(A)$. □

Corollary 3.8. *If $\mathcal{L}$ is a bounded $\veebar$-hoop and local, then $\mathcal{B}(\mathcal{L})=\{0,1\}$.*

Proof. Let $\mathfrak{k}\in\mathcal{B}(\mathcal{L})-\{0,1\}$. Since $\mathfrak{k}\veebar\mathfrak{k}'=1$, we have $\mathfrak{k}'\in\mathcal{B}(\mathcal{L})-\{0,1\}$. Moreover, $\mathcal{L}$ is a local hoop, then it has just one maximal filter such as $\mathcal{Q}$, then $\langle\mathfrak{k}\rangle\subseteq\mathcal{Q}$ and $\langle\mathfrak{k}'\rangle\subseteq\mathcal{Q}$. Hence, $\mathfrak{k},\mathfrak{k}'\in\mathcal{Q}$, and so $0\in\mathcal{Q}$, which is a contradiction. Therefore, $\mathcal{B}(\mathcal{L})=\{0,1\}$. □

Next example shows that the converse of Corollary 3.8 does not hold.

Example 3.9. Assume $A=\{0,\mathfrak{f},\mathfrak{v},\mathfrak{i},1\}$ where $0\leqq\mathfrak{f}\leqq\mathfrak{i}\leqq1$ and $0\leqq\mathfrak{v}\leqq\mathfrak{i}\leqq1$. Define the operations $\looparrowright$ and $\rtimes$ on $\mathcal{L}$ by:

$\looparrowright$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	1
0	1	1	1	1	1
$\mathfrak{f}$	$\mathfrak{v}$	1	$\mathfrak{v}$	1	1
$\mathfrak{v}$	$\mathfrak{f}$	$\mathfrak{f}$	1	1	1
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{v}$	1	1
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	1

$\rtimes$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	1
0	0	0	0	0	0
$\mathfrak{f}$	0	$\mathfrak{f}$	0	$\mathfrak{f}$	$\mathfrak{f}$
$\mathfrak{v}$	0	0	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{i}$
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	1

Then $(\mathcal{L}, \looparrowright, \rtimes, 0, 1)$ is a bounded $\veebar$-hoop. Obviously $\mathcal{B}(\mathcal{L}) = \{0, 1\}$. But $\mathcal{G}_1 = \{\mathfrak{f}, \mathfrak{i}, 1\}$ and $\mathcal{G}_2 = \{\mathfrak{v}, \mathfrak{i}, 1\}$ are maximal.

Definition 3.10. The element $\mathfrak{k} \in \mathcal{L}$ is said to be a **regular element** if for each $\mathfrak{s} \in \mathcal{L}$, $(\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k} = \mathfrak{k}$. The set $\mathcal{R}(\mathcal{L})$ contains all regular elements of $\mathcal{L}$.

Example 3.11. Let $\mathcal{L}$ be a hoop as in Example 3.4. Then $\mathcal{R}(\mathcal{L}) = \{0, \mathfrak{v}, \mathfrak{i}, 1\}$.

Theorem 3.12. *For any $\mathfrak{k}, \mathfrak{s} \in \mathcal{L}$, the next statements are equivalents:*
(i) $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$,
(ii) $\mathfrak{k}' \looparrowright \mathfrak{k} = \mathfrak{k}$,
(iii) $\mathfrak{k} = \mathfrak{k}''$ *and* $\mathfrak{k} \doublebarwedge \mathfrak{k}' = 0$.

Proof. $(i) \Rightarrow (ii)$ Consider $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Then, for any $\mathfrak{s} \in \mathcal{L}$, $(\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k} = \mathfrak{k}$. It is enough to choose $\mathfrak{s} = 0$. Thus, $\mathfrak{k}' \looparrowright \mathfrak{k} = \mathfrak{k}$.
$(ii) \Rightarrow (i)$ By Proposition 2.2(iv), $\mathfrak{k} \leqq (\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k}$. Also, by Proposition 2.3(iii), $\mathfrak{k}' \leqq \mathfrak{k} \looparrowright \mathfrak{s}$, then by Proposition 2.2(viii), $(\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k} \leqq \mathfrak{k}' \looparrowright \mathfrak{k}$. Thus, by (ii), $(\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k} \leqq \mathfrak{k}$. Hence, for any $\mathfrak{s} \in \mathcal{L}$, $(\mathfrak{k} \looparrowright \mathfrak{s}) \looparrowright \mathfrak{k} = \mathfrak{k}$. Therefore, $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$.
$(ii) \Rightarrow (iii)$ Suppose $\mathfrak{k} \in \mathcal{L}$. Then by Proposition 2.3(i), $\mathfrak{k} \leqq \mathfrak{k}''$. Also, by (ii),

$$\mathfrak{k}'' \looparrowright \mathfrak{k} = \mathfrak{k}'' \looparrowright (\mathfrak{k}' \looparrowright \mathfrak{k}) = (\mathfrak{k}'' \rtimes \mathfrak{k}') \looparrowright \mathfrak{k}.$$

By Proposition 2.3(ii), $(\mathfrak{k}'' \rtimes \mathfrak{k}') \looparrowright \mathfrak{k} = 0 \looparrowright \mathfrak{k} = 1$. Then $\mathfrak{k}'' \looparrowright \mathfrak{k} = 1$, and so $\mathfrak{k}'' = \mathfrak{k}$. Moreover,

$$\mathfrak{k} \doublebarwedge \mathfrak{k}' = \mathfrak{k}' \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k}) = \mathfrak{k}' \rtimes \mathfrak{k} = 0.$$

$(iii) \Rightarrow (ii)$ Assume $\mathfrak{k} \in \mathcal{L}$. Then by Proposition 2.3(iii), $\mathfrak{k} \leqq \mathfrak{k}' \looparrowright \mathfrak{k}$. Thus, by (iii),

$$(\mathfrak{k}' \looparrowright \mathfrak{k}) \looparrowright \mathfrak{k} = (\mathfrak{k}' \looparrowright \mathfrak{k}) \looparrowright \mathfrak{k}'' = (\mathfrak{k}' \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k})) \looparrowright 0 = (\mathfrak{k}' \doublebarwedge \mathfrak{k}) \looparrowright 0 = 1.$$

Hence, $\mathfrak{k}' \looparrowright \mathfrak{k} = \mathfrak{k}$. □

All elements of $\mathfrak{k} \in \mathcal{L}$ that $\mathfrak{k}'' = \mathfrak{k}$ is called an **MV-center** of $\mathcal{L}$ and showed by $\mathcal{MV}(\mathcal{L})$.

Corollary 3.13. *Let $\mathcal{L}$ be a bounded $\veebar$-hoop. Then:*
(i) $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$ *iff* $\mathfrak{k} \boxplus \mathfrak{k} = \mathfrak{k}$.
(ii) *if* $\mathfrak{k}, \mathfrak{s} \in \mathcal{R}(\mathcal{L})$, *then* $\mathfrak{k} \boxplus \mathfrak{s} \in \mathcal{MV}(\mathcal{L})$.
(iii) $\mathcal{B}(\mathcal{L}) \subseteq \mathcal{R}(\mathcal{L})$.
(iv) $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L})$.
(v) $\mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$ *iff* $\mathfrak{k} \doublebarwedge \mathfrak{k}' = 0$, *for each* $\mathfrak{k} \in \mathcal{L}$.

Proof. (*i*) Consider $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Then by Theorem 3.12, $\mathfrak{k} \boxplus \mathfrak{k} = \mathfrak{k}' \looparrowright \mathfrak{k} = \mathfrak{k}$. Proof of converse is straightforward.
(*ii*) Assume $\mathfrak{k}, \mathfrak{s} \in \mathcal{R}(\mathcal{L})$. Then by Theorem 3.12, $\mathfrak{s}'' = \mathfrak{s}$. Thus,

$$(\mathfrak{k} \boxplus \mathfrak{s})'' = (\mathfrak{k}' \looparrowright \mathfrak{s})'' = (\mathfrak{k}' \looparrowright \mathfrak{s}'')''.$$

By (HP3),

$$(\mathfrak{k}' \looparrowright \mathfrak{s}'')'' = ((\mathfrak{k}' \rtimes \mathfrak{s}') \looparrowright 0)'' = (\mathfrak{k}' \rtimes \mathfrak{s}')'''.$$

By Proposition 2.3(ii),

$$(\mathfrak{k}' \rtimes \mathfrak{s}')''' = (\mathfrak{k}' \rtimes \mathfrak{s}')' = \mathfrak{k}' \looparrowright \mathfrak{s}'' = \mathfrak{k}' \looparrowright \mathfrak{s} = \mathfrak{k} \boxplus \mathfrak{s}.$$

Hence, $(\mathfrak{k} \boxplus \mathfrak{s})'' = \mathfrak{k} \boxplus \mathfrak{s}$. Therefore, $\mathfrak{k} \boxplus \mathfrak{s} \in \mathcal{MV}(\mathcal{L})$.
(*iii*) Suppose $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$. Then by Proposition 3.5, $\mathfrak{k}' \looparrowright \mathfrak{k} = \mathfrak{k}$. Thus, by Theorem 3.12, $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Hence, $\mathcal{B}(\mathcal{L}) \subseteq \mathcal{R}(\mathcal{L})$.
(*iv*) Let $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Then by Theorem 3.12, $\mathfrak{k} = \mathfrak{k}''$, so $\mathfrak{k} \in \mathcal{MV}(\mathcal{L})$. Hence, $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L})$.
(*v*) By (iv), $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L})$. If $\mathfrak{k} \in \mathcal{MV}(\mathcal{L})$ and $\mathfrak{k} \doublebarwedge \mathfrak{k}' = 0$, then by Theorem 3.12, $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Hence, $\mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$. The proof of other side is clear. $\square$

Converse of Corollary 3.13(iii) and (iv) do not hold.

Example 3.14. (*i*) Assume $\mathcal{L} = \{0, \mathfrak{f}, \mathfrak{v}, \mathfrak{i}, \mathfrak{o}, 1\}$ is a set. Define the operations $\rtimes$ and $\looparrowright$ on $\mathcal{L}$ as follows:

$\looparrowright$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	1	1	1	1	1	1
$\mathfrak{f}$	$\mathfrak{f}$	1	1	1	1	1
$\mathfrak{v}$	0	$\mathfrak{f}$	1	1	1	1
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{o}$	1	$\mathfrak{o}$	1
$\mathfrak{o}$	0	$\mathfrak{f}$	$\mathfrak{i}$	$\mathfrak{i}$	1	1
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

$\rtimes$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	0	0	0	0	0	0
$\mathfrak{f}$	0	0	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$
$\mathfrak{v}$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{v}$	$\mathfrak{i}$
$\mathfrak{o}$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{o}$	$\mathfrak{o}$
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

Clearly $(\mathcal{L}, \rtimes, \looparrowright, 0, 1)$ is a bounded hoop, $\mathcal{MV}(\mathcal{L}) = \{0, \mathfrak{f}, 1\}$ and $\mathcal{R}(\mathcal{L}) = \{0, 1\}$. Hence, $\mathcal{MV}(\mathcal{L}) \nsubseteq \mathcal{R}(\mathcal{L})$.
(*ii*) According to Example 3.9, obviously $\mathcal{R}(\mathcal{L}) = \{0, \mathfrak{f}, \mathfrak{v}, 1\}$ and $\mathcal{B}(\mathcal{L}) = \{0, 1\}$, because $\mathfrak{f} \veebar \mathfrak{f}' = \mathfrak{v} \veebar \mathfrak{v}' = \mathfrak{i} \neq 1$. Hence, $\mathcal{R}(\mathcal{L}) \nsubseteq \mathcal{B}(\mathcal{L})$.

Proposition 3.15. *Let $\mathfrak{k} \in \mathcal{L}$. Then:*
(*i*) *if $\mathfrak{k}^2 = \mathfrak{k}$, then $\mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$.*
(*ii*) *if $\mathcal{L}$ is a bounded $\veebar$-hoop and for any $\mathfrak{k} \in \mathcal{L}$,*

$$\mathfrak{k}^2 = \mathfrak{k} \quad \textit{and} \quad (\mathfrak{k} \looparrowright \mathfrak{k}')' \veebar (\mathfrak{k}' \looparrowright \mathfrak{k})' = 1,$$

then $\mathcal{R}(\mathcal{L}) = \mathcal{B}(\mathcal{L})$.

Proof. (i) By Corollary 3.13(iv), we prove that $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L})$. Now, let $\mathfrak{k} \in \mathcal{MV}(\mathcal{L})$. Then $\mathfrak{k} = \mathfrak{k}''$. By assumption and Proposition 2.2(vii), we have

$$\mathfrak{k} \doublebarwedge \mathfrak{k}' = \mathfrak{k}' \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k}) = (\mathfrak{k}')^2 \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k}) = \mathfrak{k}' \rtimes \mathfrak{k}' \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k}) \leqq \mathfrak{k}' \rtimes \mathfrak{k} = 0.$$

Thus, $\mathfrak{k} \doublebarwedge \mathfrak{k}' = \mathfrak{k}' \rtimes (\mathfrak{k}' \looparrowright \mathfrak{k}) = 0$. Since $\mathfrak{k} = \mathfrak{k}''$, by Theorem 3.12, $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$. Therefore, $\mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$.
(ii) By Proposition 3.7, Theorem 3.12 and Corollary 3.13, proof is complete. □

Notation. In any bounded hoop $\mathcal{L}$, we consider the subset

$$\mathcal{Q}_1(\mathcal{L}) = \{\mathfrak{p} \in \mathcal{L} \setminus \{0,1\} \mid \mathfrak{p} = \mathfrak{k} \boxplus \mathfrak{s},\ \text{for some } \mathfrak{k}, \mathfrak{s} \in \mathcal{L} \setminus \{0,1\}\},$$

and we denote by $\mathcal{M}(\mathcal{L}) = M_1(\mathcal{L}) \cup \{0,1\}$. According to Corollary 3.13(i), clearly $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{M}(\mathcal{L})$.

Example 3.16. According to Example 3.4, we obtain $\mathcal{M}(\mathcal{L}) = \{0, \mathfrak{v}, \mathfrak{i}, \mathfrak{o}, 1\}$, where $\mathfrak{v} = \mathfrak{f} \boxplus \mathfrak{v}$, $\mathfrak{o} = \mathfrak{f} \boxplus \mathfrak{i}$, $\mathfrak{i} = \mathfrak{i} \boxplus \mathfrak{i}$.

Next example shows that $\mathcal{MV}(\mathcal{L}) \nsubseteq \mathcal{M}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L}) \nsubseteq \mathcal{MV}(\mathcal{L})$.

Example 3.17. (i) By Example 3.4, $\mathcal{MV}(\mathcal{L}) = \mathcal{L}$ but $\mathcal{M}(\mathcal{L}) = \{0, \mathfrak{v}, \mathfrak{i}, \mathfrak{o}, 1\}$. It proves $\mathcal{MV}(\mathcal{L}) \nsubseteq \mathcal{M}(\mathcal{L})$.
(ii) Suppose $\mathcal{L} = \{0, \mathfrak{f}, \mathfrak{v}, \mathfrak{i}, \mathfrak{o}, 1\}$ where $0 \leqq \mathfrak{f} \leqq \mathfrak{i} \leqq 1$, $0 \leqq \mathfrak{v} \leqq \mathfrak{o} \leqq 1$ and $0 \leqq \mathfrak{v} \leqq \mathfrak{i} \leqq 1$. Define the operations $\looparrowright$ and $\rtimes$ by

$\looparrowright$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	1	1	1	1	1	1
$\mathfrak{f}$	$\mathfrak{o}$	1	$\mathfrak{o}$	1	$\mathfrak{o}$	1
$\mathfrak{v}$	$\mathfrak{f}$	$\mathfrak{f}$	1	1	1	1
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{o}$	1	$\mathfrak{o}$	1
$\mathfrak{o}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{i}$	$\mathfrak{i}$	1	1
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

$\rtimes$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1
0	0	0	0	0	0	0
$\mathfrak{f}$	0	$\mathfrak{f}$	0	$\mathfrak{f}$	0	$\mathfrak{f}$
$\mathfrak{v}$	0	0	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{v}$
$\mathfrak{i}$	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{v}$	$\mathfrak{i}$
$\mathfrak{o}$	0	0	$\mathfrak{v}$	$\mathfrak{v}$	$\mathfrak{o}$	$\mathfrak{o}$
1	0	$\mathfrak{f}$	$\mathfrak{v}$	$\mathfrak{i}$	$\mathfrak{o}$	1

Then $(\mathcal{L}, \looparrowright, \rtimes, 0, 1)$ is a bounded hoop, $\mathcal{MV}(\mathcal{L}) = \{0, \mathfrak{f}, \mathfrak{o}, 1\}$ and $\mathcal{M}(\mathcal{L}) = \{0, \mathfrak{f}, \mathfrak{i}, \mathfrak{o}, 1\}$, where $\mathfrak{f} = \mathfrak{f} \boxplus \mathfrak{f}$, $\mathfrak{i} = \mathfrak{f} \boxplus \mathfrak{v}$, $\mathfrak{o} = \mathfrak{o} \boxplus \mathfrak{v}$. Clearly, $\mathcal{M}(\mathcal{L}) \nsubseteq \mathcal{MV}(\mathcal{L})$.

Theorem 3.18. *If $\mathcal{L}$ is a bounded $\veebar$-hoop, then the next statements are equivalent:*
(i) $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$,
(ii) $\mathfrak{k} \in \mathcal{MV}(\mathcal{L})$ *and* $\mathfrak{k} \veebar \mathfrak{k}' = 1$,
(iii) $\mathfrak{k} \in \mathcal{R}(\mathcal{L})$ *and* $\mathfrak{k}^2 = \mathfrak{k}$, $(\mathfrak{k} \looparrowright \mathfrak{k}') \veebar (\mathfrak{k}' \looparrowright \mathfrak{k}) = 1$,
(iv) $\mathfrak{k} \in \mathcal{M}(\mathcal{L})$ *and* $\mathfrak{k} \boxplus \mathfrak{k} = \mathfrak{k}$, $\mathfrak{k}^2 = \mathfrak{k}$, $\mathfrak{k}'' = \mathfrak{k}$, $(\mathfrak{k} \looparrowright \mathfrak{k}')' \veebar (\mathfrak{k}' \looparrowright \mathfrak{k})' = 1$.

Proof. According to Propositions 3.15 and 3.7 and Corollary 3.13, the proof is clear. □

Proposition 3.19. *Let $\mathcal{L}$ be a bounded $\veebar$-hoop. Then the following statements are equivalent:*
(i) $\mathfrak{k}' = 0$*, for any* $\mathfrak{k} \in \mathcal{L} \setminus \{0\}$*,*
(ii) $\mathcal{B}(\mathcal{L}) = \mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L}) = \{0,1\}$*.*

Proof. $(i) \Rightarrow (ii)$ By Corollary 3.13, obviously,

$$\{0,1\} \subseteq \mathcal{B}(\mathcal{L}) \subseteq \mathcal{R}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L}).$$

So, we prove $\mathcal{MV}(\mathcal{L}) \subseteq \{0,1\}$. Let $\mathfrak{k} \in \mathcal{L} \setminus \{0,1\}$. Then by (i), $\mathfrak{k}' = 0$, so $\mathfrak{k}'' = 1$. Thus, $\mathfrak{k} \notin \mathcal{MV}(\mathcal{L})$. Hence, $\mathcal{MV}(\mathcal{L}) \subseteq \{0,1\}$, and so

$$\mathcal{B}(\mathcal{L}) = \mathcal{R}(\mathcal{L}) = \mathcal{MV}(\mathcal{L}) = \{0,1\}.$$

$(ii) \Rightarrow (i)$ Let $\mathfrak{k} \in \mathcal{L} \setminus \{0\}$. By Proposition 2.3(ii), $\mathfrak{k}' = (\mathfrak{k}')''$ and so $\mathfrak{k}' \in \mathcal{MV}(\mathcal{L})$. Thus, $\mathfrak{k}' = 0$ or $\mathfrak{k}' = 1$. If $\mathfrak{k}' = 1$, then $\mathfrak{k}'' = 0$, and so $\mathfrak{k} = 0$, a contradiction. Hence, $\mathfrak{k}' = 0$. □

By an example we show that a hoop, satisfing in condition (i) of Proposition 3.19, exists.

Example 3.20. Suppose $\mathcal{L} = \{0, \mathfrak{f}, \mathfrak{v}, 1\}$ is a chain we have $\rtimes$ and $\looparrowright$ on $\mathcal{L}$ as follows,

$\looparrowright$	0	$\mathfrak{f}$	$\mathfrak{v}$	1
0	1	1	1	1
$\mathfrak{f}$	0	1	1	1
$\mathfrak{v}$	0	$\mathfrak{v}$	1	1
1	0	$\mathfrak{f}$	$\mathfrak{v}$	1

$\rtimes$	0	$\mathfrak{f}$	$\mathfrak{v}$	1
0	0	0	0	0
$\mathfrak{f}$	0	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$
$\mathfrak{v}$	0	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{v}$
1	0	$\mathfrak{f}$	$\mathfrak{v}$	1

Then $(\mathcal{L}, \rtimes, \looparrowright, 0, 1)$ is a bounded $\veebar$-hoop and easily, for any $\mathfrak{k} \in \mathcal{L} - \{0\}$, $\mathfrak{k}' = 0$.

Definition 3.21. [11] A *Stone algebra* is an algebraic structure $S = (S, \veebar, \bar{\bar{\wedge}}, ', 0, 1)$ such that for any $\mathfrak{k} \in S$, it satisfies in the following conditions:
$(S1)$ $S = (S, \veebar, \bar{\bar{\wedge}}, 0, 1)$ is a bounded distributive lattice.
$(S2)$ $\mathfrak{k} \,\bar{\bar{\wedge}}\, \mathfrak{k}' = 0$.
$(S3)$ $\mathfrak{k}'' \veebar \mathfrak{k}' = 1$.

Theorem 3.22. *Let $\mathcal{L}$ be a bounded $\veebar$-hoop. Then $\mathcal{L}$ is a Stone algebra iff* $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$.

Proof. ($\Rightarrow$) Since $\mathcal{L}$ is a bounded $\veebar$-hoop, by Remark 2.6, $\mathcal{L}$ is a bounded distributive lattice. If $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L}) = \{0, 1\}$, then by Proposition 3.19, for any $\mathfrak{k} \in \mathcal{L} \setminus \{0\}$, $\mathfrak{k}' = 0$. Thus, $\mathfrak{k} \mathbin{\overline{\overline{\wedge}}} \mathfrak{k}' = 0$, for any $\mathfrak{k} \in \mathcal{L}$. Now, assume $\mathfrak{k} \in \mathcal{L}$ where $\mathfrak{k} = 0$, then $\mathfrak{k}' = 1$, and so $\mathfrak{k}' \veebar \mathfrak{k}'' = 1$. If $\mathfrak{k} \neq 0$, then by Proposition 3.19, $\mathfrak{k}' = 0$, thus, $\mathfrak{k}'' = 1$, and so $\mathfrak{k}' \veebar \mathfrak{k}'' = 1$. Hence, for any $\mathfrak{k} \in \mathcal{L}$, $\mathfrak{k}'' \veebar \mathfrak{k}' = 1$. If $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L}) \neq \{0, 1\}$, then for an element $\mathfrak{k} \in \mathcal{L} \setminus \{0, 1\}$, there are two possibilities: $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$ or $\mathfrak{k} \notin \mathcal{B}(\mathcal{L})$. If $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$, then $\mathfrak{k} \mathbin{\overline{\overline{\wedge}}} \mathfrak{k}' = 0$ and since $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$, we obtain $\mathfrak{k}'' \veebar \mathfrak{k}' = \mathfrak{k}' \veebar \mathfrak{k} = 1$. If $\mathfrak{k} \notin \mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$, then $\mathfrak{k}'' \neq \mathfrak{k}$. By Proposition 2.3(ii), $\mathfrak{k}' = \mathfrak{k}''' = (\mathfrak{k}')''$. Thus, $\mathfrak{k}' \in \mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$. So, $\mathfrak{k}' \veebar \mathfrak{k}'' = 1$. Moreover, by Propositions 2.7(i) and 2.3(i), we have $\mathfrak{k} \mathbin{\overline{\overline{\wedge}}} \mathfrak{k}' = 0$. Hence, for any $\mathfrak{k} \in \mathcal{L}$, $\mathfrak{k}' \mathbin{\overline{\overline{\wedge}}} \mathfrak{k} = 0$ and $\mathfrak{k}' \veebar \mathfrak{k}'' = 1$. Therefore, $\mathcal{L}$ is a Stone algebra.

($\Leftarrow$) Suppose $\mathcal{L}$ is a Stone algebra. By Corollary 3.13(iii) and (iv), $\mathcal{B}(\mathcal{L}) \subseteq \mathcal{MV}(\mathcal{L})$. Let $\mathfrak{k} \in \mathcal{MV}(\mathcal{L})$. Then $\mathfrak{k} = \mathfrak{k}''$. Since $\mathfrak{k}' \veebar \mathfrak{k}'' = 1$, we consequence that $\mathfrak{k}' \veebar \mathfrak{k} = 1$. Thus, $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$. Hence, $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$. □

Theorem 3.23. *Consider $\mathcal{L}$ is a bounded $\veebar$-hoop where for any $\mathfrak{k} \in \mathcal{L}$, $\mathfrak{k}^2 = \mathfrak{k}$. Then $\mathcal{L}$ is a Boolean algebra iff $\mathcal{L}$ is a Stone algebra with (DNP).*

Proof. If $\mathcal{L}$ is a Boolean algebra, then by Proposition 3.15, $\mathcal{B}(\mathcal{L}) = \mathcal{MV}(\mathcal{L})$, and so by Theorem 3.22, $\mathcal{L}$ is a Stone algebra with (DNP). Conversely, since $\mathcal{L}$ is a Stone algebra with (DNP), then, for any $\mathfrak{k} \in \mathcal{L}$, $\mathfrak{k}' \veebar \mathfrak{k}'' = \mathfrak{k}' \veebar \mathfrak{k} = 1$ and $\mathfrak{k}' \mathbin{\overline{\overline{\wedge}}} \mathfrak{k} = 0$. Thus, $\mathfrak{k} \in \mathcal{B}(\mathcal{L})$. Therefore, $\mathcal{L}$ is a Boolean algebra. □

4 Regular filters in hoops

Now, we define regular filters of hoop and $\veebar$-hoop with RF-property and study some properties of them. Also, we show that every $\veebar$-hoop with RF-property, is a Boolean algebra and $\mathcal{L}$ with RF-property such that $\mathcal{B}(\mathcal{L}) = \{0, 1\}$, is a local hoop.

Notation. From now, $\mathcal{L}$ is a bounded $\veebar$-hoop.

Definition 4.1. A filter $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ is **regular** if $\mathcal{G}^{\ddagger\ddagger} = \mathcal{G}$. The set of all regular filters is denoted by $R^{\ddagger}(\mathcal{F}(A))$.

Example 4.2. By Example 3.4, $\mathcal{G} = \{1, \mathfrak{o}\}$ is a regular filter of $\mathcal{L}$ and $R^{\ddagger}(\mathcal{F}(\mathcal{L})) = \{\{1\}, \{1, \mathfrak{o}\}\}$.

Definition 4.3. $\mathcal{L}$ has **regular filter property**, **RF-property** for short, if every proper filter $\mathcal{G} \in \mathcal{F}(\mathcal{L})$, is regular.

Example 4.4. According to Example 3.14, $\mathcal{L}$ has RF-property.

Theorem 4.5. *Consider $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ is proper. Then $\mathcal{G}$ is regular iff $\mathcal{G} \veebar \mathcal{G}^{\ddagger} = \mathcal{L}$.*

Proof. Assume $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ is regular. By Proposition 2.10, $\mathcal{G} \cap \mathcal{G}^{\ddagger} = \{1\}$. Thus, by Theorems 2.14 and 2.15, we consequence that

$$\mathcal{L} = \{1\}^{\ddagger} = (\mathcal{G} \mathbin{\bar{\bar{\wedge}}} \mathcal{G}^{\ddagger})^{\ddagger} = (\mathcal{G} \cap \mathcal{G}^{\ddagger})^{\ddagger} = \mathcal{G}^{\ddagger} \cup \mathcal{G}^{\ddagger\ddagger} = \mathcal{G}^{\ddagger} \veebar \mathcal{G}^{\ddagger\ddagger} = \mathcal{G}^{\ddagger} \veebar \mathcal{G}.$$

Hence, $\mathcal{G} \veebar \mathcal{G}^{\ddagger} = \mathcal{L}$.

Conversely, let $\mathcal{G} \veebar \mathcal{G}^{\ddagger} = \mathcal{L}$. Then by Theorem 2.14,

$$\{1\} = \mathcal{L}^{\ddagger} = \mathcal{G}^{\ddagger\ddagger} \mathbin{\bar{\bar{\wedge}}} \mathcal{G}^{\ddagger} = \mathcal{G}^{\ddagger\ddagger} \cap \mathcal{G}^{\ddagger}.$$

So $\{1\} = \mathcal{G}^{\ddagger\ddagger} \mathbin{\bar{\bar{\wedge}}} \mathcal{G}^{\ddagger}$. Moreover, by Theorem 2.15, $\mathcal{G}$ is pseudocomplement of $\mathcal{G}$ and $\mathcal{G} \cap \mathcal{G}^{\ddagger} = \{1\}$. Then $\mathcal{G} = \mathcal{G}^{\ddagger\ddagger}$. Hence, $\mathcal{G}$ is regular. □

Theorem 4.6. *If $\mathcal{L}$ has RF-property, then the set $(R^{\ddagger}(\mathcal{F}(\mathcal{L})), \subseteq, \bar{\bar{\wedge}}, \veebar, \{1\}, \mathcal{L})$ is a Boolean algebra, where $\mathcal{G} \mathbin{\bar{\bar{\wedge}}} \mathcal{V} = \mathcal{G} \cap \mathcal{V}$, $\mathcal{G} \veebar \mathcal{V} = \langle \mathcal{G} \cup \mathcal{V} \rangle$ and $\mathcal{G}^{\ddagger}$ is pseudo-complement of $\mathcal{G}$, for any $\mathcal{G}, \mathcal{V} \in \mathcal{F}(\mathcal{L})$.*

Proof. Using Theorems 2.14, 2.15 and 4.5. □

Proposition 4.7. *The next assertions are equivalent:*
(i) $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$,
(ii) $\langle \mathfrak{e} \rangle^{\ddagger} = \langle \mathfrak{e}' \rangle$,
(iii) $\langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle$.

Proof. $(i) \Rightarrow (ii)$ Let $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$. Since $\mathfrak{e} \veebar \mathfrak{e}' = 1$, and by Definition 2.9, $\langle \mathfrak{e} \rangle^{\ddagger} = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{e} \veebar \mathfrak{k} = 1\}$, we consequence that $\mathfrak{e}' \in \langle \mathfrak{e} \rangle^{\ddagger}$, and so $\langle \mathfrak{e}' \rangle \subseteq \langle \mathfrak{e} \rangle^{\ddagger}$. If $\mathfrak{k} \in \langle \mathfrak{e} \rangle^{\ddagger}$, since $\mathfrak{e} \veebar \mathfrak{k} = 1$ and $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, by Remark 2.6, and by Proposition 2.7(i) we have

$$\mathfrak{e}' = \mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} 1 = \mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} (\mathfrak{e} \veebar \mathfrak{k}) = (\mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}) \veebar (\mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} \mathfrak{k}) = \mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} \mathfrak{k}.$$

Then $\mathfrak{e}' \leqq \mathfrak{k}$, and so $\mathfrak{k} \in \langle \mathfrak{e}' \rangle$. Hence, $\langle \mathfrak{e} \rangle^{\ddagger} \subseteq \langle \mathfrak{e}' \rangle$. Therefore, $\langle \mathfrak{e} \rangle^{\ddagger} = \langle \mathfrak{e}' \rangle$.

$(ii) \Rightarrow (i)$ Since $\langle \mathfrak{e} \rangle^{\ddagger} = \langle \mathfrak{e}' \rangle$, we have $\mathfrak{e}' \in \langle \mathfrak{e} \rangle^{\ddagger}$. Then, $\mathfrak{e}' \veebar \mathfrak{e} = 1$. Also, by Proposition 2.7(i), $0 = (\mathfrak{e}' \veebar \mathfrak{e})' = \mathfrak{e}'' \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}'$. Then by Proposition 2.3(i), $\mathfrak{e} \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}' \leqq \mathfrak{e}' \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}'' = 0$, thus, $\mathfrak{e} \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}' = 0$. Hence, $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$.

$(i) \Rightarrow (iii)$ Let $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$. Then by Proposition 3.5, $\mathfrak{e} = \mathfrak{e}''$, and so $\langle \mathfrak{e} \rangle = \langle \mathfrak{e}'' \rangle$. Also, by (ii), $\langle \mathfrak{e} \rangle^{\ddagger} = \langle \mathfrak{e}' \rangle$, and so $\langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e}' \rangle^{\ddagger} = \langle \mathfrak{e}'' \rangle = \langle \mathfrak{e} \rangle$. Hence, $\langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle$.

$(iii) \Rightarrow (i)$ By Definition 2.9,

$$\langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k} \veebar \mathfrak{s} = 1, \text{ for any } \mathfrak{s} \in \langle \mathfrak{e} \rangle^{\ddagger}\}.$$

Then by (ii),

$$\langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k} \veebar \mathfrak{s} = 1, \text{ for any } \mathfrak{s} \in \langle \mathfrak{e}' \rangle\} = \{\mathfrak{k} \in \mathcal{L} \mid \mathfrak{k} \veebar \mathfrak{s} = 1, \text{ for any } \mathfrak{s} \geqq \mathfrak{e}'\}.$$

Since $\mathfrak{e} \in \langle \mathfrak{e} \rangle = \langle \mathfrak{e} \rangle^{\ddagger\ddagger}$, we consequence that $\mathfrak{e} \veebar \mathfrak{e}' = 1$. Thus, by Remark 2.6, Propositions 2.7(i) and 2.3(i), we have $\mathfrak{e} \mathbin{\bar{\bar{\wedge}}} \mathfrak{e}' = 0$. Hence, $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$. □

Corollary 4.8. *If* $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$*, then* $\langle \mathfrak{e} \rangle \in R^{\ddagger}(\mathcal{F}(\mathcal{L}))$*.*

Proposition 4.9. *Consider* $\mathcal{G} \in \mathcal{F}(\mathcal{L})$*. Then* $\mathcal{G} \in R^{\ddagger}(\mathcal{F}(\mathcal{L}))$ *iff there exists* $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ *such that* $\mathcal{G} = \langle \mathfrak{e} \rangle$*.*

Proof. ($\Rightarrow$) Assume $\mathcal{G} \in R^{\ddagger}(\mathcal{F}(\mathcal{L}))$, $\mathfrak{f} \in \mathcal{G}^{\ddagger}$ and $\mathfrak{e} \in \mathcal{G}$. Then $\mathfrak{f} \veebar \mathfrak{e} = 1$. By Proposition 2.4, $1 = \mathfrak{f} \veebar \mathfrak{e} \leqq (\mathfrak{f} \looparrowright \mathfrak{e}) \looparrowright \mathfrak{e}$. Thus, by Proposition 2.2(iv), $\mathfrak{e} \leqq \mathfrak{f} \looparrowright \mathfrak{e} \leqq \mathfrak{e}$, and so $\mathfrak{f} \looparrowright \mathfrak{e} = \mathfrak{e}$. Hence, by Theorem 4.6,

$$\mathfrak{f} \doublebarwedge \mathfrak{e} = \mathfrak{f} \rtimes (\mathfrak{f} \looparrowright \mathfrak{e}) = \mathfrak{f} \rtimes \mathfrak{e} = 0.$$

Let $\mathfrak{k} \in \mathcal{G}$. Then by Remark 2.6,

$$\mathfrak{e} \doublebarwedge \mathfrak{k} = 0 \veebar (\mathfrak{e} \doublebarwedge \mathfrak{k}) = (\mathfrak{f} \doublebarwedge \mathfrak{e}) \veebar (\mathfrak{k} \doublebarwedge \mathfrak{e}) = (\mathfrak{f} \veebar \mathfrak{k}) \doublebarwedge \mathfrak{e} = \mathfrak{e}.$$

Thus, $\mathfrak{e} \leqq \mathfrak{k}$, and so $\mathcal{G} = \langle \mathfrak{e} \rangle$.

($\Leftarrow$) Let $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$. By Proposition 4.7, $\mathcal{G}^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle = \mathcal{G}$. Hence, $\mathcal{G} \in R^{\ddagger}(\mathcal{F}(\mathcal{L}))$. □

Corollary 4.10. $\mathcal{L}$ *has RF-property iff, for any proper filter* $\mathcal{G} \in \mathcal{F}(\mathcal{L})$*, there is* $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ *wheret* $\mathcal{G} = \langle \mathfrak{e} \rangle$*.*

Proposition 4.11. *Any prime filter of* $\mathcal{L}$ *with RF-property is included in a unique maximal filter.*

Proof. Assume $\mathcal{P} \in Spec(\mathcal{L})$. By Zorn's Lemma, obviously there is $\mathcal{Q} \in Max(\mathcal{L})$ where $\mathcal{P} \subseteq \mathcal{Q}$. Now, suppose $\mathcal{Q}_1 \in Max(\mathcal{L})$ and $\mathcal{Q}_2 \in Max(\mathcal{L})$ such that $\mathcal{P} \subseteq \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{Q}_1 \neq \mathcal{Q}_2$. Since $\mathcal{Q}_1 \neq \mathcal{Q}_2$, there exists $\mathfrak{f} \in \mathcal{Q}_1$ such that $\mathfrak{f} \notin \mathcal{Q}_2$. By Proposition 2.11, there exists $n \in \mathbb{N}$ such that $(\mathfrak{f}^n)' \in \mathcal{Q}_2$. If $(\mathfrak{f}^n)' \in \mathcal{Q}_1$, then by Proposition 2.3(ii), $0 = \mathfrak{f}^n \rtimes (\mathfrak{f}^n)' \in \mathcal{Q}_1$, a contradiction. So, $(\mathfrak{f}^n)' \notin \mathcal{Q}_1$. Since $\mathfrak{f} \notin \mathcal{Q}_2$ and $(\mathfrak{f}^n)' \notin \mathcal{Q}_1$, we consequence $\mathfrak{f}, (\mathfrak{f}^n)' \notin \mathcal{P}$. Moreover, from $\mathcal{Q}_1 \in \mathcal{F}(\mathcal{L})$ and $\mathcal{L}$ has RF-property, then by Corollary 4.10, there is $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ such that $\mathcal{Q}_1 = \langle \mathfrak{e} \rangle$. Since $\mathfrak{e} \rtimes \mathfrak{e}' = 0$, obviously, $\mathfrak{e}' \notin \mathcal{Q}_1$. Also, if $\mathfrak{f} \in \mathcal{Q}_1 = \langle \mathfrak{e} \rangle$, then there exists $n \in \mathbb{N}$ such that $\mathfrak{e}^n \leqq \mathfrak{f}$, also, from $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$, $\mathfrak{e} \veebar \mathfrak{e}' = 1 \in \mathcal{P}$. From $\mathcal{P} \in Spec(\mathcal{L})$, we obtain $\mathfrak{e} \in \mathcal{P}$ or $\mathfrak{e}' \in \mathcal{P}$. If $\mathfrak{e} \in \mathcal{P}$, since $\mathfrak{e}^n \leqq \mathfrak{f}$, then $\mathfrak{f} \in \mathcal{P}$, a contradiction. If $\mathfrak{e}' \in \mathcal{P}$, since $\mathcal{P} \subseteq \mathcal{Q}_1 \cap \mathcal{Q}_2$, then $\mathfrak{e}' \in \mathcal{Q}_1$, a contradiction. Hence, $\mathfrak{e}, \mathfrak{e}' \notin \mathcal{P}$, a contradiction. □

Proposition 4.12. *If* $\mathcal{L}$ *has RF-property and* $\mathcal{B}(\mathcal{L}) = \{0, 1\}$*, then* $\mathcal{L}$ *is local.*

Proof. Suppose $\mathcal{L}$ has RF-property and $\mathcal{Q} \in Max(\mathcal{L})$. Then by Corollary 4.10, there exists $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ such that $\mathcal{Q} = \langle \mathfrak{e} \rangle$. Thus, $\mathcal{Q} = \langle 0 \rangle = \mathcal{L}$ or $\mathcal{Q} = \langle 1 \rangle = \{1\}$. So, $\mathcal{L}$ has just one maximal filter. Hence, $\mathcal{L}$ is local. □

Definition 4.13. $\mathfrak{f} \in \mathcal{L}$ is an **archimedean element** if there exists $n \in \mathbb{N}$ such that $\mathfrak{f}^n \in \mathcal{B}(\mathcal{L})$. $\mathcal{L}$ is an **archimedean hoop**, if all its elements are archimedean.

Example 4.14. According to Example 3.4, $\mathcal{B}(\mathcal{L}) = \{0, \mathfrak{v}, \mathfrak{i}, 1\}$. From $\mathfrak{f}^2 = 0$ and $\mathfrak{o}^2 = \mathfrak{i}$ and $\mathfrak{f}^2, \mathfrak{o}^2 \in \mathcal{B}(\mathcal{L})$, we consequence that every elements of $\mathcal{L}$ is archimedean, and so $\mathcal{L}$ is an archimedean hoop.

Proposition 4.15. *$\mathcal{L}$ is an archimedean hoop iff, for any $\mathfrak{k} \in \mathcal{L}$, there is $n \in \mathbb{N}$ where $\mathfrak{k} \veebar (\mathfrak{k}^n)' = 1$.*

Proof. ($\Rightarrow$) Since $\mathcal{L}$ is an archimedean hoop, for any $\mathfrak{k} \in \mathcal{L}$, there exists $n \in \mathbb{N}$ such that $\mathfrak{k}^n \in \mathcal{B}(\mathcal{L})$. Then $\mathfrak{k}^n \veebar (\mathfrak{k}^n)' = 1$. Since $\mathfrak{k}^n \leqq \mathfrak{k}$, we have

$$1 = \mathfrak{k}^n \veebar (\mathfrak{k}^n)' \leqq \mathfrak{k} \veebar (\mathfrak{k}^n)'$$

Thus, $\mathfrak{k} \veebar (\mathfrak{k}^n)' = 1$.
($\Leftarrow$) Consider $\mathfrak{k} \in \mathcal{L}$ and for some $n \in \mathbb{N}$, $\mathfrak{k} \veebar (\mathfrak{k}^n)' = 1$. In order to prove that $\mathfrak{k}^n \in \mathcal{B}(\mathcal{L})$, we have to prove that $\mathfrak{k}^n \veebar (\mathfrak{k}^n)' = 1$ and $\mathfrak{k}^n \mathbin{\bar{\bar{\wedge}}} (\mathfrak{k}^n)' = 0$. For this, by Proposition 2.7(ii), we have

$$\begin{aligned}
1 &= 1 \rtimes 1 \\
&= (\mathfrak{k} \veebar (\mathfrak{k}^n)') \rtimes (\mathfrak{k} \veebar (\mathfrak{k}^n)') \\
&= (\mathfrak{k} \rtimes (\mathfrak{k} \veebar (\mathfrak{k}^n)')) \veebar ((\mathfrak{k}^n)' \rtimes (\mathfrak{k} \veebar (\mathfrak{k}^n)')) \\
&= \mathfrak{k}^2 \veebar (\mathfrak{k} \rtimes (\mathfrak{k}^n)') \veebar ((\mathfrak{k}^n)')^2 \leqq \mathfrak{k}^2 \veebar (\mathfrak{k}^n)'
\end{aligned}$$

So, $\mathfrak{k}^2 \veebar (\mathfrak{k}^n)' = 1$. By continuing this method, we consequence that $\mathfrak{k}^n \veebar (\mathfrak{k}^n)' = 1$. Moreover, by Propositions 2.7(i) and 2.3(i), we have $\mathfrak{k}^n \mathbin{\bar{\bar{\wedge}}} (\mathfrak{k}^n)' = 0$. Hence, $\mathfrak{k}^n \in \mathcal{B}(\mathcal{L})$. Therefore, $\mathcal{L}$ is an archimedean hoop. ☐

Theorem 4.16. *Let $\mathcal{L}$ has RF-property, where every filter has just one generator. Then $Spec(\mathcal{L}) = Max(\mathcal{L})$.*

Proof. By Theorem 2.12, clearly $Max(\mathcal{L}) \subseteq Spec(\mathcal{L})$. We have to show $Spec(\mathcal{L}) \subseteq Max(\mathcal{L})$. For this, let $\mathcal{Q} \in Spec(\mathcal{L})$ and $\mathfrak{k} \notin \mathcal{Q}$. If $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ is proper such that $\mathfrak{k} \in \mathcal{G}$, since $\mathcal{L}$ has RF-property, by Corollary 4.10, there exist $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ such that $\langle \mathfrak{k} \rangle = \langle \mathfrak{e} \rangle = \mathcal{G}$. Thus, there is $n \in \mathbb{N}$ such that $\mathfrak{k}^n = \mathfrak{e}$. Since $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$,

$$1 = \mathfrak{e} \veebar \mathfrak{e}' \leqq \mathfrak{k}^n \veebar (\mathfrak{k}^n)'$$

So, $\mathfrak{k}^n \veebar (\mathfrak{k}^n)' = 1$. Moreover, $\mathcal{Q} \in Spec(\mathcal{L})$, $\mathfrak{k} \notin \mathcal{Q}$ and $\mathfrak{k}^n \veebar (\mathfrak{k}^n)' = 1 \in \mathcal{Q}$, then $(\mathfrak{k}^n)' \in \mathcal{Q}$. Thus, by Proposition 2.11, $\mathcal{Q} \in Max(\mathcal{L})$. If there is not any $\mathcal{G} \in \mathcal{F}(\mathcal{L})$ such that $\mathfrak{k} \in \mathcal{G}$, then let $\langle \mathfrak{k} \rangle = \langle 0 \rangle = \mathcal{L}$. Thus, there is $n \in \mathbb{N}$ such that $\mathfrak{k}^n = 0$, and so $(\mathfrak{k}^n)' = 1 \in \mathcal{Q}$. So, for any $\mathfrak{k} \in \mathcal{L}$ that $\mathfrak{k} \notin \mathcal{Q}$, there exists $n \in \mathbb{N}$ such that $(\mathfrak{k}^n)' \in \mathcal{Q}$. Hence, by Proposition 2.11, $\mathcal{Q} \in Max(\mathcal{L})$. Therefore, $Spec(\mathcal{L}) = Max(\mathcal{L})$. ☐

Theorem 4.17. *$\mathcal{L}$ is archimedean iff $Spec(\mathcal{L}) = Max(\mathcal{L})$.*

Proof. ($\Rightarrow$) By Theorem 2.12, $Max(\mathcal{L}) \subseteq Spec(\mathcal{L})$. So, we show $Spec(\mathcal{L}) \subseteq Max(\mathcal{L})$. For this, let $\mathcal{P} \in Spec(\mathcal{L})$ and $\mathfrak{k} \notin \mathcal{P}$. Since $\mathcal{L}$ is archimedean, we have $n \in \mathbb{N}$ such that $\mathfrak{k}^n \in \mathcal{B}(\mathcal{L})$. By Proposition 4.15, $\mathfrak{k} \veebar (\mathfrak{k}^n)' = 1$. Moreover, $\mathcal{P} \in Spec(\mathcal{L})$ and $1 \in \mathcal{P}$, we obtain $(\mathfrak{k}^n)' \in \mathcal{P}$. By Proposition 2.11, $\mathcal{P} \in Max(\mathcal{L})$. Hence, $Spec(\mathcal{L}) = Max(\mathcal{L})$.
($\Leftarrow$) First of all, since $Spec(\mathcal{L}) = Max(\mathcal{L})$, we prove $\mathcal{P} \in Spec(\mathcal{L})$ is a minimal prime. For this, assume $\mathcal{P}, \mathcal{N} \in Spec(\mathcal{L})$ such that $\mathcal{P} \subseteq \mathcal{N}$. Since $\mathcal{P} \in Max(\mathcal{L})$, we have $\mathcal{P} = \mathcal{N}$. Thus, $\mathcal{P}$ is minimal prime. Consider $\mathfrak{f} \in \mathcal{L} - \{1\}$. We prove $\mathfrak{f}$ is an archimedean element. We denote $\mathcal{G} = \langle \mathfrak{f} \rangle^{\ddagger}$, by Proposition 2.10, $\mathcal{G} \in \mathcal{F}(\mathcal{L})$. Since $\mathfrak{f} \neq 1$ and $\mathfrak{f} \veebar \mathfrak{f} \neq 1$, we have $\mathfrak{f} \notin \mathcal{G}$. Now, suppose $\mathcal{G}^* = \langle \mathcal{G} \cup \{\mathfrak{f}\} \rangle$. If we suppose $\mathcal{G}^*$ is a proper filter of $\mathcal{L}$, then by Remark 2.12, there $\mathcal{P} \in Spec(\mathcal{L})$ such that $\mathcal{G}^* \subseteq \mathcal{P}$. Also, we show that since $Spec(\mathcal{L}) = Max(\mathcal{L})$, $\mathcal{P}$ is a minimal prime. Suppose if there is $\mathfrak{k} \in \mathcal{L} - \mathcal{P}$ such that $\mathfrak{f} \veebar \mathfrak{k} = 1$, then $\mathfrak{k} \in \mathcal{G} \subseteq \mathcal{G}^* \subseteq \mathcal{P}$, and so $\mathfrak{k} \in \mathcal{P}$, a contradiction. So, $\mathcal{G}^* \notin Spec(\mathcal{L})$. So, $\mathcal{G}^* = \mathcal{L}$. Thus, $0 \in \mathcal{G}^*$. Then there is $n \in \mathbb{N}$ and $\mathfrak{o} \in \mathcal{G}$ such that $\mathfrak{o} \rtimes \mathfrak{f}^n = 0$, and so $\mathfrak{o} \leqq (\mathfrak{f}^n)'$. Hence, $1 = \mathfrak{f} \veebar \mathfrak{o} \leqq \mathfrak{f} \veebar (\mathfrak{f}^n)'$. So, $\mathfrak{f} \veebar (\mathfrak{f}^n)' = 1$. Therefore, by Proposition 4.15, $\mathcal{L}$ is archimedean. $\square$

Theorem 4.18. *If $\mathcal{L}$ is archimedean, then $\mathcal{L}$ has RF-property.*

Proof. For $\mathfrak{k} \in \mathcal{L}$, there exists $n \in \mathbb{N}$ such that $\mathfrak{k}^n \in \mathcal{B}(\mathcal{L})$. Thus, there is $\mathfrak{e} \in \mathcal{B}(\mathcal{L})$ such that $\mathfrak{k}^n = \mathfrak{e}$. Hence, $\langle \mathfrak{k} \rangle = \langle \mathfrak{k}^n \rangle = \langle \mathfrak{e} \rangle$, and so $\langle \mathfrak{k} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle^{\ddagger\ddagger}$. By Proposition 4.7, $\langle \mathfrak{k} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle^{\ddagger\ddagger} = \langle \mathfrak{e} \rangle = \langle \mathfrak{k} \rangle$. So, for any $\mathfrak{k} \in \mathcal{L}$, $\langle \mathfrak{k} \rangle^{\ddagger\ddagger} = \langle \mathfrak{k} \rangle$. Therefore, $\mathcal{L}$ has RF-property. $\square$

Corollary 4.19. *The following statements are equivalent:*
(i) *$\mathcal{L}$ has RF-property,*
(ii) *$Spec(\mathcal{L}) = Max(\mathcal{L})$,*
(iii) *$\mathcal{L}$ is archimedean.*

5 Conclusions and future works

We study hoops in order to give some new characterizations for regular and Boolean elements in hoops and we investigate the relation between them. Specially, we show that any bounded $\veebar$-hoop is a Stone algebra iff MV-center set and Boolean elements set are equal. Then we introduce the concept of regular filter in hoops and $\veebar$-hoops with RF-property and investigate some properties of them. Moreover, we prove that every $\veebar$-hoop with RF-property, is a Boolean algebra and any hoop $\mathcal{L}$ with RF-property such that $\mathcal{B}(\mathcal{L}) = \{0, 1\}$, is a local hoop. Finally, we prove that any hoop $\mathcal{L}$ has RF-property iff $Spec(\mathcal{L}) = Max(\mathcal{L})$ and iff $\mathcal{L}$ is a hyperarchimedean.

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