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Construction of reversible cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$

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Abstract

Let q be a power of prime p. In this article, we investigate the reversible cyclic codes of arbitrary length n over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$, where $u^3 = 0 \mod q$. Further, we find a unique set of generators for cyclic codes over R and classify the reversible cyclic codes with their generators. Moreover, it is shown that the dual of reversible cyclic code over R is reversible. Finally, some examples of reversible cyclic codes are provided to justify the importance of these results.

1 Introduction

Initially, linear codes were studied over finite fields, but in the early 1970s, these codes were discussed over finite rings. Because of its new importance in algebraic coding theory and extensive applications, linear codes over finite rings have received much attention since the 1990s. In 1994, Hammons et al. [7] calculated non-linear binary codes over $\mathbb{Z}_4$ under the Gray map. This motivated the study of linear codes over finite rings. After that, in the last three decades, cyclic codes and their properties have become a hot topic of research, and a large number of researchers have examined the various properties of the cyclic codes.

In 1964, Massey [11] first defined the characteristic properties of reversible cyclic codes. After that, the results of the formation of reversible cyclic codes over $\mathbb{Z}_4$ were presented by Siap and Abualrub [2] in 2007. In 2015, Srinivasulu

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and Bhaintwal [15] investigated reversible cyclic codes over $\mathbb{F}_4 + u\mathbb{F}_4$, $u^2 = 0$ and their implications for DNA codes. Recently, Islam and Prakash [8] studied reversible cyclic codes over the ring of integer modulo p^k. In 2021, Prakash et al. [13] studied the reversible cyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q$, $u^2 = 0 \mod q$, where q is the power of prime p.

In this article, we investigate reversible cyclic codes of arbitrary length n over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$, $u^3 = 0 \mod q$. Further, we show that the dual of reversible cyclic codes over the ring R is reversible. Finally, we provide some examples in support of our results.

2 Preliminaries

A code C of length n over the ring R is defined as a non-empty subset of R^n. A code C is called a linear code, if C forms an R-submodule of R^n. By $w_H(C)$, we denote the Hamming weight of a code C, which is defined as the smallest Hamming weight of all of its non-zero codewords. Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1}) \in C$. Then the Hamming distance between $\mathbf{a}$ and $\mathbf{b}$ is defined as $d_H(\mathbf{a}, \mathbf{b}) = |\{i : \quad a_i \neq b_i\}|$ i.e., $d_H(\mathbf{a}, \mathbf{b}) = w_H(\mathbf{a} - \mathbf{b})$. The Hamming distance of a code C is defined as $d_H(C) = \min\{d_H(\mathbf{a}, \mathbf{b}) : \ \mathbf{a}, \mathbf{b} \in C, \ \mathbf{a} \neq \mathbf{b}\}$. The Euclidean inner product of $\mathbf{a}, \mathbf{b} \in R^n$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_0b_0 + a_1b_1 + \cdots + a_{n-1}b_{n-1}$. The codewords $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$. The dual code $C^\perp$ of C is defined as $C^\perp = \{\mathbf{a} \in R^n : \ \mathbf{a} \cdot \mathbf{b} = 0, \ \forall \ \mathbf{b} \in C\}$. A code C is said to be self-dual if $C = C^\perp$, and self-orthogonal if $C \subseteq C^\perp$. A linear code C is said to be reversible if $(a_{n-1}, a_{n-2}, \ldots, a_0) \in C$, whenever $(a_0, a_1, \ldots, a_{n-1}) \in C$.

A linear code C of length n over R is called a cyclic code if $(a_0, a_1, \ldots, a_{n-1}) \in C$, then its cyclic shift $(a_{n-1}, a_0, \ldots, a_{n-2}) \in C$. Also, a cyclic code over R can be viewed as an ideal in $R_n = R[\kappa]/\langle \kappa^n - 1\rangle$, identifying $(a_0, a_1, \ldots, a_{n-1})$ by $a_0 + a_1\kappa + \cdots + a_{n-1}\kappa^{n-1}$.

For each polynomial $f(\kappa) = a_0 + a_1\kappa + \cdots + a_{n-1}\kappa^{n-1}$ with $a_{n-1} \neq 0$, the reciprocal of $f(\kappa)$ is defined as $f^*(\kappa) = \kappa^{n-1} f(\frac{1}{\kappa}) = a_{n-1} + a_{n-2}\kappa + \cdots + a_1\kappa^{n-2} + a_0\kappa^{n-1}$. Note that, $\deg(f^*(\kappa)) \leq \deg f(\kappa)$ and if $a_0 \neq 0$, then $\deg f^*(\kappa) = \deg f(\kappa)$. The polynomial $f(\kappa)$ is called self-reciprocal iff $f^*(\kappa) = f(\kappa)$.

3 Construction of cyclic codes over R

Let $\mathbb{F}_q$ be a finite field with q elements, where $q = p^r$, p is a prime and $r > 1$. Throughout the paper, $R = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q = \{a + ub + u^2c : a, b, c \in \mathbb{F}_q\}$, where $u^3 = 0 \mod q$. Thus, R is a commutative ring having q^3 elements. The structure of cyclic codes over R depends on the ring $R_1 = \mathbb{F}_q + u\mathbb{F}_q =$

$\{a + ub : a, b \in \mathbb{F}_q\}$ with $u^2 = 0 \mod q$ and the structure of cyclic codes over R_1 depends on the ring $\mathbb{F}_q$. Let C_1 be a cyclic code in $R_{1,n} = R_1[\kappa]/\langle \kappa^n - 1\rangle$. Define $\sigma_1 : R_1 \longrightarrow \mathbb{F}_q$ by $\sigma_1(a + ub) = a$. Then σ_1 is a ring homomorphism, which is extended to a ring homomorphism $\Phi_1 : C_1 \longrightarrow \mathbb{F}_q[\kappa]/\langle \kappa^n - 1\rangle$ defined by

$$\Phi_1(a_0 + a_1\kappa + \cdots + a_{n-1}\kappa^{n-1}) = \sigma_1(a_0) + \sigma_1(a_1)\kappa + \cdots + \sigma_1(a_{n-1})\kappa^{n-1}. \quad (1)$$

Now, $\ker \Phi_1 = \{ur(\kappa) : r(\kappa) \in \mathbb{F}_q[\kappa]\}$. Let $I_u = \{r(\kappa) : ur(\kappa) \in \ker \Phi_1\}$ is an ideal in $F_q[\kappa]/\langle \kappa^n - 1\rangle$. Thus, $I_u = \langle a(\kappa)\rangle$ for some polynomial $a(\kappa) \in \mathbb{F}_q[\kappa]$ and $\ker \Phi_1 = \langle ua(\kappa)\rangle$ with $a(\kappa)/(\kappa^n - 1)$. The image of C_1 under Φ_1 is also an ideal and hence a cyclic code in $\mathbb{F}_q[\kappa]/\langle \kappa^n - 1\rangle$, which is generated by $g(\kappa)$ with $g(\kappa)/(\kappa^n - 1)$. Hence

$$C_1 = \langle g(\kappa) + ul(\kappa), ua(\kappa)\rangle,$$

for some polynomial $l(\kappa) \in \mathbb{F}_q[\kappa]$.

Let C be a cyclic code in $R_n = R[\kappa]/\langle \kappa^n - 1\rangle$. Define a map $\sigma : R \longrightarrow R_1$ by $\sigma(a+ub+u^2c) = a+ub$. Then σ is a ring homomorphism, which is extended to a ring homomorphism $\Phi : C \longrightarrow R_1[\kappa]/\langle \kappa^n - 1\rangle$ defined by

$$\Phi(a_0 + a_1\kappa + \cdots + a_{n-1}\kappa^{n-1}) = \sigma(a_0) + \sigma(a_1)\kappa + \cdots + \sigma(a_{n-1})\kappa^{n-1}. \quad (2)$$

Now, $\ker \Phi = \{u^2 r(\kappa) : r(\kappa) \in \mathbb{F}_q[\kappa]\}$. Let $I_{u^2} = \{r(\kappa) : u^2 r(\kappa) \in \ker \Phi\}$. Then, I_{u^2} is an ideal in $\mathbb{F}_q[\kappa]/\langle \kappa^n - 1\rangle$ and hence a cyclic code in $\mathbb{F}_q[\kappa]/\langle \kappa^n - 1\rangle$. Thus, $I_{u^2} = \langle a_2(\kappa)\rangle$ for some polynomial $a_2(\kappa) \in \mathbb{F}_q[\kappa]$ and hence $\ker \Phi = \langle u^2 a_2(\kappa)\rangle$. The image $\Phi(C)$ is an ideal and hence a cyclic code in $R_1[\kappa]/\langle \kappa^n - 1\rangle$. Thus, by using the pullback method in [2], the cyclic code of length n over R is given by

$$C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), ua_1(\kappa) + u^2 l_3(\kappa), u^2 a_2(\kappa)\rangle,$$

where $l_1(\kappa)$, $l_2(\kappa)$, $l_3(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are polynomials over $\mathbb{F}_q$.

Throughout the article, we use $g(\kappa)$, $l_1(\kappa)$, $l_2(\kappa)$, $l_3(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ as mentioned above. Now, we state the special case of the Theorem 3.6 of [6], which gives the structure of a cyclic code of an arbitrary length n over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$, where $u^3 = 0 \mod q$.

Theorem 3.1. *Let C be a cyclic code of length n over R. Then the following hold:*

1. *If $gcd(n, q) = 1$, then $\frac{R[\kappa]}{\langle \kappa^n - 1\rangle}$ is a principal ideal ring and $C = \left\langle g(\kappa), ua_1(\kappa), u^2 a_{2(\kappa)}\right\rangle$, where $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$.*

2. *If* $\gcd(n,q) \neq 1$, *then*

   (a) $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\rangle$, *where* $g(\kappa)$, $l_1(\kappa)$ *and* $l_2(\kappa)$ *are polynomials over* $\mathbb{F}_q$ *with* $g(\kappa) \mid (\kappa^n - 1) \mod q$, $(g(\kappa) + ul_1(\kappa)) \mid (\kappa^n - 1) \mod q$ *in* R_1, $(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)) \mid (\kappa^n - 1) \mod q$ *in* R *and* $\deg l_2(\kappa) < \deg l_1(\kappa)$, *provided* $g(\kappa) = a_2(\kappa)$.

   (b) $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), u^2 a_2(\kappa)\rangle$, *where* $a_2(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$, $g(\kappa)$ *divides* $l_1(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$ *and* $a_2(\kappa)$ *divides* $l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{g(\kappa)})$, *also* $\deg(g(\kappa)) > \deg(a_2(\kappa)) > \deg(l_2(\kappa))$, *provided* $g(\kappa) = a_1(\kappa)$.

   (c) $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), ua_1(\kappa) + u^2 l_3(\kappa), u^2 a_2(\kappa)\rangle$, *where* $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$, $a_1(\kappa) \mid l_1(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$, $a_2(\kappa) \mid l_3(\kappa)(\frac{\kappa^n-1}{a_1(\kappa)})$ *and* $a_2(\kappa) \mid l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{a_1(\kappa)})$. *Moreover,* $\deg(g(\kappa)) > \deg(a_1(\kappa)) > \deg(a_2(\kappa))$, $\deg(a_i(\kappa)) > \deg(l_i(\kappa))$ *for* $i = 1, 2$ *and* $\deg(a_2(\kappa)) > \deg(l_3(\kappa))$.

By using the same technique with the necessary variations as used in the proof of [2, Theorem 6], one can obtain the following result.

Theorem 3.2. *Let* $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), ua_1(\kappa) + u^2 l_3(\kappa), u^2 a_2(\kappa)\rangle$ *be a cyclic code of length* n *over* $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. *Then,* $I_{u^2} = \langle a_2(\kappa)\rangle$ *and* $w_H(C) = w_H(I_{u^2})$.

4 Reversible cyclic codes over R

In this section, we discuss the reversible cyclic codes of arbitrary length n over R and find the necessary and sufficient conditions for a cyclic code C over R to be reversible. Here, we assume that $g(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are monic polynomials over $\mathbb{F}_q$. For any codeword $a = (a_0, a_1, \ldots, a_{n-1}) \in C$, the reverse of a is defined as $a^r = (a_{n-1}, a_{n-2}, \ldots, a_0)$. A linear code C of length n over the ring R is said to be reversible if $a^r \in C$, $\forall\ a \in C$.

Lemma 4.1. *Let* $f_1(\kappa)$ *and* $f_2(\kappa)$ *be any two polynomials over* R *with* $\deg(f_1(\kappa)) > \deg(f_2(\kappa))$. *Then the following hold:*

1. $(f_1(\kappa) \cdot f_2(\kappa))^* = f_1^*(\kappa) \cdot f_2^*(\kappa)$ *and*
2. $(f_1(\kappa) + f_2(\kappa))^* = f_1^*(\kappa) + \kappa^{\deg f_1(\kappa) - \deg f_2(\kappa)} f_2^*(\kappa)$.

Theorem 4.1. *[11, Theorem 1] The cyclic code* C *over* $\mathbb{F}_q$ *generated by the monic polynomial* $g(\kappa)$ *is reversible iff* $g(\kappa)$ *is self-reciprocal.*

Lemma 4.2. *Let* C *be a reversible cyclic code of length* n *over* R *and* $\Phi : C \longrightarrow \frac{R_1[\kappa]}{\langle \kappa^n - 1\rangle}$ *be a ring homomorphism as defined in Section (3). Then* $\Phi(C)$ *is also a reversible cyclic code.*

Proof. Let $a = (a_0, a_1, \ldots, a_{n-1})$ be any codeword in C and $\Phi(a) \in \Phi(C)$ from Section (3). Then $\Phi(a) = (\sigma(a_0), \sigma(a_1), \ldots, \sigma(a_{n-1}))$. Since C is reversible, so $a^r = (a_{n-1}, a_{n-2}, \ldots, a_0) \in C$. Now,

$$(\Phi(a))^r = (\sigma(a_{n-1}), \sigma(a_{n-2}), \ldots, \sigma(a_0)) = (\Phi(a^r)) \in \Phi(C).$$

Therefore, $\Phi(C)$ is reversible whenever C is reversible. □

Theorem 4.2. *Let $C = \big\langle g(\kappa), ua_1(\kappa), u^2a_2(\kappa)\big\rangle$ be a cyclic code of length n over R with $gcd(n,q) = 1$, where $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$. Also, $g(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are polynomials over $\mathbb{F}_q$. Then C is reversible iff $g(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are self-reciprocal polynomials.*

Proof. Suppose C is a reversible cyclic code of length n over R. From the construction of cyclic code C over R, we have $\Phi(C) = \langle g(\kappa), ua_1(\kappa)\rangle$ and by Lemma 4.2, $\Phi(C)$ is a reversible code over R_1. Again from the construction of C_1, $\Phi_1(C_1) = \langle g(\kappa)\rangle$ and hence by Lemma 4.2, $\Phi_1(C_1)$ is a reversible code over $\mathbb{F}_q$. Therefore, $g(\kappa)$ is a self-reciprocal polynomial over $\mathbb{F}_q$.

Since $\ker \Phi_1 = \{ur(\kappa) : r(\kappa) \in \mathbb{F}_q[\kappa]\}$ and $I_u = \{r(\kappa) : ur(\kappa) \in \ker \Phi_1\} = \langle a_1(\kappa)\rangle$ is a cyclic code. It is enough to show that I_u is reversible. Let $r(\kappa) = r_0 + r_1\kappa + \cdots + r_{n-1}\kappa^{n-1} \in I_u$ be an arbitrary polynomial. Then $r(\kappa) \in \mathbb{F}_q[\kappa]$ is also in C_1. Since C_1 is a reversible cyclic code over R_1, so $r^*(\kappa)$ is also in C_1. Therefore, $ur^*(\kappa) \in \ker \Phi_1$ i.e., $r^*(\kappa) \in I_u$ and hence I_u is reversible, which implies that $a_1(\kappa)$ is a self-reciprocal polynomial.

Also, we have $\ker \Phi = \{u^2r(\kappa) : r(\kappa) \in \mathbb{F}_q[\kappa]$ is a polynomial in $C\}$ and $I_{u^2} = \{r(\kappa) : u^2r(\kappa) \in \ker \Phi\} = \big\langle a_2(\kappa)\big\rangle$ is a cyclic code. It is sufficient to show that I_{u^2} is reversible. For this, let $r(\kappa) = r_0 + r_1\kappa + \cdots + r_{n-1}\kappa^{n-1} \in I_{u^2}$ be an arbitrary polynomial. Then $r(\kappa) \in \mathbb{F}_q[\kappa]$ is a polynomial in C. Since C is reversible over R. So, $r^*(\kappa) \in C$ and hence $u^2r^*(\kappa) \in \ker \Phi$ i.e., $r^*(\kappa) \in I_{u^2}$. Thus, I_{u^2} is reversible, which implies that $a_2(\kappa)$ is a self-reciprocal polynomial. Conversely, let $g(\kappa), a_1(\kappa)$ and $a_2(\kappa)$ be self-reciprocal polynomials over $\mathbb{F}_q$. Let $c(\kappa) \in C$ i.e., $c(\kappa) = g(\kappa)m_1(\kappa) + ua_1(\kappa)m_2(\kappa) + u^2a_2(\kappa)m_3(\kappa)$ for some polynomials $m_1(\kappa), m_2(\kappa)$ and $m_3(\kappa)$ over $\mathbb{F}_q$. Consider

$$\begin{aligned}
c^*(\kappa) &= \big(g(\kappa)m_1(\kappa) + ua_1(\kappa)m_2(\kappa) + u^2a_2(\kappa)m_3(\kappa)\big)^* \\
&= \big(g(\kappa)m_1(\kappa)\big)^* + u\kappa^i\big(a_1(\kappa)m_2(\kappa)\big)^* + u^2\kappa^j\big(a_2(\kappa)m_3(\kappa)\big)^* \\
&= g^*(\kappa)m_1^*(\kappa) + u\kappa^i a_1^*(\kappa)m_2^*(\kappa) + u^2\kappa^j a_2^*(\kappa)m_3^*(\kappa) \\
&= g(\kappa)m_1^*(\kappa) + ua_1(\kappa)\kappa^i m_2^*(\kappa) + u^2a_2(\kappa)\kappa^j m_3^*(\kappa),
\end{aligned}$$

where $m_1^*(\kappa)$, $\kappa^i m_2^*(\kappa)$ and $\kappa^j m_3^*(\kappa)$ are polynomials over $\mathbb{F}_q$, $\deg\big(g(\kappa)m_1(\kappa)\big) - \deg\big(a_1(\kappa)m_2(\kappa)\big) = i$ and $\deg\big(g(\kappa)m_1(\kappa)\big) - \deg\big(a_2(\kappa)m_3(\kappa)\big) = j$, which implies that $c^*(\kappa) \in \langle g(\kappa), ua_1(\kappa), u^2a_2(\kappa)\rangle$. Therefore, C is a reversible cyclic code over R. □

Theorem 4.3. *Let $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\rangle$ be a cyclic code of length n over R with $gcd(n, q) \neq 1$, $g(\kappa) \mid (\kappa^n - 1) \mod q$, $\deg(l_2(\kappa)) < \deg(l_1(\kappa)) < \deg(g(\kappa))$, where $g(\kappa)$, $l_1(\kappa)$ and $l_2(\kappa)$ are polynomials over $\mathbb{F}_q$. Then, C is reversible iff*

1. *$g(\kappa)$ is a self-reciprocal polynomial and*
2. *if $\deg(g(\kappa)) - \deg(l_1(\kappa)) = i$ and $\deg(g(\kappa)) - \deg(l_2(\kappa)) = j$, then*
   (a) *$\kappa^i l_1^*(\kappa) = l_1(\kappa)$ and $\kappa^j l_2^*(\kappa) = l_2(\kappa)$, or*
   (b) *$g(\kappa) = \frac{1}{r_2 - r_1^2}(\kappa^j l_2^*(\kappa) - r_1\kappa^i l_1(\kappa) - l_2(\kappa))$, if $r_1 \neq \sqrt{r}_2, r_1 \neq 0$, $r_2 \neq 0$.*

Proof. Suppose C is a reversible cyclic code of length n over R. Then, it is obvious that $\langle g(\kappa)\rangle$ is a reversible cyclic code and hence $g(\kappa)$ is a self-reciprocal polynomial. As C is a reversible, so $(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))^* = g^*(\kappa) + u\kappa^i l_1^* + u^2\kappa^j l_2^* = g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) \in C$, which implies that there exists a polynomial $r(\kappa) \in R[\kappa]$ such that

$$g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))r(\kappa).$$

Comparing the highest degree on both sides of equality, we get $r(\kappa) \in R[\kappa]$ is a constant polynomial, say $r(\kappa) = r_0 + ur_1 + u^2 r_2$, where r_0, r_1, $r_2 \in \mathbb{F}_q$. Then

$$g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))(r_0 + ur_1 + u^2 r_2). \quad (3)$$

Multiplying the above equation with u^2 and using $u^3 = 0 \mod q$, we have $u^2 g(\kappa) = u^2 g(\kappa) r_0$, which implies that $r_0 = 1$ and hence by Equation (3), we get

$$u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = g(\kappa)(ur_1 + u^2 r_2) + ul_1(\kappa)(1 + ur_1) + u^2 l_2(\kappa).$$

Comparing the coefficients of u and u^2, we find that

$$\kappa^i l_1^*(\kappa) = r_1 g(\kappa) + l_1(\kappa), \quad (4)$$

$$\kappa^j l_2^*(\kappa) = r_2 g(\kappa) + r_1 l_1(\kappa) + l_2(\kappa). \quad (5)$$

Case(i) If $r_1 = r_2 = 0$, then $\kappa^i l_1^*(\kappa) = l_1(\kappa)$ and $\kappa^j l_2^*(\kappa) = l_2(\kappa)$.
Case(ii) If $r_1 \neq 0$, $r_2 \neq 0$ and $r_1 \neq \sqrt{r}_2$, then, from Equation (4), we have $l_1(\kappa) = \kappa^i l_1^*(\kappa) - r_1 g(\kappa)$. Therefore, by Equation (5), we obtain

$$\kappa^j l_2^*(\kappa) = r_2 g(\kappa) + r_1(\kappa^i l_1^*(\kappa) - r_1 g(\kappa)) + l_2(\kappa) = (r_2 - r_1^2) g(\kappa) + r_1\kappa^i l_1^*(\kappa) + l_2(\kappa).$$

$$\implies g(\kappa) = \frac{1}{r_2 - r_1^2}[\kappa^j l_2^*(\kappa) - r_1\kappa^i l_1(\kappa) - l_2(\kappa)].$$

Conversely, assume that conditions (1) and (2) hold. Let $p(\kappa) \in C$. Then $p(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))r(\kappa)$ for some polynomial $r(\kappa) \in R[\kappa]$. Now,

$$\begin{aligned} p^*(\kappa) &= (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))^* r^*(\kappa) \\ &= (g^*(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))r^*(\kappa) \\ &= (g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))r^*(\kappa). \end{aligned}$$

Using Equation (4) and Equation (5), we have

$$\begin{aligned} p^*(\kappa) &= \big(g(\kappa) + u(r_1 g(\kappa) + l_1(\kappa)) + u^2(r_2 g(\kappa) + r_1 l_1(\kappa) + l_2(\kappa)\big)r^*(\kappa) \\ &= \big(g(\kappa)(1 + ur_1 + u^2 r_2) + ul_1(\kappa)(1 + ur_1) + u^2 l_2(\kappa)\big)r^*(\kappa) \\ &= \big(g(\kappa)(1 + ur_1 + u^2 r_2) + ul_1(\kappa)(1 + ur_1 + u^2 r_2) \\ &\quad + u^2 l_2(\kappa)(1 + ur_1 + u^2 r_2)\big)r^*(\kappa) \\ &= \big(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\big)(1 + ur_1 + u^2 r_2)r^*(\kappa). \end{aligned}$$

Therefore, $p^*(\kappa) \in C$. Hence, C is a reversible cyclic code over R. □

Corollary 4.1. *In Theorem 4.3, if* $r_1 \neq 0$, $r_2 = 0$, *then* $g(\kappa) = \frac{1}{r_1^2}\big(l_2(\kappa) - \kappa^j l_2^*(\kappa) + r_1\kappa^i l_1(\kappa)\big)$ *and if* $r_1 = 0$, $r_2 \neq 0$, *then* $g(\kappa) = \frac{1}{r_2}\big(\kappa^j l_2^*(\kappa) - l_2(\kappa)\big)$.

Theorem 4.4. *Let* $C = \big\langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), u^2 a_2(\kappa)\big\rangle$ *be a cyclic code of length* n *over* R *with* $gcd(n, q) \neq 1$, $a_2(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$, $\deg(l_2(\kappa)) < \deg(a_2(\kappa)) < \deg(g(\kappa))$, *where* $g(\kappa)$, $l_1(\kappa)$, $l_2(\kappa)$ *and* $a_2(\kappa)$ *are polynomials over* $\mathbb{F}_q$. *Then,* C *is reversible iff*

1. *$g(\kappa)$ and $a_2(\kappa)$ are self-reciprocal polynomials and*
2. *if $\deg(g(\kappa)) - \deg(l_1(\kappa)) = i$ and $\deg(g(\kappa)) - \deg(l_2(\kappa)) = j$, then*
   (a) $\kappa^i l_1^*(\kappa) = l_1(\kappa)$ *and* $a_2(\kappa) \mid \kappa^j l_2^*(\kappa) - l_2(\kappa)$, *or*
   (b) $g(\kappa) = \frac{1}{r_2 - r_1^2}(\kappa^j l_2^*(\kappa) - r_1\kappa^i l_1^*(\kappa) - l_2(\kappa) - a_2(\kappa)s(\kappa))$, *if* $r_1 \neq \sqrt{r}_2$ *and* $r_1 \neq 0$, $r_2 \neq 0$.

Proof. Suppose C is a reversible cyclic code over R. Then, by Theorem 4.2, $\langle g(\kappa)\rangle$ and $\langle a_2(\kappa)\rangle$ are reversible cyclic codes over $\mathbb{F}_q$. Therefore, $g(\kappa)$ and $a_2(\kappa)$ are self-reciprocal polynomials over $\mathbb{F}_q$. For the second proof, given that $\deg(g(\kappa)) - \deg(l_1(\kappa)) = i$ and $\deg(g(\kappa)) - \deg(l_2(\kappa)) = j$. So, we have

$$\begin{aligned} \big(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\big)^* &= g^*(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) \\ &= g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa). \end{aligned}$$

Since C is reversible, $g(\kappa)+u\kappa^i l_1^*(\kappa)+u^2\kappa^j l_2^*(\kappa) \in C$. This implies that there exist polynomials $r(\kappa)$ and $s(\kappa)$ in $R[\kappa]$ such that

$$g(\kappa)+u\kappa^i l_1^*(\kappa)+u^2\kappa^j l_2^*(\kappa) = (g(\kappa)+ul_1(\kappa)+u^2 l_2(\kappa))r(\kappa)+u^2 a_2(\kappa)s(\kappa). \quad (6)$$

As the degree of $g(\kappa)$ is maximum. By comparing the highest degree on both sides of equality, we get $r(\kappa)$ must be a constant polynomial over R, say $r(\kappa) = r_0+ur_1+u^2 r_2$, where $r_0, r_1, r_2 \in \mathbb{F}_q$. Therefore, Equation (6) becomes

$$\begin{aligned} g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))(r_0 + ur_1 + u^2 r_2) \\ + u^2 a_2(\kappa)s(\kappa). \end{aligned}$$

Multiplying the above expression by u^2 and applying $u^3 = 0 \mod q$, we have $u^2 g(\kappa) = u^2 g(\kappa) r_0$, hence $r_0 = 1$. So, we have

$$u\kappa^i l_1^*(\kappa)+u^2\kappa^j l_2^*(\kappa) = u(r_1 g(\kappa)+l_1(\kappa))+u^2(r_2 g(\kappa)+r_1 l_1(\kappa)+l_2(\kappa)+a_2(\kappa)s(\kappa)).$$

Comparing the coefficients of u and u^2, we have

$$\kappa^i l_1^*(\kappa) = r_1 g(\kappa) + l_1(\kappa), \quad (7)$$

$$\kappa^j l_2^*(\kappa) = r_2 g(\kappa) + r_1 l_1(\kappa) + l_2(\kappa) + a_2(\kappa)s(\kappa). \quad (8)$$

Case (i) If $r_1 = r_2 = 0$. Then, we have $\kappa^i l_1^*(\kappa) = l_1(\kappa)$ and $a_2(\kappa) \mid (\kappa^j l_2^*(\kappa) - l_2(\kappa))$.

Case (ii) If $r_1 \neq 0, r_2 \neq 0$ and $r_1 \neq \sqrt{r_2}$, then from Equation (7) $l_1(\kappa) = \kappa^i l_1^*(\kappa) - r_1 g(\kappa)$. By Equation (8), we get $\kappa^j l_2^*(\kappa) = r_2 g(\kappa) + r_1(\kappa^i l_1^*(\kappa) - r_1 g(\kappa)) + l_2(\kappa) + a_2(\kappa)s(\kappa) = (r_2 - r_1^2)g(\kappa) + r_1\kappa^i l_1^*(\kappa) + l_2(\kappa) + a_2(\kappa)s(\kappa)$. Thus,

$$g(\kappa) = \frac{1}{r_2 - r_1^2}[\kappa^j l_2^*(\kappa) - r_1\kappa^i l_1^*(\kappa) - l_2(\kappa) - a_2(\kappa)s(\kappa)].$$

Conversely, assume that conditions (1) and (2) hold. Let $p(\kappa) \in C$, then, $p(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))m(\kappa) + u^2 a_2(\kappa)n(\kappa)$ for some polynomials $m(\kappa)$, $n(\kappa) \in R[\kappa]$. By Lemma 4.1, we have

$$\begin{aligned} p^*(\kappa) &= (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))^* m^*(\kappa) + u^2\kappa^k a_2^*(\kappa)n^*(\kappa) \\ &= (g^*(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))m^*(\kappa) + u^2\kappa^k a_2^*(\kappa)n^*(\kappa) \\ &= (g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))m^*(\kappa) + u^2\kappa^k a_2(\kappa)n^*(\kappa), \end{aligned}$$

where $\deg(g(\kappa)) - \deg(l_1(\kappa)) = i$, $\deg(g(\kappa)) - \deg(l_2(\kappa)) = j$ and $\deg(g(\kappa)m(\kappa)) - \deg(a_2(\kappa)n(\kappa)) = k$. Now, using Equation (7) and Equation (8), we have

$$\begin{aligned}
p^*(\kappa) &= (g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))m^*(\kappa) + u^2\kappa^k a_2(\kappa)n^*(\kappa) \\
&= (g(\kappa) + u(r_1 g(\kappa) + l_1(\kappa)) + u^2(r_2 g(\kappa) + r_1 l_1(\kappa)(\kappa) + l_2(\kappa) \\
&\quad + a_2(\kappa)s(\kappa)))m^*(\kappa) + u^2\kappa^k a_2(\kappa)n^*(\kappa) \\
&= (g(\kappa)(1 + ur_1 + u^2 r_2) + ul_1(\kappa)(1 + ur_1) + u^2 l_2(\kappa))m^*(\kappa) \\
&\quad + u^2 a_2(\kappa)(s(\kappa)m^*(\kappa) + \kappa^k n^*(\kappa)) \\
&= (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))(1 + ur_1 + u^2 r_2)m^*(\kappa) \\
&\quad + u^2 a_2(\kappa)(s(\kappa)m^*(\kappa) + \kappa^k n^*(\kappa)) \\
&= (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))m_1(\kappa) + u^2 a_2(\kappa)n_1(\kappa) \in C,
\end{aligned}$$

where $m_1(\kappa) = (1 + ur_1 + u^2 r_2)m^*(\kappa)$ and $n_1(\kappa) = s(\kappa)m^*(\kappa) + \kappa^k n^*(\kappa)$. Hence, C is a reversible cyclic code over R. $\square$

Theorem 4.5. *Let $C = \langle g(\kappa)+ul_1(\kappa)+u^2 l_2(\kappa), ua_1(\kappa)+u^2 l_3(\kappa), u^2 a_2(\kappa)\rangle$ be a cyclic code of length n over R with $gcd(n, q) \neq 1$, $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1)$* mod q, *$\deg(a_2(\kappa)) \leq \deg(a_1(\kappa)) \leq \deg(g(\kappa))$, $\deg(l_t(\kappa)) \leq \deg(a_t(\kappa))$ for $t = 1, 2$, $\deg(l_3(\kappa)) \leq \deg(a_2(\kappa))$, where $g(\kappa)$, $l_1(\kappa)$, $l_2(\kappa)$, $l_3(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are polynomials over $\mathbb{F}_q$. Then, C is reversible iff*

1. *$g(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are self-reciprocal polynomials and*
2. *if $\deg(g(\kappa)) - \deg(l_1(\kappa)) = i$, $\deg(g(\kappa)) - \deg(l_2(\kappa)) = j$ and $\deg(a_1(\kappa)) - \deg(l_3(\kappa)) = k$, then*
   (a) *$a_1(\kappa) \mid \kappa^i l_1^*(\kappa) - l_1(\kappa)$,*
   (b) *$a_2(\kappa) \mid \kappa^k l_3^*(\kappa) - l_3(\kappa)$ and*
   (c) *$a_2(\kappa) \mid \kappa^j l_2^*(\kappa) - l_2(\kappa) - r_1 l_1(\kappa) - l_3(\kappa)s_0(\kappa)$, where $s_0(\kappa) \in \mathbb{F}_q[\kappa]$.*

Proof. Let C be a reversible cyclic code over R. Then, by Theorem 4.2, $\langle g(\kappa)\rangle$, $\langle a_1(\kappa)\rangle$ and $\langle a_2(\kappa)\rangle$ are reversible cyclic codes. Therefore, $g(\kappa)$, $a_1(\kappa)$ and $a_2(\kappa)$ are self-reciprocal polynomials. If C is reversible, then by Lemma 4.1, we have

$$\begin{aligned}
(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))^* &= g^*(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) \\
&= g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) \in C.
\end{aligned}$$

This implies that there exist polynomials $r(\kappa)$, $s(\kappa)$ and $t(\kappa)$ over R such that

$$\begin{aligned}
g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) &= (g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))r(\kappa) + (ua_1(\kappa) \\
&\quad + u^2 l_3(\kappa))s(\kappa) + u^2 a_2(\kappa)t(\kappa).
\end{aligned}$$

Comparing the highest degree on both sides of equality, we get $r(\kappa)$ is a constant polynomial over R, say $r(\kappa) = r_0 + ur_1 + u^2r_2$, where $r_0, r_1, r_2 \in \mathbb{F}_q$. Then

$$\begin{aligned} g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = &(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa))(r_0 + ur_1 + u^2r_2) \\ &+ (ua_1(\kappa) + u^2 l_3(\kappa))s(\kappa) + u^2 a_2(\kappa)t(\kappa). \end{aligned}$$

Multiplying the above expression by u^2 on both sides and using $u^3 = 0 \mod q$, we get $u^2g(\kappa) = u^2g(\kappa)r_0$, which implies that $r_0 = 1$. Therefore, we have

$$\begin{aligned} u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) = &g(\kappa)(ur_1 + u^2r_2) + (ul_1(\kappa) + u^2 l_2(\kappa))(1 + ur_1 + u^2r_2) \\ &+ (ua_1(\kappa) + u^2 l_3(\kappa))s(\kappa) + u^2 a_2(\kappa)t(\kappa). \end{aligned}$$

As $s(\kappa) = s_0(\kappa) + us_1(\kappa) + u^2s_2(\kappa)$ and $t(\kappa) = t_0(\kappa) + ut_1(\kappa) + u^2t_2(\kappa)$ are polynomials over R, where $s_i(\kappa), t_i(\kappa) \in \mathbb{F}_q[\kappa]$ for $i = 0, 1, 2$.

$$\begin{aligned} u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa) &= g(\kappa)(ur_1 + u^2r_2) + ul_1(\kappa)(1 + ur_1) + u^2 l_2(\kappa) + (ua_1(\kappa) \\ &\quad + u^2 l_3(\kappa))(s_0(\kappa) + us_1(\kappa)) + u^2 a_2(\kappa)t_0(\kappa) \\ &= u(r_1g(\kappa) + l_1(\kappa) + a_1(\kappa)s_0(\kappa)) + u^2(r_2g(\kappa) + r_1l_1(\kappa) \\ &\quad + l_2(\kappa) + a_1(\kappa)s_1(\kappa) + l_3(\kappa)s_0(\kappa) + a_2(\kappa)t_0(\kappa)). \end{aligned}$$

Now, comparing the coefficients of u and u^2, we get

$$\kappa^i l_1^*(\kappa) = r_1g(\kappa) + l_1(\kappa) + a_1(\kappa)s_0(\kappa), \tag{9}$$

$$\kappa^j l_2^*(\kappa) = r_2g(\kappa) + r_1l_1(\kappa) + l_2(\kappa) + a_1(\kappa)s_1(\kappa) + l_3(\kappa)s_0(\kappa) + a_2(\kappa)t_0(\kappa). \tag{10}$$

Since $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa)$, so by Equation (9) and Equation (10), we find that $a_1(\kappa) \mid (\kappa^i l_1^*(\kappa) - l_1(\kappa))$ and $a_2(\kappa) \mid (\kappa^j l_2^*(\kappa) - l_2(\kappa) - r_1l_1(\kappa) - l_3(\kappa)s_0(\kappa))$. Since $ua_1(\kappa) + u^2l_3(\kappa) \in C$ and C is a reversible code, $(ua_1(\kappa) + u^2l_3(\kappa))^* = ua_1^*(\kappa) + u^2\kappa^k l_3^*(\kappa) = ua_1(\kappa) + u^2\kappa^k l_3^*(\kappa) \in C$, where $\deg(a_1(\kappa)) - \deg(l_3(\kappa)) = k$ and $a_1(\kappa)$ is a self-reciprocal polynomial. Therefore, there exist polynomials $p(\kappa)$ and $q(\kappa)$ over R such that

$$ua_1(\kappa) + u^2\kappa^k l_3^*(\kappa) = (ua_1(\kappa) + u^2l_3(\kappa))p(\kappa) + u^2a_2(\kappa)q(\kappa). \tag{11}$$

On comparing the highest degree on both sides of equality, we get $p(\kappa)$ is a constant polynomial, say $p(\kappa) = p_0 + up_1 + u^2p_2$, where $p_0, p_1, p_2 \in \mathbb{F}_q$. Substituting the value of $p(\kappa)$ in Equation (11) we find that $ua_1(\kappa) + u^2\kappa^k l_3^*(\kappa) = (ua_1(\kappa) + u^2l_3(\kappa))(p_0 + up_1) + u^2a_2(\kappa)q(\kappa)$, which implies that $p_0 = 1$. Therefore, we obtain $q(\kappa) = q_0(\kappa) + uq_1(\kappa) + u^2q_2(\kappa) \in R[\kappa]$, and hence Equation

(11) becomes $ua_1(\kappa)+u^2\kappa^k l_3^*(\kappa) = (ua_1(\kappa)+u^2l_3(\kappa))(1+up_1)+u^2a_2(\kappa)q_0(\kappa)$. Therefore,

$$\begin{aligned} u^2\kappa^k l_3^*(\kappa) &= u^2p_1a_1(\kappa) + u^2l_3(\kappa) + u^2a_2(\kappa)q_0(\kappa), \\ \implies \kappa^k l_3^*(\kappa) &= p_1a_1(\kappa) + l_3(\kappa) + a_2(\kappa)q_0(\kappa), \end{aligned} \tag{12}$$

where $p_1 \in \mathbb{F}_q$ and $q_0(\kappa) \in \mathbb{F}_q[\kappa]$. Since $a_2(\kappa) \mid a_1(\kappa)$, we have $a_2(\kappa) \mid (\kappa^k l_3^*(\kappa) - l_3(\kappa))$.

Conversely, assume that conditions (1) and (2) hold. Let $p(\kappa)$ be any polynomial in C. Then, there exist polynomials $m_1(\kappa)$, $m_2(\kappa)$ and $m_3(\kappa)$ over R such that $p(\kappa) = (g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa))m_1(\kappa) + (ua_1(\kappa) + u^2l_3(\kappa))m_2(\kappa) + u^2a_2(\kappa)m_3(\kappa)$. Consider

$$\begin{aligned} p^*(\kappa) = {} & (g(\kappa) + u\kappa^i l_1^*(\kappa) + u^2\kappa^j l_2^*(\kappa))m_1^*(\kappa) + \kappa^\alpha(ua_1(\kappa) + u^2\kappa^k l_3^*(\kappa))m_2^*(\kappa) \\ & + u^2\kappa^\beta a_2(\kappa)m_3^*(\kappa), \end{aligned}$$

where $\alpha = \deg(g(\kappa)m_1(\kappa)) - \deg(a_1(\kappa)m_2(\kappa))$ and $\beta = \deg(g(\kappa)m_1(\kappa)) - \deg(a_2(\kappa)m_3(\kappa))$. By using Equation (10), Equation (11) and Equation (12), we find that

$$\begin{aligned} p^*(\kappa) &= (g(\kappa) + u(r_1g(\kappa) + l_1(\kappa) + a_1(\kappa)s_0(\kappa)) + u^2(r_2g(\kappa) + r_1l_1(\kappa) + l_2(\kappa) \\ &\quad + a_1(\kappa)s_1(\kappa) + l_3(\kappa)s_0(\kappa) + a_2(\kappa)t_0(\kappa)))m_1^*(\kappa) + \kappa^\alpha(ua_1(\kappa) \\ &\quad + u^2(p_1a_1(\kappa) + l_3(\kappa) + a_2(\kappa)q_0(\kappa)))m_2^*(\kappa) + u^2\kappa^\beta a_2(\kappa)m_3^*(\kappa) \\ &= (g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa))n_1(\kappa) + (ua_1(\kappa) + u^2l_3(\kappa))n_2(\kappa) \\ &\quad + u^2a_2(\kappa)n_3(\kappa), \end{aligned}$$

where $n_1(\kappa) = (1+ur_1+u^2r_2)m_1^*(\kappa)$, $n_2(\kappa) = (s_0(\kappa)+us_1(\kappa))m_1^*(\kappa)+\kappa^\alpha(1+up_1)m_2^*(\kappa)$ and $n_3(\kappa) = t_0(\kappa)m_1^*(\kappa)+\kappa^\alpha q_0(\kappa)m_2^*(\kappa)+\kappa^\beta m_3^*(\kappa)$ are polynomials over R. Therefore, $p^*(\kappa) \in C$. Hence, C is a reversible cyclic code over R. □

5 Dual of reversible cyclic codes over R

In this section, we discuss the dual of reversible cyclic code of arbitrary length n over R. Let C be a cyclic code of length n with parity-check polynomial $h(\kappa) = h_0 + h_1\kappa + \cdots + h_k\kappa^k$ and $\bar{h}(\kappa) = h^*(\kappa)$. Then the dual code $C^\perp$ of a cyclic code C has the following characterization.

Theorem 5.1. *[13, Theorem 4.1] Let C be a cyclic code of length n over $\mathbb{F}_q$. Then, the dual code $C^\perp = \langle \bar{h}(\kappa) \rangle$ of C is reversible if and only if $h(\kappa) \in C^\perp$.*

Definition 5.2. Let I be an ideal in R. Then, the annihilator of I is defined as

$$A(I) = \{b \ : \ b.a = 0, \ \forall \ a \in I\}.$$

Let C be a cyclic code with an associated ideal I. Then the associated ideal of the dual code $C^{\perp}$ is defined as

$$A(I)^* = \{b^* \ : \ b \in A(I)\},$$

where b^* stand for the reverse of a polynomial b. First, we discuss the annihilator and the dual of a cyclic code over R_1 and then over R in the following results.

Theorem 5.3. *Let $C = \big\langle g(\kappa), ua(\kappa)\big\rangle$ be a cyclic code of length n over $\mathbb{F}_q + u\mathbb{F}_q$ with $gcd(n,q) = 1$. Then $A(C) = \big\langle \frac{\kappa^n-1}{a(\kappa)}, u\frac{\kappa^n-1}{g(\kappa)}\big\rangle$ and $C^{\perp} = \big\langle \big(\frac{\kappa^n-1}{a(\kappa)}\big)^*, u\big(\frac{\kappa^n-1}{g(\kappa)}\big)^*\big\rangle$.*

Proof. The proof is similar to the proof of [13, Theorem 7]. $\square$

Theorem 5.4. *Let C be a cyclic code of length n over $\mathbb{F}_q+u\mathbb{F}_q$ with $gcd(n,q) \neq 1$. If $C = \big\langle g(\kappa) + ul(\kappa), ua(\kappa)\big\rangle$ with $\deg(g(\kappa)) > \deg(a(\kappa)) > \deg(l(\kappa))$, $g(\kappa) = a(\kappa)m_1(\kappa)$ and $l(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big) = a(\kappa)m_2(\kappa)$, where $m_1(\kappa)$ and $m_2(\kappa)$ are polynomials over $\mathbb{F}_q$. Then*

1. *$A(C) = \big\langle \frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa), u\frac{\kappa^n-1}{g(\kappa)}\big\rangle$ and*
2. *$C^{\perp} = \big\langle \big(\frac{\kappa^n-1}{a(\kappa)}\big)^* - u\kappa^i m_2^*(\kappa), u\big(\frac{\kappa^n-1}{g(\kappa)}\big)^*\big\rangle$, where $i = \deg(\frac{\kappa^n-1}{a(\kappa)}) - \deg(m_2(\kappa))$.*

Proof. Let $C = \big\langle g(\kappa)+ul(\kappa), ua(\kappa)\big\rangle$ be a cyclic code of length n over $\mathbb{F}_q+u\mathbb{F}_q$ with $a(\kappa) \mid g(\kappa) \mid (\kappa^n-1) \mod q$ and $a(\kappa) \mid l(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$. Therefore, there exist polynomials $m_1(\kappa)$, $m_2(\kappa) \in \mathbb{F}_q[\kappa]$ such that $g(\kappa) = a(\kappa)m_1(\kappa)$ and $l(\kappa)(\frac{\kappa^n-1}{g(\kappa)}) = a(\kappa)m_2(\kappa)$. If $M = \big\langle \frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa), u\frac{\kappa^n-1}{g(\kappa)}\big\rangle$. Then

$$\begin{aligned}
\big(\frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa)\big)(g(\kappa)+ul(\kappa)) &= (\frac{\kappa^n-1}{a(\kappa)})g(\kappa) + ul(\kappa)(\frac{\kappa^n-1}{a(\kappa)}) - um_2(\kappa)g(\kappa) \\
&= (\kappa^n-1)m_1(\kappa) + ug(\kappa)m_2(\kappa) - um_2(\kappa)g(\kappa) \\
&= 0, \\
\big(\frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa)\big)ua(\kappa) &= 0, \\
u\big(\frac{\kappa^n-1}{g(\kappa)}\big)(g(\kappa)+ul(\kappa)) &= 0, \\
u\big(\frac{\kappa^n-1}{g(\kappa)}\big)ua(\kappa) &= 0,
\end{aligned}$$

which implies that $\frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa)$, $u(\frac{\kappa^n-1}{g(\kappa)}) \in A(C)$ i.e., $M \subseteq A(C)$.

Next, we show that $A(C) \subseteq M$. Let $A(C) = \langle h(\kappa) + ur(\kappa), us(\kappa)\rangle$ be an annihilator of C. Then $us(\kappa)(g(\kappa) + ul(\kappa)) = us(\kappa)g(\kappa) = 0$, which implies that $s(\kappa) = (\frac{\kappa^n-1}{g(\kappa)})f_1(\kappa)$ for some polynomial $f_1(\kappa)$ over $\mathbb{F}_q$. Now, $(h(\kappa) + ur(\kappa))ua(\kappa) = uh(\kappa)a(\kappa) = 0$, which implies that $h(\kappa) = \frac{\kappa^n-1}{a(\kappa)}$. Also, $(h(\kappa) + ur(\kappa))(g(\kappa) + ul(\kappa)) = 0$. Since $h(\kappa)g(\kappa) = 0$ so, $r(\kappa) = -m_2(\kappa)$. Hence, we have $A(C) \subseteq M$. Therefore, we conclude that $A(C) = \langle \frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa), u\frac{\kappa^n-1}{g(\kappa)}\rangle$.

If we take $\deg(\frac{\kappa^n-1}{a(\kappa)}) - \deg(m_2(\kappa)) = i$ and $(\frac{\kappa^n-1}{a(\kappa)} - um_2(\kappa))^* = (\frac{\kappa^n-1}{a(\kappa)})^* - u\kappa^i m_2^*(\kappa)$, then, we get $C^\perp = \langle \big(\frac{\kappa^n-1}{a(\kappa)}\big)^* - u\kappa^i m_2^*(\kappa), u\big(\frac{\kappa^n-1}{g(\kappa)}\big)^*\rangle$. □

Corollary 5.1. *In Theorem 5.4, if $a(\kappa) = g(\kappa)$. Then*

1. $A(C) = \langle \frac{\kappa^n-1}{g(\kappa)} - um_2(\kappa)\rangle$ *and*
2. $C^\perp = \langle \big(\frac{\kappa^n-1}{g(\kappa)}\big)^* - u\kappa^i m_2^*(\kappa)\rangle$, *where* $\deg(\frac{\kappa^n-1}{g(\kappa)}) - \deg(m_2(\kappa)) = i$.

Theorem 5.5. *Let $C = \langle g(\kappa)+ul(\kappa), ua(\kappa)\rangle$ be a cyclic code of length n over $\mathbb{F}_q + u\mathbb{F}_q$ and $C^\perp$ be the dual code of C as defined in Theorem 5.4. Then, $C^\perp$ is a reversible code if C is reversible.*

Proof. The proof is similar to the proof of [13, Theorem 10]. □

Theorem 5.6. *Let $C = \langle g(\kappa), ua_1(\kappa), u^2a_2(\kappa)\rangle = \langle g(\kappa) + ua_1(\kappa) + u^2a_2(\kappa)\rangle$ be a cyclic code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with $gcd(n, q) = 1$, where $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$. Then*

1. $A(C) = \langle \frac{\kappa^n-1}{a_2(\kappa)}, u\frac{\kappa^n-1}{a_1(\kappa)}, u^2\frac{\kappa^n-1}{g(\kappa)}\rangle$ *and*
2. $C^\perp = \langle \big(\frac{\kappa^n-1}{a_2(\kappa)}\big)^*, u\big(\frac{\kappa^n-1}{a_1(\kappa)}\big)^*, u^2\big(\frac{\kappa^n-1}{g(\kappa)}\big)^*\rangle$.

Proof. Straightforward. □

Theorem 5.7. *Let $C = \langle g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa), ua_1(\kappa) + u^2l_3(\kappa), u^2a_2(\kappa)\rangle$ be a cyclic code of length n over R with $gcd(n, q) \neq 1$, $a_2(\kappa) \mid a_1(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$, $a_1(\kappa) \mid l_1(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$, $a_2(\kappa) \mid l_3(\kappa)(\frac{\kappa^n-1}{a_1(\kappa)})$ and $a_2(\kappa) \mid l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{a_1(\kappa)})$. If there exist polynomial $m_i(\kappa)$ over $\mathbb{F}_q$ for $i = 0, 1, 2, 3, 4$ such that $\kappa^n - 1 = a_1(\kappa)m_0(\kappa)$, $l_1(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big) = a_1(\kappa)m_1(\kappa) = a_2(\kappa)m_2(\kappa)$, $l_3(\kappa)\big(\frac{\kappa^n-1}{a_1(\kappa)}\big) = a_2(\kappa)m_3(\kappa)$ and $l_2(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big)\big(\frac{\kappa^n-1}{a_1(\kappa)}\big) = a_2(\kappa)m_4(\kappa)$. Then*

1. $A(C) = \langle \frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}\big(m_1(\kappa)m_3(\kappa) - m_4(\kappa)\big), u\big(\frac{\kappa^n-1}{a_1(\kappa)}\big) - u^2m_1(\kappa), u^2\big(\frac{\kappa^n-1}{g(\kappa)}\big)\rangle$ *and*

2. $C^{\perp} = \big\langle \frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}\big(m_1(\kappa)m_3(\kappa) - m_4(\kappa)\big)^*, \big(u\frac{\kappa^n-1}{a_1(\kappa)} - u^2m_1(\kappa)\big)^*,$ $u^2\big(\frac{\kappa^n-1}{g(\kappa)}\big)^*\big\rangle,$

provided $m_2(\kappa) - m_3(\kappa) = 0$.

Proof. Let C be a cyclic code of length n over R as defined above and $M = \big\langle \frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}\big(m_1(\kappa)m_3(\kappa) - m_4(\kappa)\big), u\frac{\kappa^n-1}{a_1(\kappa)} - u^2m_1(\kappa), u^2\frac{\kappa^n-1}{g(\kappa)}\big\rangle$. Then

$$\big(\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\big)(u^2a_2(\kappa)) = 0,$$

$$\begin{aligned}\big(\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\big)&(ua_1(\kappa) + u^2l_3(\kappa))\\ &= u^2a_1(\kappa)m_3(\kappa) - u^2a_1(\kappa)m_3(\kappa)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\big(\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}&(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\big)(g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa))\\ &= ug(\kappa)m_2(\kappa) + u^2g(\kappa)\frac{a_1(\kappa)}{\kappa^n-1}m_4(\kappa) - ug(\kappa)m_3(\kappa)\\ &\quad - u^2l_1(\kappa)m_3(\kappa) + u^2\frac{g(\kappa)}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa).\end{aligned}$$

Since $m_2(\kappa) - m_3(\kappa) = 0$, so the right-hand side of the above expression becomes

$$\begin{aligned}u^2g(\kappa)(\frac{a_1(\kappa)}{\kappa^n-1})m_4(\kappa) - u^2l_1(\kappa)m_3(\kappa) &+ u^2\frac{g(\kappa)}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\\ &= -u^2l_1(\kappa)m_3(\kappa) + u^2\frac{g(\kappa)}{m_0(\kappa)}m_1(\kappa)m_3(\kappa)\\ &= -u^2g(\kappa)\frac{a_1(\kappa)}{\kappa^n-1}m_1(\kappa)m_3(\kappa) + u^2g(\kappa)\frac{a_1(\kappa)}{\kappa^n-1}m_1(\kappa)m_3(\kappa)\\ &= 0.\end{aligned}$$

Therefore, $\big(\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\big) \in A(C)$. Now,

$$\big(u(\frac{\kappa^n-1}{a_1(\kappa)}) - u^2m_1(\kappa)\big)(u^2a_2(\kappa)) = 0,$$
$$\big(u(\frac{\kappa^n-1}{a_1(\kappa)}) - u^2m_1(\kappa)\big)(ua_1(\kappa) + u^2l_3(\kappa)) = 0,$$
$$\big(u(\frac{\kappa^n-1}{a_1(\kappa)}) - u^2m_1(\kappa)\big)(g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa)) = 0.$$

Thus, $u\big(\frac{\kappa^n-1}{a_1(\kappa)}\big) - u^2m_1(\kappa) \in A(C)$. Now,

$$\big(u^2\frac{\kappa^n-1}{g(\kappa)}\big)(u^2a_2(\kappa)) = 0,$$
$$\big(u^2\frac{\kappa^n-1}{g(\kappa)}\big)(ua_1(\kappa) + u^2l_3(\kappa)) = 0,$$
$$\big(u^2\frac{\kappa^n-1}{g(\kappa)}\big)(g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa)) = 0,$$

which implies that $u^2\big(\frac{\kappa^n-1}{g(\kappa)}\big) \in A(C)$. Hence, $M \subseteq A(C)$.

Next, we show that $A(C) \subseteq M$. For this, we consider $A(C) = \big\langle h(\kappa) + ur_1(\kappa) + u^2r_2(\kappa), ub_1(\kappa) + u^2r_3(\kappa), u^2b_2(\kappa)\big\rangle$ be an annihilator of C. Then $u^2b_2(\kappa)\big(g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa)\big) = u^2b_2(\kappa)g(\kappa) = 0$, which implies that $b_2(\kappa) = \big(\frac{\kappa^n-1}{g(\kappa)}\big)t_1(\kappa)$ for some polynomial $t_1(\kappa)$ over $\mathbb{F}_q$. Now, we consider $0 = (ub_1(\kappa) + u^2r_3(\kappa))(ua_1(\kappa) + u^2l_3(\kappa)) = u^2b_1(\kappa)a_1(\kappa)$, this implies that $b_1(\kappa) = \frac{\kappa^n-1}{a_1(\kappa)}$. Now, we have $0 = (u\frac{\kappa^n-1}{a_1(\kappa)} + u^2r_3(\kappa))(g(\kappa) + ul_1(\kappa) + u^2l_2(\kappa)) = u^2l_1(\kappa)\frac{\kappa^n-1}{a_1(\kappa)} + u^2r_3(\kappa)g(\kappa) = u^2m_1(\kappa)g(\kappa) + u^2r_3(\kappa)g(\kappa)$ and hence $r_3(\kappa) = -m_1(\kappa)$. Thus, $ub_1(\kappa) + u^2r_3(\kappa) = u\frac{\kappa^n-1}{a_1(\kappa)} - u^2m_1(\kappa)$.

Assume that $0 = (h(\kappa) + ur_1(\kappa) + u^2r_2(\kappa))(u^2a_2(\kappa)) = u^2h(\kappa)a_2(\kappa)$, which implies that $h(\kappa) = \frac{\kappa^n-1}{a_2(\kappa)}$. Further, $0 = (\frac{\kappa^n-1}{a_2(\kappa)} + ur_1(\kappa) + u^2r_2(\kappa))(ua_1(\kappa) + u^2l_3(\kappa)) = u^2l_3(\kappa)(\frac{\kappa^n-1}{a_2(\kappa)}) + u^2r_1(\kappa)a_1(\kappa) = u^2a_1(\kappa)m_3(\kappa) + u^2a_1(\kappa)r_1(\kappa)$, since

$r_1(\kappa) = -m_3(\kappa)$. Thus,

$$\begin{aligned}
0 =& \big(\frac{\kappa^n - 1}{a_2(\kappa)} - um_3(\kappa) + u^2 r_2(\kappa)\big)(g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)) \\
=& ul_1(\kappa)\frac{\kappa^n - 1}{a_2(\kappa)} + u^2 l_2(\kappa)\frac{\kappa^n - 1}{a_2(\kappa)} - um_3(\kappa)g(\kappa) - u^2 l_1(\kappa)m_3(\kappa) + u^2 r_2(\kappa)g(\kappa) \\
=& ug(\kappa)m_2(\kappa) + u^2 g(\kappa)m_4(\kappa)\frac{a_1(\kappa)}{\kappa^n - 1} - ug(\kappa)m_3(\kappa) \\
& - u^2 g(\kappa)m_1(\kappa)m_3(\kappa)\frac{a_1(\kappa)}{\kappa^n - 1} + u^2 r_2(\kappa)g(\kappa) \\
=& ug(\kappa)(m_2(\kappa) - m_3(\kappa)) + u^2\big(\frac{a_1(\kappa)g(\kappa)}{\kappa^n - 1}(m_4(\kappa) - m_1(\kappa)m_3(\kappa)) + r_2(\kappa)g(\kappa)\big).
\end{aligned}$$

Comparing the coefficients of u and u^2, we have $m_2(\kappa) - m_3(\kappa) = 0$ and $\frac{a_1(\kappa)g(\kappa)}{\kappa^n-1}(m_4(\kappa) - m_1(\kappa)m_3(\kappa)) + r_2(\kappa)g(\kappa) = 0$. Thus, $r_2(\kappa) = \frac{1}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))$ and hence we conclude that $A(C) \subseteq M$. □

Corollary 5.2. *With the same conditions in Theorem 5.7, if we take* $\frac{\kappa^n-1}{a_1(\kappa)} = m_0(\kappa)$ *and* $(\frac{\kappa^n-1}{a_2(\kappa)}) = m(\kappa)$. *Then* $A(C) = \big\langle(\frac{\kappa^n-1}{a_2(\kappa)}) - um_3(\kappa) + u^2\big(\frac{m_1(\kappa)m_3(\kappa)}{m_0(\kappa)} - \frac{m_4(\kappa)}{m(\kappa)}\big), u\big(\frac{\kappa^n-1}{a_1(\kappa)}\big) - u^2 m_1(\kappa), u^2\big(\frac{\kappa^n-1}{g(\kappa)}\big)\big\rangle$.

Corollary 5.3. *Let* $C = \big\langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), u^2 a_2(\kappa)\big\rangle$ *be a cyclic code of length* n *over* R *with* $gcd(n,q) \neq 1$, $a_2(\kappa) \mid g(\kappa) \mid (\kappa^n - 1) \mod q$, $a_2(\kappa) \mid l_3(\kappa)(\frac{\kappa^n-1}{a_1(\kappa)})$ *and* $a_2(\kappa) \mid l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{a_1(\kappa)})$. *If there exist polynomial* $m_i(\kappa)$ *over* $\mathbb{F}_q$ *for* $i = 0,1,2,3,4$ *such that* $(\kappa^n - 1) = g(\kappa)m_0(\kappa)$, $l_1(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big) = g(\kappa)m_1(\kappa) = a_2(\kappa)m_2(\kappa)$, $l_3(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big) = a_2(\kappa)m_3(\kappa)$ *and* $l_2(\kappa)\big(\frac{\kappa^n-1}{g(\kappa)}\big)\big(\frac{\kappa^n-1}{g(\kappa)}\big) = a_2(\kappa)m_4(\kappa)$, *then*

1. $A(C) = \big\langle(\frac{\kappa^n-1}{a_2(\kappa)}) - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa)), u^2(\frac{\kappa^n-1}{g(\kappa)})\big\rangle$ *and*

2. $C^\perp = \big\langle(\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa)))^*, u^2(\frac{\kappa^n-1}{g(\kappa)})^*\big\rangle$,

provided $m_2(\kappa) - m_3(\kappa) = 0$ *and* $a_1(\kappa) = g(\kappa)$.

Corollary 5.4. *Let* $C = \big\langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\big\rangle$ *be a cyclic code of length* n *over* R *with* $gcd(n,q) \neq 1$, $g(\kappa) \mid (\kappa^n - 1) \mod q$, $g(\kappa) \mid l_1(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$, $g(\kappa) \mid l_3(\kappa)(\frac{\kappa^n-1}{g(\kappa)})$ *and* $g(\kappa) \mid l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{g(\kappa)})$. *If there exist polynomial* $m_i(\kappa)$ *over* $\mathbb{F}_q$ *for* $i = 0,1,2,3,4$ *such that* $(\kappa^n - 1) = g(\kappa)m_0(\kappa)$, $l_1(\kappa)(\frac{\kappa^n-1}{g(\kappa)}) = g(\kappa)m_1(\kappa) = g(\kappa)m_2(\kappa)$, $l_3(\kappa)(\frac{\kappa^n-1}{g(\kappa)}) = g(\kappa)m_3(\kappa)$ *and* $l_2(\kappa)(\frac{\kappa^n-1}{g(\kappa)})(\frac{\kappa^n-1}{g(\kappa)}) = g(\kappa)m_4(\kappa)$. *Then*

1. $A(C) = \left\langle (\frac{\kappa^n-1}{a_2(\kappa)}) - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa))\right\rangle$ *and*

2. $C^{\perp} = \left\langle (\frac{\kappa^n-1}{a_2(\kappa)} - um_3(\kappa) + \frac{u^2}{m_0(\kappa)}(m_1(\kappa)m_3(\kappa) - m_4(\kappa)))^*\right\rangle$,

provided $m_2(\kappa) - m_3(\kappa) = 0$ *and* $a_2(\kappa) = g(\kappa)$.

6 Examples

Non-zero generator polynomials of C	Distance of C d
1 or $1 + u + u^2$	1
f_1^i, $i = 1, 2$	2
f_2^i, $i = 1, 2$	3
$f_1 f_2$	2
$f_1^2 f_2$	4
$f_1 f_2^2$	6

Table 1: Generator polynomials and distance of corresponding code C.

Example 6.1. Let $C = \left\langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\right\rangle$ be a reversible cyclic code of length 6 over $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. Then, we have $(\kappa^6 - 1) = (\kappa + 1)^2(\kappa^2 + \kappa + 1)^2 = f_1^2 f_2^2$ over $\mathbb{Z}_2$, where $f_1 = \kappa + 1$ and $f_2 = \kappa^2 + \kappa + 1$ are self-reciprocal polynomials. Then, using Theorem 4.3 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 1.

Example 6.2. Let $C = \left\langle g(\kappa)+ul_1(\kappa)+u^2 l_2(\kappa), u^2 a_2(\kappa)\right\rangle$ be a reversible cyclic code of length 6 over $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. We know that $(\kappa^6 - 1) = f_1^2 f_2^2$ over $\mathbb{Z}_2$, as in Example 6.1. Then, using Theorem 4.4 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 2, where $\alpha \in \mathbb{Z}_2$.

Example 6.3. Let $C = \left\langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), ua_1(\kappa) + u^2 l_3(\kappa), u^2 a_2(\kappa)\right\rangle$ be a reversible cyclic code of length 6 over $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. We know that $(\kappa^6 - 1) = f_1^2 f_2^2$ over $\mathbb{Z}_2$, where $f_1 = \kappa + 1$ and $f_2 = \kappa^2 + \kappa + 1$. Then, using Theorem 4.5 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 3, where $\alpha \in \mathbb{Z}_2$.

Non-zero generator polynomials of C	Distance of C d
f_1, u^2	1
$f_1^2, u^2 f_1^i$, where $i = 0, 1$	1, 2
f_2, u^2	1
$f_1 f_2 + u^2\alpha, u^2 f_i$, where $i = 1, 2$	2, 3
$f_2^2, u^2 f_2^i$, where $i = 0, 1$	1, 3
$f_1^2 f_2, u^2 f_1^i$, where $i = 0, 1, 2$	1, 2, 2
$f_1^2 f_2 + u^2\alpha f_1, u^2 f_1^i f_2$, where $i = 0, 1$	3, 2
$f_1 f_2^2 + u^2\alpha, u^2 f_1$	2
$f_1 f_2^2 + u\alpha f_2^2, u^2 f_2^i$, where $i = 0, 1, 2$	1, 3, 3
$f_1 f_2^2 + u^2\alpha f_2, u^2 f_1 f_2$	2

Table 2: Generator polynomials and distance of corresponding code C.

Example 6.4. Let $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa)\rangle$ be a reversible cyclic code of length 6 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$. Then, we have $(\kappa^6 - 1) = (\kappa+1)^3(\kappa+2)^3 = f_1^3 f_2^3$ over $\mathbb{Z}_3$, where $f_1 = \kappa + 1$ and $f_2 = \kappa + 2$. Here, f_1^i for $i = 1, 2, 3$ and f_2^2 are self-reciprocal polynomials. Then, using Theorem 4.3 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 4.

Example 6.5. Let $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), u^2 a_2(\kappa)\rangle$ be a reversible cyclic code of length 6 over $\mathbb{Z}_3 + u\mathbb{Z}_3 + u^2\mathbb{Z}_3$. Then $(\kappa^6 - 1) = f_1^3 f_2^3$ over $\mathbb{Z}_3$, as in Example 6.4. Then, using Theorem 4.4 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 5, where $\alpha \in \mathbb{Z}_3$.

Example 6.6. Let $C = \langle g(\kappa) + ul_1(\kappa) + u^2 l_2(\kappa), ua_1(\kappa) + u^2 l_3(\kappa), u^2 a_2(\kappa)\rangle$ be a reversible cyclic code of length 6 over $Z_3 + uZ_3 + u^2 Z_3$. Clearly, $(\kappa^6 - 1) = f_1^3 f_2^3$ over Z_3, as in Example 6.4. Then, using Theorem 4.5 and Theorem 3.2, the non-zero generator polynomials and distance of corresponding code C are given in table 6, where $\alpha \in \mathbb{Z}_3$.

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Non-zero generator polynomials of C	Distance of C d
f_1^2, uf_1, u^2	1
$f_1f_2 + u\alpha, uf_i, u^2$, where $i = 1, 2$	1
$f_1^2f_2, uf_1, u^2$	1
$f_1^2f_2 + u\alpha f_1, uf_1^2, u^2f_1^i$, where $i = 0, 1$	1, 2
$f_1^2f_2 + u\alpha f_1, uf_2, u^2$	1
$f_1^2f_2 + u\alpha f_1, uf_1f_2, u^2$	1
$f_1^2f_2 + u\alpha f_1, uf_1f_2 + u^2\alpha, u^2f_1$	2
$f_1^2f_2, uf_1f_2, u^2f_2$	3
$f_1^2f_2 + u^2\alpha, uf_1f_2 + u^2f_1, u^2f_2$	3
$f_1^if_2^2, uf_2, u^2$, where $i = 0, 1$	1
$f_1f_2^2 + u\alpha, uf_1, u^2$	1
$f_1f_2^2, uf_2^2, u^2f_2^i$, where $i = 0, 1$	1, 3
$f_1f_2^2 + u\alpha f_2, uf_1f_2, u^2$	1
$f_1f_2^2, uf_1f_2 + u^2\alpha, u^2f_1$	2
$f_1f_2^2 + u\beta f_2, uf_1f_2 + u^2\alpha, u^2f_2$	3

Table 3: Generator polynomials and distance of corresponding code C.

Non-zero generator polynomials of C	Distance of C d
1 or $1 + u + u^2$	1
f_1^i, where $i = 1, 2, 3$	2, 3, 2
$f_1^if_2^2$, where $i = 0, 1, 2, 3$	3, 4, 3, 6

Table 4: Generator polynomials and distance of corresponding code C.

that are relevant to the content of this article.

Non-zero generator polynomials of C	Distance of C d
f_1^i, u^2, where $i = 1, 2$	1
$f_1^2 + u^2\alpha, u^2 f_1$	2
$f_1^3, u^2 f_1^i$, where $i = 0, 1, 2$	1, 2, 3
$f_1^i f_2^2, u^2$, where $i = 0, 1$	1
$f_1 f_2^2, u^2 f_1$	2
$f_1 f_2^2 + u^2\alpha, u^2 f_2^2$	3
$f_1^2 f_2^2, u^2$	1
$f_1^2 f_2^2, u^2 f_i^2$, where $i = 1, 2$	3
$f_1^2 f_2^2 + u^2\alpha, u^2 f_1$	2
$f_1^2 f_2^2 + u^2 f_1, u^2 f_2^2$	3
$f_1^2 f_2^2 + u^2 f_2^2, u^2 f_1 f_2^2$	1
$f_1^3 f_2^2, u^2 f_1^i$, where $i = 0, 1, 2, 3$	1, 2, 3, 2
$f_1^3 f_2^2, u^2 f_1^i f_2^2$, where $i = 0, 1$	3, 4
$f_1^3 f_2^2 + u^2\alpha f_1^2, u^2 f_1^2 f_2^2$	3

Table 5: Generator polynomials and distance of corresponding code C.

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Non-zero generator polynomials of C	Distance of C d
$f_1^2 + u\alpha, uf_1, u^2$	1
f_1^3, uf_1^i, u^2, where $i = 0, 1, 2$	1
$f_1^3, uf_1^2, u^2 f_1$	2
$f_1 f_2^2, uf_i^i, u^2$, where $i = 1, 2$	1
$f_1^2 f_2^2 + u\alpha, uf_1, u^2$	1
$f_1^2 f_2^2, uf_i^2, u^2$, where $i = 1, 2$	1
$f_1^2 f_2^2, uf_1^2 + u^2\alpha, u^2 f_1$	2
$f_1^2 f_2^2, uf_1 f_2^2, u^2 f_2^i$, where $i = 0, 2$	1, 3
$f_1^2 f_2^2, uf_1 f_2^2 + u^2\alpha, u^2 f_1$	2
$f_1^2 f_2^2 + u^2 f_1, uf_1 f_2^2, u^2 f_2^2$	3
$f_1^3 f_2^2, uf_i^i, u^2$, where $i = 1, 2$	1
$f_1^3 f_2^2, uf_1^2, u^2 f_1^i$, where $i = 0, 1$	1, 2
$f_1^3 f_2^2, uf_1^3, u^2 f_1^i$, where $i = 0, 1, 2$	1, 2, 3
$f_1^3 f_2^2, uf_1 f_2^2, u^2 f_1^i$, where $i = 0, 1$	1, 2
$f_1^3 f_2^2, uf_1 f_2^2, u^2 f_2^2$	3
$f_1^3 f_2^2, uf_1^2 f_2^2, u^2 f_1^i$, where $i = 0, 1, 2$	1, 2, 3
$f_1^3 f_2^2, uf_1^2 f_2^2 + u^2 f_1, u^2 f_1^2$	3

Table 6: Generator polynomials and distance of corresponding code C.

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