DOI: 10.2478/auom-2023-0026
An. Şt. Univ. Ovidius Constanţa **Vol. 31(2),2023, 177–189**

Computing the total H-irregularity strength of edge comb product of graphs

Mohamad Fahruli Wahyujati and Yeni Susanti

Abstract

A simple undirected graph $\Gamma = (V_\Gamma, E_\Gamma)$ admits an H-covering if every edge in E_Γ belongs to at least one subgraph of Γ that is isomorphic to a graph H. For any graph Γ admitting H-covering, a total labelling $\beta : V_\Gamma \cup E_\Gamma \longrightarrow \{1, 2, \ldots, p\}$ is called an H-irregular total p-labelling of Γ if every two different subgraphs H_1 and H_2 of Γ isomorphic to H have distinct weights where the weight $w_\beta(K)$ of subgraph K of Γ is defined as $w_f(K) := \sum_{v \in V_K} f(v) + \sum_{e \in E_K} f(e)$. The smallest number p for which a graph Γ admits an H-irregular total p-labelling is called the total H-irregularity strength of Γ and is denoted by $ths(\Gamma)$. In this paper, we determine the total H-irregularity strength of edge comb product of two graphs.

1 Introduction

Each graph throughout this paper is simple, finite and undirected. Notation $\Gamma = (V_\Gamma, E_\Gamma)$ denotes a graph Γ with vertex set and edge set V_Γ and E_Γ, respectively. A *labelling* means any mapping that maps a set of graphs elements (vertices or edges, or both) to a set of numbers, called *labels*. If the domain of the mapping is the set of vertices (or edges) then the labelling is called a *vertex labelling* (or an *edge labelling*). If the domain is both vertices and edges then the labelling is called a *total labelling*.

Key Words: Total H-Irregular Strength, Edge Comb Product Graph, Cycle, Path.
2010 Mathematics Subject Classification: Primary 05C78, 05C76; Secondary 05C38.
Received: 24.08.2022
Accepted: 28.12.2022

According to Chartrand G. et al. [1] for any positive integer p, an *edge irregular p-labelling* of a graph $\Gamma = (V_\Gamma, E_\Gamma)$ is a map $\beta : E_\Gamma \longrightarrow \{1, 2, \ldots, p\}$ such that $w_\beta(x) \neq w_\beta(y)$ for every two distinct vertices $x, y \in V_\Gamma$ where $w_\beta(x)$ is defined as $w_\beta(x) = \sum_{xy \in E_\Gamma} \beta(xy)$. In [2] Bača M. et al. introduced a total edge irregularity strength of graphs. For a graph Γ they defined a labelling $\gamma : V_\Gamma \cup E_\Gamma \longrightarrow \{1, 2, \ldots, p\}$ to be an *edge irregular total p-labelling* of Γ if for every two different edges xy and $x'y'$ in E_Γ it follows that $w_\gamma(xy) \neq w_\gamma(x'y')$, where $w_\gamma(xy) = \gamma(x) + \gamma(xy) + f(y)$. The minimum p for which the graph Γ has an edge irregular total p-labelling is called *the total edge irregularity strength* of Γ, denoted by $tes(\Gamma)$.

A family of subgraphs $H_1, H_2, \ldots, H_t$ of Γ such that each of edge in E_Γ belongs to at least one of the subgraph H_i, $i \in \{1, 2, \ldots, t\}$, is called an edge-covering of Γ and it is said that Γ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If each subgraph H_i is isomorphic to a given graph H, then Γ admits an H-covering.

In 2017, Ashraf F. et al. [3] developed the irregularity strength concept for graph admitting H-covering. Let Γ be a graph admitting H-covering and α be a total p-labelling on Γ. The weight of subgraph K of Γ respect to α, denoted by $wt_\alpha(K)$ is defined as

$$wt_\alpha(K) = \sum_{v \in V_K} \alpha(v) + \sum_{e \in E_K} \alpha(e).$$

Any weight of subgraph K that isomorphic to H is called *H-weight*. A total p-labelling α is called an *H-irregular total p-labelling* of Γ if each two distinct subgraphs K_1 and K_2 isomorphic to H have distinct weights, that is $wt_\alpha(K_1) \neq wt_\alpha(K_2)$. The minimum positive integer p such that Γ has an H-irregular total p-labelling is called *total H-irregularity strength* of Γ and denoted by $ths(\Gamma, H)$. Furthermore, the authors of [3] also found the lower bound of $ths(\Gamma, H)$ for any graph Γ admitting H-covering, as follows:

$$ths(\Gamma, H) \geq \left\lceil 1 + \frac{t-1}{V_H + E_H} \right\rceil$$

where t is the number of subgraphs of Γ that are isomorphic to H.

Ashraf F. et al. [3] found the total P_m-irregularity strength of P_n with m, n both integers, $2 \leq m \leq n$, the total C_m-irregularity strength of ladder graph L_n with $n \geq 3$ and $m = 4$ or $m = 6$, and the total C_3-irregularity strength of fan graph F_n. Ashraf F. et al. [4] determined the exact value of total H-irregularity strengths of $\mathcal{G}$-amalgamation of graphs. Agustin I.H. et al. [5] found the total H-irregularity strength of shackle and vertex amalgamation of some graphs, namely $ths(Shack(H, v, n))$, $ths(c(Shack(H, v, n)))$,

$ths(Amal(H, v, n))$ and $ths(c(Amal(H, v, n)))$. In [6], Nisviasari R. et al. have given the total H-irregularity strengths of some triangular ladder and grid graphs. Labane H. et al. in [7] found the exact value of total H-irregularity strength of cycles graphs whenever H is a path graph.

In this paper, we investigate the lower bound of the total H-irregularity strength of edge comb product of two graphs and the exact values of the total H-irregularity strength of edge comb product of two classes of graph, including edge comb product of two cycles, edge comb product of path and any graph H, and edge comb product of star and any graph H.

2 Results

Let Γ be two connected graphs. Let e be an edge of H. The edge comb product between Γ and H, denoted by $\Gamma \trianglerighteq H$, is a graph obtained by taking one copy of Γ and $|E_\Gamma|$ copies of H and then identifying the i-th copy of H at the edge e to the i-th edge of Γ.

By the definition of edge comb product above, it is clear that the edge comb product $H \trianglerighteq H$ between H and its self admits an H-covering. The following theorem provides a sharp lower bound for the total H-irregularity strength of edge comb product of two connected graphs.

Theorem 2.1. *Let Γ and H be two connected graphs. Then*
(i)

$$ths(\Gamma \trianglerighteq H, H) \geq \left\lceil 1 + \frac{|E_\Gamma| - 1}{|V_H| + |E_H|} \right\rceil$$

if Γ does not admit H-covering.
(ii)

$$ths(\Gamma \trianglerighteq H, H) \geq \left\lceil 1 + \frac{t + |E_\Gamma| - 1}{|V_H| + |E_H|} \right\rceil$$

if Γ admits H-covering given by t subgraphs that are isomorphic to H.

Proof.
(i.) Let Γ be a graph that does not admit H-covering. It is obvious that $\Gamma \trianglerighteq H$ admits H-covering given by $|E_\Gamma|$ subgraphs isomorphic to H. Assume that α is an H-irregular total p-labelling of $\Gamma \trianglerighteq H$ with $ths(\Gamma \trianglerighteq H, H) = p$.

The minimum possible weights for all subgraphs of $\Gamma \trianglerighteq H$ that are isomorphic to H are $|V_H| + |E_H|$, $|V_H| + |E_H| + 1$, $|V_H| + |E_H| + 2, \ldots$, $|V_H| + |E_H| + |E_\Gamma| - 1$. On the other hand, the maximum possible weight for any subgraph of $\Gamma \trianglerighteq H$ isomorphic to H is at most $p(|V_H| + |E_H|)$. Hence

$$p(|V_H| + |E_H|) \geq |V_H| + |E_H| + |E_\Gamma| - 1$$
$$p \geq \frac{|V_H| + |E_H| + |E_\Gamma| - 1}{|V_H| + |E_H|}.$$

Since $ths(\Gamma \trianglerighteq H, H)$ should be an integer and we need a sharpest lower bound, it implies

$$ths(\Gamma \trianglerighteq H, H) \geq \left\lceil 1 + \frac{|E_\Gamma| - 1}{|V_H| + |E_H|} \right\rceil.$$

(ii.) Let Γ be a graph that admits an H-covering given by t subgraphs isomorphic to H. We have that $\Gamma \trianglerighteq H$ is H-covering given by at least $t + |E_\Gamma|$ subgraphs isomorphic to H. Similar to above, we get

$$ths(\Gamma \trianglerighteq H, H) \geq \left\lceil 1 + \frac{t + |E_\Gamma| - 1}{|V_H| + |E_H|} \right\rceil.$$

□

By considering $\Gamma \cong H$, from Theorem 2.1 (ii) we obtain Corollary 2.2 below.

Corollary 2.2. *Let H be a connected graph. Then*

$$ths(H \trianglerighteq H, H) \geq 2.$$

The following theorem shows the sharpness of bound in Theorem 2.1. We provide Theorem 2.3 for the existence of edge comb product of graphs whose H-irregularity strength satisfies the lower bound in Theorem 2.1.

Theorem 2.3. *Let $m, n \geq 3$ be positive integers. Then:*

$$ths(C_m \trianglerighteq C_n, C_n) = \begin{cases} 2 & \textit{for } m \leq 2n+1, \\ \left\lceil 1 + \frac{m-1}{2n} \right\rceil & \textit{for } m > 2n+1. \end{cases}$$

Proof. Let C_m and C_n be two cycles of m and n vertices, . We define the vertex set and edge set of $C_m \trianglerighteq C_n$ as follows.

$V_{C_m \trianglerighteq C_n} = \{v_i : 1 \leq i \leq m\} \cup \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n-2\}$
and
$E_{C_m \trianglerighteq C_n} = \{v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{v_1 v_m\} \cup \{v_i u_{i,1} : 1 \leq i \leq m\} \cup$

$\{v_{i+1}u_{i,n-2} : 1 \leq i \leq m-1\} \cup \{v_1 u_{m,n-2}\} \cup$
$\{u_{i,j}u_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-3\}$.

For $m \neq n$, $C_m \trianglerighteq C_n$ admits C_n-covering given by m subgraphs of $C_m \trianglerighteq C_n$ isomorphic to C_n. By applying Theorem 2.1 (i), we get

$$ths(C_m \trianglerighteq C_n, C_n) \geq \left\lceil 1 + \frac{m-1}{2n} \right\rceil. \tag{1}$$

Furthermore, if $m \neq n$ and $m \leq 2n+1$, we obtain

$$ths(C_m \trianglerighteq C_n, C_n) \geq 2. \tag{2}$$

For $m = n$, we have that $C_n \trianglerighteq C_n$ admits a C_n-covering given by $n+1$ subgraphs of $C_n \trianglerighteq C_n$ isomorphic to C_n. According to Corrolary 2.2, we have

$$ths(C_n \trianglerighteq C_n, C_n) \geq 2. \tag{3}$$

Therefore, the Equation (1) is the lower bound of the H-irregularity strength of $C_m \trianglerighteq C_n$ for $m > 2n+1$, meanwhile Equation (2) and (3) give the lower bound of the H-irregularity strength of $C_m \trianglerighteq C_n$ for $m \leq 2n+1$.

Now, we consider two cases for m and n.

Case 1. $m \leq 2n+1$

We define a C_n-irregular total 2-labelling $\delta : V(C_m \trianglerighteq C_n) \cup E(C_m \trianglerighteq C_n) \rightarrow \{1, 2\}$ as follows

$$\delta(v_i) = \begin{cases} 1 & \text{for } i = 1, 2 \\ 2 & \text{for } 3 \leq i \leq m \end{cases}$$

$$\delta(u_{i,j}) = \begin{cases} 1 & \text{for } i = 1, 2, 3, m \text{ and} \\ & \quad 1 \leq j \leq n-2 \\ \left\lfloor \frac{i-4}{2n-1} \right\rfloor + 1 & \text{for } 4 \leq i \leq m-1 \text{ and} \\ & \quad j > i-3 \\ \left\lfloor \frac{i-4}{2n-1} \right\rfloor + 2 & \text{for } 4 \leq i \leq m-1 \text{ and} \\ & \quad j \leq i-3 \end{cases}$$

$$\delta(v_i v_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \\ 2 & \text{for } 2 \leq i \leq m-1 \end{cases}$$

$$\delta(v_m v_1) = 1$$

$$\delta(v_i u_{i,1}) = \begin{cases} 1 & \text{for } 1 \leq i \leq n+1 \text{ or } i = m \\ \left\lfloor \frac{i-(n+2)}{2n-1} \right\rfloor + 2 & \text{for } n+2 \leq i \leq m-1 \end{cases}$$

$$\delta(v_{i+1}u_{i,n-2}) = \left\lfloor \frac{i-1}{2n-1} \right\rfloor + 1 \quad \text{for } 1 \le i \le m-1$$
$$\delta(v_1 u_{m,n-2}) = 1$$
$$\delta(u_{i,j}u_{i,j+1}) = \begin{cases} 1 & \text{for } 1 \le i \le n+2 \text{ and } 1 \le j \le n-3, \text{ or} \\ & \quad i = m \text{ and } 1 \le j \le n-3 \\ \left\lfloor \frac{i-(n+3)}{2n-1} \right\rfloor + 1 & \text{for } n+3 \le i \le m-1 \text{ and} \\ & \quad j > i-(n+2) \\ \left\lfloor \frac{i-(n+3)}{2n-1} \right\rfloor + 2 & \text{for } n+3 \le i \le m-1 \text{ and} \\ & \quad j \le i-(n+2). \end{cases}$$

It is readily seen that all the labels of the vertices and edges of $C_m \trianglerighteq C_n$ under δ labelling above are at most 2. Let C_n^i be a subgraph of $C_m \trianglerighteq C_n$ isomorphic to C_n with $V_{C_n^i} = \{v_i, v_{i+1 \pmod m}\} \cup \{u_{i,j} : 1 \le j \le n-2\}$ for any $1 \le i \le m$. For the case $m = n$, there is an addition subgraph isomorphic to C_n, namely C_n^0 with $V_{C_n^0} = \{v_i : 1 \le i \le m\}$.

For the weight of subgraphs C_n^0 and C_n^i, $i = 1, 2, \ldots, n$, we get the following

$$wt_\delta(C_n^0) = \sum_{v \in V_{C_n^0}} \delta(v) + \sum_{e \in E_{C_n^0}} \delta(e) = 4n - 4$$

and

$$\begin{aligned} wt_\delta(C_n^i) &= \sum_{v \in V_{C_n^i}} \delta(v) + \sum_{e \in E_{C_n^i}} \delta(e) \\ &= \begin{cases} 2n & \text{for } i = 1 \\ 2n + i & \text{for } 2 \le i \le m-1 \\ 2n + 1 & \text{for } i = m. \end{cases} \end{aligned}$$

We have $wt_\delta(C_n^{i+1}) = wt_\delta(C_n^i) + 1$ for every $2 \le i \le m-2$. Since the weight of every subgraph C_n^i in $C_m \trianglerighteq C_n$ is different under labeling δ, it follows that the labeling δ is the desired C_n-irregular total 2-labelling for $C_m \trianglerighteq C_n$ with $m \le 2n+1$.

Case 2. $m > 2n + 1$

Let $p = \left\lceil 1 + \frac{m-1}{2n} \right\rceil$. Since $m > 2n+1$, we have $p \ge 3$. In order to show the converse of inequality, we define a C_n-irregular total p-labelling $\alpha : V(C_m \trianglerighteq C_n) \cup E(C_m \trianglerighteq C_n) \to \{1, 2, \ldots, p\}$ as follows

$$\alpha(v_i) = \begin{cases} 1 & \text{for } i = 1, 2 \\ \left\lfloor \frac{i-3}{2n-1} \right\rfloor + 2 & \text{for } 3 \le i \le m-p+1 \text{ and} \\ p & \text{for } i = m-p+2 \text{ and } m \not\equiv 2 \pmod{2n} \\ p-1 & \text{for } i = m-p+2 \text{ and } m \equiv 2 \pmod{2n} \\ m-i+2 & \text{for } m-p+3 \le i \le m \end{cases}$$

$$\alpha(u_{i,j}) = \begin{cases} 1 & \text{for } i = 1,2,3 \text{ and } 1 \le j \le n-2 \\ \left\lfloor \frac{i-4}{2n-1} \right\rfloor + 1 & \text{for } 4 \le i \le m-p+1 \text{ and} \\ & \quad j > i - \lfloor \frac{i-4}{2n-1} \rfloor (2n-1) - 3 \\ \left\lfloor \frac{i-4}{2n-1} \right\rfloor + 2 & \text{for } 4 \le i \le m-p+1 \text{ and} \\ & \quad j \le i - \lfloor \frac{i-4}{2n-1} \rfloor (2n-1) - 3 \\ m-i+1 & \text{for } m-p+1 < i \le m \text{ and} \\ & \quad 1 \le j \le n-2 \end{cases}$$

$$\alpha(v_i v_{i+1}) = \begin{cases} 1 & \text{for } i = 1 \\ \left\lfloor \frac{i-2}{2n-1} \right\rfloor + 2 & \text{for } 2 \le i \le m-p+1 \\ p-1 & \text{for } i = m-p+2 \text{ and } m \not\equiv 2 \pmod{2n} \\ p & \text{for } i = m-p+2 \text{ and } m \equiv 2 \pmod{2n} \\ m-i+1 & \text{for } m-p+3 \le i \le m-1 \end{cases}$$

$$\alpha(v_m v_1) = 1$$

$$\alpha(v_i u_{i,1}) = \begin{cases} 1 & \text{for } 1 \le i \le n+1 \text{ or } i = m \\ \left\lfloor \frac{i-(n+2)}{2n-1} \right\rfloor + 2 & \text{for } n+2 \le i \le m-p+1 \\ m-i+1 & \text{for } m-p+2 \le i \le m-1 \end{cases}$$

$$\alpha(v_{i+1} u_{i,n-2}) = \begin{cases} \left\lfloor \frac{i-1}{2n-1} \right\rfloor + 1 & \text{for } 1 \le i \le m-p+1 \\ m-i+1 & \text{for } m-p+2 \le i \le m-1 \end{cases}$$

$$\alpha(v_1 u_{m,n-2}) = 1$$

$$\alpha(u_{i,j} u_{i,j+1}) = \begin{cases} 1 & \text{for } 1 \le i \le n+2 \text{ and} \\ & \quad 1 \le j \le n-3 \\ \left\lfloor \frac{i-(n+3)}{2n-1} \right\rfloor + 1 & \text{for } n+3 \le i \le m-p+1 \text{ and} \\ & j > i - \lfloor \frac{i-(n+3)}{2n-1} \rfloor (2n-1) - (n+2) \\ \left\lfloor \frac{i-(n+3)}{2n-1} \right\rfloor + 2 & \text{for } n+3 \le i \le m-p+1 \text{ and} \\ & j \le i - \lfloor \frac{i-(n+3)}{2n-1} \rfloor (2n-1) - (n+2) \\ m-i+1 & \text{for } m-p+1 < i \le m \text{ and} \\ & \quad 1 \le j \le n-3 \end{cases}$$

We can see that under labelling α, all vertices and edges of $C_m \trianglerighteq C_n$ are at most p. Let C_n^i be a subgraph of $C_m \trianglerighteq C_n$ isomorphic to C_n with $V_{C_n^i} = \{v_i, v_{i+1 \pmod m}\} \cup \{u_{i,j} : 1 \leq j \leq n-2\}$ for any $1 \leq i \leq m$.

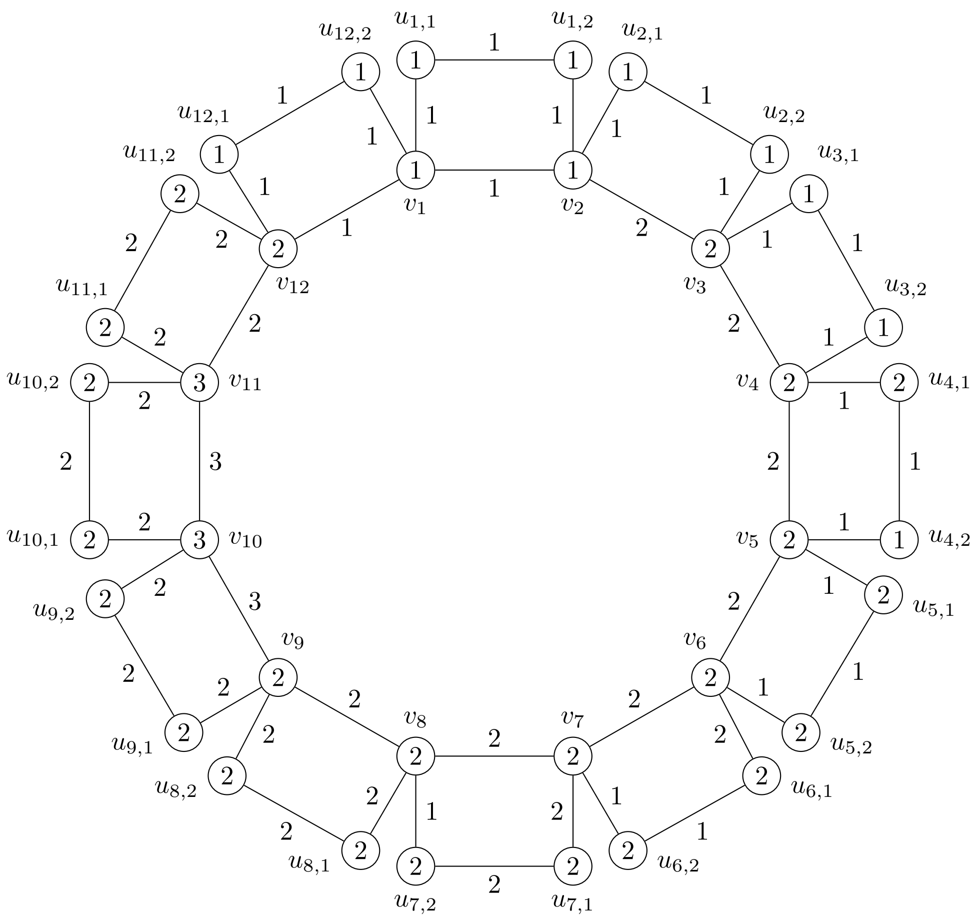

Figure 1: An C_4-irregular total 3-labelling of $C_{12} \trianglerighteq C_4$

For the weight of subgraphs C_n^i, $i = 1, 2, \ldots, n$, we get the following

$$wt_\alpha(C_n^i) = \sum_{v \in V_{C_n^i}} \alpha(v) + \sum_{e \in E_{C_n^i}} \alpha(e)$$

$$= \begin{cases} 2n & \text{for } i = 1 \\ i - \left\lfloor \frac{i-2}{2n-1} \right\rfloor (2n-1) + \left\lceil \frac{i-1}{2n-1} \right\rceil 2n & \text{for } 2 \leq i \leq m-p \\ (m-i+1)2n+1 & \text{for } m-p+1 \leq i \leq m. \end{cases}$$

Under the labelling α above, it is easily to check that $wt_\alpha(C_n^r) \neq wt_\alpha(C_n^s)$ for any $1 \leq r, s \leq m$, $r \neq s$. Thus, $ths(C_m \trianglerighteq C_n) \leq \left\lceil 1 + \frac{m-1}{2n} \right\rceil$. This completes the proof.

□

Figure 1 illustrates an C_4-irregular total 3-labelling of $C_{12} \trianglerighteq C_4$.

In the next theorem, we determine the exact value of the total H-irregularity strength of path graph and an arbitrary 2-connected graph H.

Theorem 2.4. *Let P_m be a path graph of order $m \geq 2$ and H be any 2-connected graph. Then*

$$ths(P_m \trianglerighteq H, H) = 1 + \left\lceil \frac{m-2}{|V_H| + |E_H|} \right\rceil.$$

Proof. As H is a 2-connected graph, we have $P_m \trianglerighteq H$ admits H-covering and it contains exactly $m-1$ subgraphs of $P_m \trianglerighteq H$ isomorphic to H. Let us denote the vertex set and edge set of subgraph P_m of $P_m \trianglerighteq H$ by $V_{P_m} = \{v_i : 1 \leq i \leq n\}$ and $E_{P_m} = \{v_i v_{i+1} : 1 \leq i \leq m-1\}$, respectively. Let us also denote the elements (both vertices and edges) of the graph $P_m \trianglerighteq H$ from the i-th copy of H that are not vertices of the P_m by the symbol a_i^j, for any $1 \leq i \leq m-1$ and $1 \leq j \leq s-2$, where $s = |V_H| + |E_H|$. We define an H-irregular total labelling β in the following way:

$$\beta(v_i) = \begin{cases} 1 & \text{for } i = 1, 2 \\ \left\lfloor \frac{i-3}{s} \right\rfloor + 2 & \text{for } 3 \leq i \leq m \end{cases}$$

$$\beta(a_i^j) = \begin{cases} \left\lfloor \frac{i-1}{s} \right\rfloor + 1 & \text{for } i \equiv 1 \pmod{s} \text{ or } i \equiv 2 \pmod{s} \text{ or } i \equiv 3 \pmod{s}, \\ & 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq s-2 \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 2 & \text{for } i \not\equiv 1 \pmod{s} \text{ or } i \not\equiv 2 \pmod{s} \text{ or } i \not\equiv 3 \pmod{s}, \\ & 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq i - \lfloor \frac{i-4}{s} \rfloor s - 3 \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 1 & \text{for } i \not\equiv 1 \pmod{s} \text{ or } i \not\equiv 2 \pmod{s} \text{ or } i \not\equiv 3 \pmod{s}, \\ & 1 \leq i \leq m-1 \text{ and } i - \lfloor \frac{i-4}{s} \rfloor s - 2 \leq j \leq s-2. \end{cases}$$

If $m - 1 \equiv 1 \pmod{s}$, then the maximal used label is

$$\left\lfloor \frac{(m-1)-1}{s} \right\rfloor + 1 = \frac{(m-1)-1}{s} + 1 = \frac{m-2}{s} + 1.$$

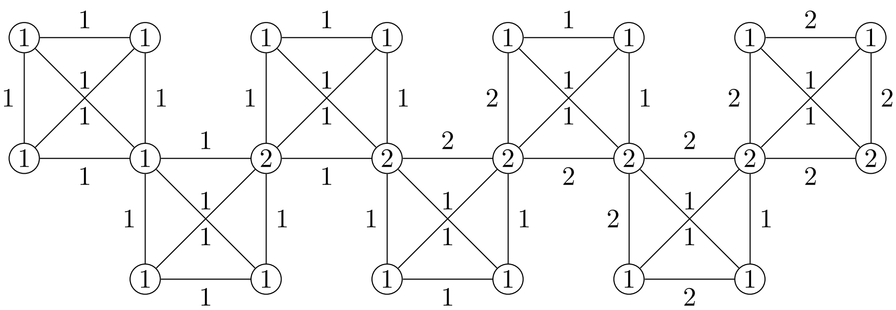

Figure 2: An K_4-irregular total 2-labelling of $P_8 \trianglerighteq K_4$

If $m-1 \equiv 2 \pmod s$, then the maximal used label is

$$\left\lfloor \frac{m-3}{s} \right\rfloor + 2 = \left(\frac{m-3}{s} + 1 \right) + 1 = \left\lceil \frac{m-2}{s} \right\rceil + 1.$$

If $m-1 \equiv 3 \pmod s$, then the maximal used label is

$$\left\lfloor \frac{m-3}{s} \right\rfloor + 2 = \left(\left\lfloor \frac{m-3}{s} \right\rfloor + 1 \right) + 1 = \left\lceil \frac{m-2}{s} \right\rceil + 1.$$

If $m-1 \not\equiv 1 \pmod s$ or $m-1 \not\equiv 2 \pmod s$ or $m-1 \equiv 3 \pmod s$, then the maximal used label is

$$\left\lfloor \frac{(m-1)-1}{s} \right\rfloor + 2 = \left(\left\lfloor \frac{m-2}{s} \right\rfloor + 1 \right) + 1 = \left\lceil \frac{m-2}{s} \right\rceil + 1.$$

Thus β is $\left(\left\lceil \frac{m-2}{s} \right\rceil + 1\right)$-labelling.

Let H_i be a subgraph of $P_m \trianglerighteq H$ that is isomorphic to H, $i = 1, 2, \ldots, m-1$, with element set as follows $V_{H_i} \cup E_{H_i} = \{v_i, v_{i+1}\} \cup \{a_i^j : 1 \leq j \leq s-2\}$. For the weight of subgraphs H_i, $i = 1, 2, \ldots m-1$, we obtain

$$wt_\beta(H_i) = |V_H| + |E_H| - 1 + i.$$

Hence, under the labelling β, all H-weights form a consecutive sequence $(|V_H| + |E_H|, |V_H| + |E_H| + 1, \ldots, |V_H| + |E_H| + m - 2)$. It implies that all H-weights are distinct.

By applying Theorem 2.1 (i), we have $ths(P_m \trianglerighteq H, H) \geq 1 + \left\lceil \frac{m-2}{|V_H|+|E_H|} \right\rceil$ as the lower bound. It proves that the irregular total labelling β has the required properties. $\square$

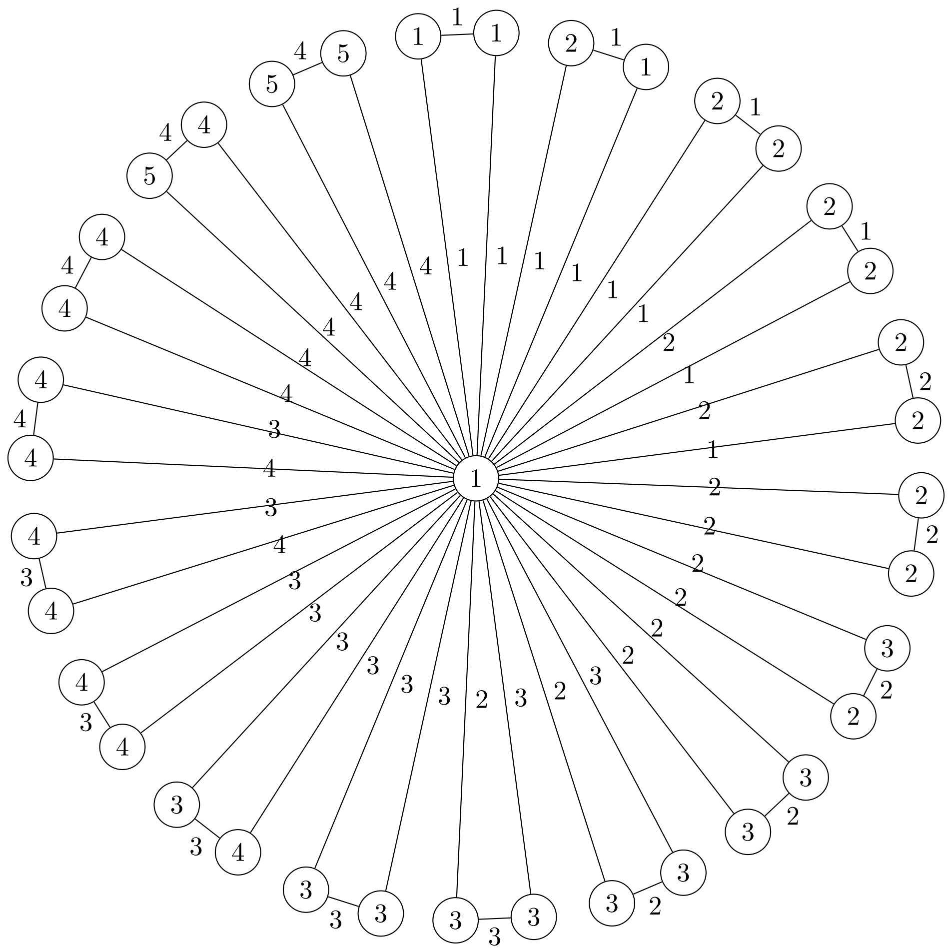

Figure 3: An C_3-irregular total 5-labelling of $S_{18} \trianglerighteq C_3$

As an example of Theorem 2.4, we give Figure 2 with H is the complete graph K_4.

Theorem 2.5. *Let S_m be a star graph of order $m+1$ with $m \geq 3$ and let H be 2-connected graph. Then*

$$ths(S_m \trianglerighteq H) = 1 + \left\lceil \frac{m-1}{|V_H| + |E_H| - 1} \right\rceil.$$

Proof. The edge comb product of star graph S_m and any connected graph

H is equivalent to the vertex-amalgamation $Amal(H, K_1, m)$. Thus,

$$ths(S_m \trianglerighteq H) = 1 + \left\lceil \frac{m-1}{|V_H| + |E_H| - |V_{K_1}| - |E_{K_1}|} \right\rceil \qquad (see[4])$$

$$= 1 + \left\lceil \tfrac{m-1}{|V_H|+|E_H|-1} \right\rceil. \qquad \square$$

Figure 3 depicts an C_3-irregular total 5-labelling of edge comb product of S_{18} and C_3.

3 Conclusions

In this paper we give the total H-irregularity strength $ths(\Gamma \trianglerighteq H, H)$ of edge comb products $\Gamma \trianglerighteq H$ of graphs Γ and H. We give an estimation on the lower bound of $ths(\Gamma \trianglerighteq H, H)$ for any two connected graphs Γ and H. Moreover, we determine the $ths(C_m \trianglerighteq C_n, C_n)$ for arbitrary positive integers $m, n \geq 3$, the $ths(P_m \trianglerighteq H, H)$ for arbitrary positive integer $m \geq 2$ and arbitrary 2-connected graph H and the $ths(S_m \trianglerighteq H, H)$ for arbitrary positive integer $m \geq 3$ and arbitrary 2-connected graph H. For future research, it will be interesting to investigate the exact value of vertex H-irregularity strength, edge H-irregularity strength and total face irregularity strength for edge comb product of graphs (see [8]). It could be investigated also some other parameters for the edge comb graphs, such as their vertex-face H-irregularity strength and edge-face H-irregularity strength (see [9]).

Acknowledgments

We would like to thank the reviewers for the valuable comment and suggestion.

References

[1] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 187-192.

[2] M. Bača, S. Jendrol, M. Miller, and J. Ryan, On irregular total labelling, *Discrete Matematics* **307** (2007), 1378-1388.

[3] F. Ashraf, M. Bača, M. Lascakova, and A. Semaničová-Feňovčíková, On H-irregularity strength of graph, *Discussiones Mathematicae: Graph Theory* **37** (2017), 1067-1078.

[4] F. Ashraf, M. Bača, A. Semaničová-Feňovčíkov; A. Shabbir, On H-irregularity strengths of $\mathcal{G}$-amalgamation of graphs, *Electronic Journal of Graph Theory and Applications* **5(2)** (2017), 325334.

[5] I.H. Agustin, Dafik, Marsidi, E.R. Albirri, On the total H-irregularity strength of graphs, *A new notion, Journal of Physics: Conf. Series* **855** (2017), 1-9.

[6] R. Nisviasari, Dafik, I.H. Agustin, The total H-irregularity strength of triangular ladder and grid graphs, *Journal of Physics: Conf. Series* **1211-012005** (2019).

[7] H. Labane, I. Bouchemakh, and A. Semaničová-Feňovčíkov; On H-irregularity strength of cycles and diamond graphs, *Discrete Mathematics, Algorithms and Applications*, **14(5)** 2150157 (2022).

[8] F. Ashraf, M. Bača, Z. Kimkov, A. Semaniov-Feovkov, On vertex and edge H-irregularity strengths of graphs, *Discrete Math. Algorithms Appl.*, **8 (4)** (2016), 13.

[9] M. Bača, N. Hinding, A. Javed, and A. Semaničová-Feňovčíkov; H-Irregularity strengths of plane graphs, *Symmetry*, **13 (2)** 229 (2021).

Mohamad Fahruli WAHYUJATI,
Department of Mathematics,
Universitas Gadjah Mada,
Sekip Utara Bulaksumur Yogyakarta Indonesia.
Email: fahruli@ugm.ac.id

Yeni SUSANTI,
Department of Mathematics,
Universitas Gadjah Mada,
Sekip Utara Bulaksumur Yogyakarta Indonesia.
Email: yeni_math@ugm.ac.id