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Solving the dynamic coloring problem for direct products of paths with fan graphs

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Abstract

This paper deals with the r-dynamic chromatic problem of the direct product of a path with a fan graph $F_{m,n}$. The problem is completely solved except for the case $n < r \in \{2m+2,\, 2m+3\}$, which is solved under certain assumptions. It enables us to determine in particular the dynamic chromatic number concerning this problem, for all $r \leq 7$, and also, for all $m \in \{1, 2\}$.

1 Introduction

Let r and ℓ be two positive integers. An *r-dynamic proper ℓ-coloring* of a finite and simple graph $G = (V(G), E(G))$ is any map $c : V(G) \to \{0, \ldots, \ell - 1\}$ such that $c(v) \neq c(w)$, for every pair of adjacent vertices $v, w \in V(G)$, and

$$|c(N(v))| \geq \min\{r, d(v)\}, \tag{1}$$

for every vertex $v \in V(G)$, where $N(v)$ and $d(v)$ denote, respectively, the neighborhood and the degree of the vertex v. The *r-dynamic chromatic number* of the graph G is the minimum positive integer k such that G has an r-dynamic proper k-coloring. It is denoted $\chi_r(G)$. Both concepts were introduced by Montgomery [19] as a natural generalization of the classical problem of graph coloring. More specifically, if $r = 1$, then these two concepts coincide with the classical ones of proper ℓ-coloring and chromatic number of a graph.

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Since the original manuscript of Montgomery, a wide amount of authors have dealt with the problem of determining the r-dynamic chromatic number of distinct types of graphs. As a first stage, they focused on the case $r = 2$ [2, 3, 5, 6], whereas the case $r > 2$ has received particular attention in the recent literature [8, 12, 20]. Of special relevance for the aim of this paper is the study of the r-dynamic coloring of different types of products of graphs [1, 4, 16, 17]. Particularly, it has recently been studied the r-dynamic chromatic number of the direct product of paths with paths, cycles, complete graphs, wheel graphs and star graphs [9, 7, 11]. This paper delves into this last topic by focusing on the direct product of a path with a fan graph. Notice that the r-dynamic coloring of fan graphs has already been dealt with [10, 18, 21].

The paper is organized as follows. In Section 2, we describe some preliminary concepts and results on Graph Theory that are used throughout the manuscript. Then, a detailed study of cases is analyzed in Section 3 for solving the r-dynamic coloring problem of the direct product of a path with a fan graph $F_{m,n}$. Particularly, this problem is completely solved except for the case $n < r \in \{2m + 2,\, 2m + 3\}$, which is solved under certain assumptions. In order to make easier the reading of the manuscript, a series of examples illustrate each case of the study. Finally, Section 4 makes use of the obtained results for establishing the exact solution of the mentioned problem, whenever $r \leq 7$ or $m \in \{1, 2\}$.

2 Preliminaries

This section deals with some preliminary concepts and results on Graph Theory that are used throughout the paper. For more details about this topic, we refer the reader to the classical manuscript of Harary [13].

Any *graph* $G = (V(G), E(G))$ is formed by a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ so that each edge joins two vertices, which are called *adjacent*. The number of vertices and edges are, respectively, the *order* and *size* of G. If both of them are finite, then the graph is *finite*. From now on, let vw denote the edge formed by two vertices $v, w \in V(G)$. If $v = w$, then the edge is a *loop*. A graph is *simple* if it contains no loops and no two edges join the same pair of vertices. All the graphs in this paper are simple and finite.

A *path* between two distinct vertices $v, w \in V(G)$ is any ordered sequence of n adjacent vertices $\langle\, v_0 = v, v_1, \ldots, v_{n-2}, v_{n-1} = w\,\rangle$ in $V(G)$, with $n > 2$, such that all the vertices under consideration are pairwise distinct. From here on, let P_n denote the path of order n. Then, the *fan graph* $F_{m,n}$ is the graph resulting after joining each one of the vertices of the path P_n with m isolated vertices. If $m = 1$ (respectively, $m = 2$), then it constitutes the *simple fan graph* $F_{1,n}$ (respectively, the *double fan graph* $F_{2,n}$).

The *direct product* of two finite simple graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \times H$, whose set of vertices is the Cartesian product $V(G) \times V(H)$, and two such vertices (v, v') and (w, w') are adjacent if and only if $vw \in E(G)$ and $v'w' \in E(H)$. Figure 1 illustrates this concept for the direct product $P_3 \times F_{1,3}$.

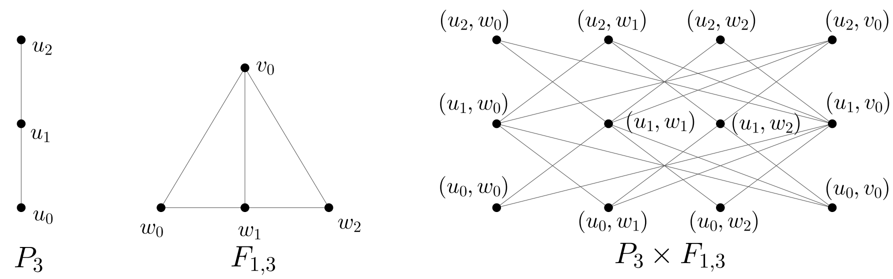

Figure 1: Construction of the direct product $P_3 \times F_{1,3}$.

The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set formed by its adjacent vertices. Its cardinality is the *degree* $d_G(v)$ of the vertex v. When there is no risk of confusion, the respective notations $N(v)$ and $d(v)$ are used. Let $\delta(G)$ and $\Delta(G)$ respectively denote the minimum and maximum vertex degree of the graph G. The following result holds readily from the previous definitions.

Lemma 1. *Let G and H be two finite simple graphs. Then,*

1. $d_{G\times H}((v, w)) = d_G(v)\, d_H(w)$, *for all* $(v, w) \in V(G \times H)$.
2. $\delta(G \times H) = \delta(G)\, \delta(H)$.
3. $\Delta(G \times H) = \Delta(G)\, \Delta(H)$.

Concerning the dynamic coloring problem described in the introductory section, the following results are known.

Lemma 2. *[14] Let G be a simple finite graph and let r be a positive integer. Then,* $\min\{r, \Delta(G)\} + 1 \leq \chi_r(G) \leq \chi_{r+1}(G)$. *Moreover,* $\chi_r(G) \leq \chi_{\Delta(G)}(G)$.

Lemma 3. *[15] Let n and r be two positive integers such that $n > 2$. Then,* $\chi_1(P_n) = 2$, *and* $\chi_r(P_n) = 3$, *whenever* $r > 1$.

Lemma 4. *[10] Let m, n and r be three positive integers such that $n > 2$. Then,*

$$\chi_r(F_{m,n}) = \begin{cases} 3, & \text{if } r \in \{1,2\},\\ 2r-1, & \text{if } 3 \leq r \leq \min\{m+1, n\},\\ n+r-1, & \text{if } n < r \leq m+1,\\ m+r, & \text{if } \max\{3, m+1\} \leq r \leq n,\\ m+n, & \text{if } r \geq \max\{m+1, n\}. \end{cases}$$

Lemma 5. *[9] Let G and H be two finite simple graphs and let r be a positive integer such that $r \leq \delta(G')$, for some $G' \in \{G, H\}$. Then, $\chi_r(G \times H) \leq \chi_r(G')$.*

3 Dynamic coloring of $P_l \times F_{m,n}$

Let $l > 2$, $n > 2$, m and r be four positive integers. This section deals with the dynamic coloring problem for the direct product of the path $P_l = \langle u_0, \ldots, u_{l-1} \rangle$ and the fan graph $F_{m,n}$, where $V(F_{m,n}) = \{v_0, \ldots, v_{m-1}, w_0, \ldots, w_{n-1}\}$ arises from m isolated vertices $v_0, \ldots, v_{m-1}$ and a path $\langle w_0, \ldots, w_{n-1} \rangle$. From Lemma 1, $\delta(P_l \times F_{m,n}) = \min\{m+1,\, n\}$ and $\Delta(P_l \times F_{m,n}) = \max\{2m+4,\, 2n\}$. The following study of cases arises.

3.1 Case $r \geq 2m+4$

Lemma 6. *If $r \geq 2m+4$, then $\chi_r(P_l \times F_{m,n}) = 2m + \min\{2n,\, r\}$.*

Proof. Let c be an r-dynamic proper $\chi_r(P_l \times F_{m,n})$-coloring of $P_l \times F_{m,n}$. Since $d(u_1w_0) = 2m+2 < 2m+4 \leq r$, Condition (1) applied to the vertex u_1w_0 implies that the set $\{c(u_0v_0), \ldots, c(u_0v_{m-1}),\, c(u_2v_0), \ldots, c(u_2v_{m-1})\}$ is formed by $2m$ distinct colors and also that no vertex in $\{u_0w_0, \ldots, u_0w_{n-1},\, u_2w_0, \ldots, u_2w_{n-1}\}$ is colored by one of these colors. Otherwise, one could find a vertex u_1w_i, with $0 \leq i < n$, such that $|c(N(u_1w_i))| < d(u_iw_i) \leq 2m+4$, which would contradict Condition (1). Thus, since $d(u_1v_0) = 2n$, the same condition applied to the vertex u_1v_0 implies that the set $\{c(u_0w_0), \ldots, c(u_0w_{n-1}),\, c(u_2w_0), \ldots, c(u_2w_{n-1})\}$ is formed by $\min\{2n,\, r\}$ extra distinct colors. Hence, $2m + \min\{2n,\, r\} \leq \chi_r(P_l \times F_{m,n})$.

In order to prove that this lower bound is tight, it is enough to consider the r-dynamic proper coloring c of the direct product $P_l \times F_{m,n}$ such that, for all $i < l$, $j < m$ and $k < n$, we have that $c(u_iv_j) = \min\{2n,\, r\} + m \cdot \left\lfloor \frac{i \bmod 4}{2} \right\rfloor + j$ and

$$c(u_iw_k) = \begin{cases} \left(2k + \left\lfloor \frac{i \bmod 4}{2} \right\rfloor\right) \bmod r, & \text{if } \begin{cases} r \geq 2n, \\ r < 2n \text{ is odd}, \\ r < 2n \text{ is even and} \\ \qquad k \notin \{\frac{r}{2}, \ldots, r-1\}, \end{cases} \\ \left(2k + 1 - \left\lfloor \frac{i \bmod 4}{2} \right\rfloor\right) \bmod r, & \text{otherwise.} \end{cases}$$

Here, it is relevant that $r \geq 6$. (Figure 2 illustrates the direct product $P_4 \times F_{2,5}$, for $r \in \{8, 9\}$.) □

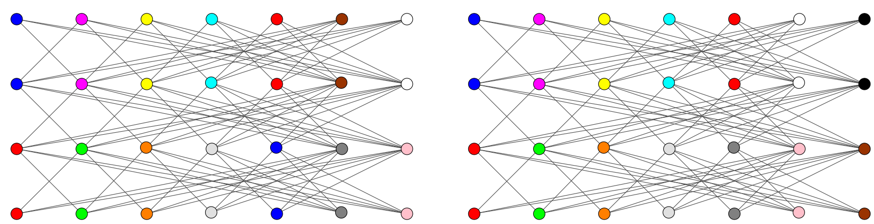

Figure 2: r-dynamic proper $(r+4)$-coloring of the direct product $P_4 \times F_{2,5}$, for $r = 8$ (left) and $r = 9$ (right).

3.2 Case $n \geq r \in \{2m+2,\, 2m+3\}$

Proposition 1. *Let $r \in \{2m+2,\, 2m+3\}$ be such that $r \leq n$. Then, $\chi_r(P_l \times F_{m,n}) = 2m + r$.*

Proof. Let c be an r-dynamic proper $\chi_r(P_l \times F_{m,n})$-coloring of $P_l \times F_{m,n}$. Condition (1) implies that the $2m$ vertices in $N(u_1w_0) \setminus \{u_0w_1,\, u_2w_1\}$ are colored with pairwise distinct colors, and also that all of these colors are different from the, at least, r distinct colors of the set $N(u_0v_0) = \{c(u_1w_0), \ldots, c(u_1w_{n-1})\}$. Thus, the result holds because, from Lemmas 2 and 6, $2m + r \leq \chi_r(P_l \times F_{m,n}) \leq \chi_{2m+4}(P_l \times F_{m,n}) = 2m + r$. □

3.3 Case $n < r \in \{2m+2,\, 2m+3\}$

Lemma 7. *Let $r \in \{2m+2,\, 2m+3\}$ be such that $n < r$. Then, $2m + n \leq \chi_r(P_l \times F_{m,n})$.*

Proof. Let c be an r-dynamic proper $\chi_r(P_l \times F_{m,n})$-coloring of $P_l \times F_{m,3}$. Condition (1) implies that the $2m+2$ vertices of $N(u_1w_0)$ are colored by pairwise distinct colors. Similarly, since $n < r$, the n vertices of $N(u_0v_0)$ are colored by pairwise distinct colors. From adjacency, these n colors are different from those ones in $c(N(u_0v_0) \setminus \{u_0w_1,\, u_2w_1\})$. □

The next result shows some cases for $m = 2$ in which the lower bound in Lemma 7 is reached.

Proposition 2. *Let $r \in \{6,7\}$ and $n \in \{r-2, r-1\}$. Then, $\chi_r(P_l \times F_{2,n}) = n+4$.*

Proof. Let us prove each case separately.

- **Case** $r = 6$ and $n \in \{4,5\}$.

  From Lemma 7, we have that $n + 4 \leq \chi_r(P_l \times F_{2,n})$. In order to prove that this lower bound is reached, it is enough to consider the 6-dynamic proper $(n+4)$-coloring c of the direct product $P_l \times F_{2,n}$ that is described so that, for all $i < l$, $j < 2$ and $k < n$,

$$c(u_iv_j) = \begin{cases} 4 + \left(\left(2 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4\right), & \text{if } j = 0,\\ \left(2 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4, & \text{if } j = 1.\end{cases}$$

$$c(u_iw_k) = \begin{cases} 8, & \text{if } n = 5 \text{ and } k = 2,\\ 4 + \left(\left(3 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4\right), & \text{if } i \text{ is even and } k = 0,\\ 4 + \left(\left(1 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4\right), & \text{if } i \text{ is odd and } k = 0,\\ \left\lfloor \frac{i}{2} \right\rfloor \bmod 4, & \text{if } k = 1,\\ 4 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 4\right), & \text{if } k = n-2,\\ \left(3 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4, & \text{if } i \text{ is even and } k = n-1,\\ \left(1 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 4, & \text{if } i \text{ is odd and } k = n-1.\end{cases}$$

  (Figure 3 illustrates the case $l = 8$.)

- **Case** $(n, r) = (5, 7)$.

  From Lemma 7, we have that $9 \leq \chi_7(P_l \times F_{2,5})$. Figure 4 illustrates that this lower bound is tight for all $l \in \{3,4,5\}$. It is also reached for $l = 6$, as it is illustrated in Figure 5 (left).

  In order to prove that the mentioned lower bound is also reached for all $l \geq 7$, it is enough to consider the 7-dynamic proper 9-coloring c of the direct product $P_l \times F_{2,5}$ that is described so that, for all $i < l$, $j < 2$ and $k < n$,

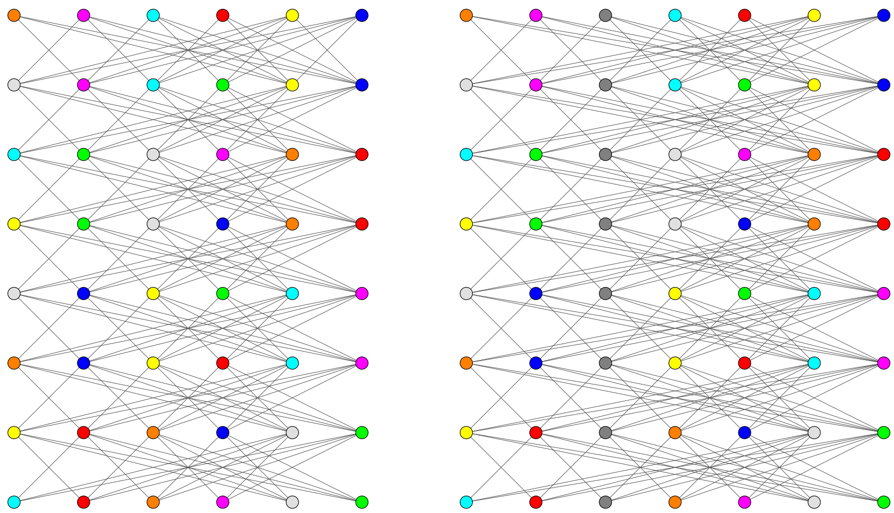

Figure 3: 6-dynamic proper 8- and 9-colorings of $P_8 \times F_{2,n}$.

$$c(u_i v_j) = \begin{cases} 7, & \text{if } (i,j) = (0,0), \\ (i+j) \bmod 9, & \text{if } (i,j) \notin \{(0,0),\, (l-1,1)\}, \\ (l+2) \bmod 9, & \text{if } (i,j) = (l-1,1). \end{cases}$$

$$c(u_i w_k) = \begin{cases} 6, & \text{if } (i,k) = (1,1), \\ i+4, & \text{if } k = 2 \text{ and } i \in \{0,1\}, \\ (l-5) \bmod 9, & \text{if } (i,k) = (l-3,4), \\ (l+1) \bmod 9, & \text{if } (i,k) = (l-2,1), \\ (l-6) \bmod 9, & \text{if } (i,k) = (l-2,4), \\ l \bmod 9, & \text{if } (i,k) \in \{(l-3,3),\, (l-2,3)\}, \\ (i-2) \bmod 9, & \text{if } k = 0, \\ (i+4) \bmod 9, & \text{if } k = 1 \text{ and } i \notin \{1, l-2\}, \\ (i-3) \bmod 9, & \text{if } k \in \{2,3\} \text{ and } (i,k) \notin \{(0,2),(1,2), \\ & \qquad (l-3,3),(l-2,3)\}, \\ (i+3) \bmod 9, & \text{if } k = 4 \text{ and } i \notin \{l-3, l-2\}. \end{cases}$$

(Figure 5 (right) illustrates the case $l = 7$.)

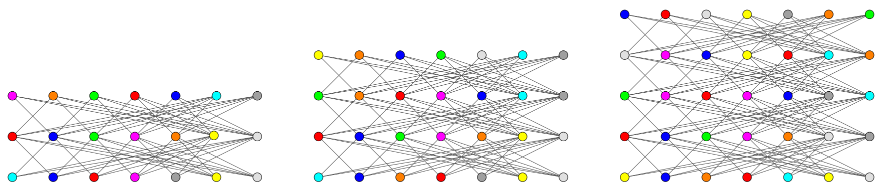

Figure 4: 7-dynamic proper 9-coloring of $P_l \times F_{2,5}$, for $l \in \{3, 4, 5\}$.

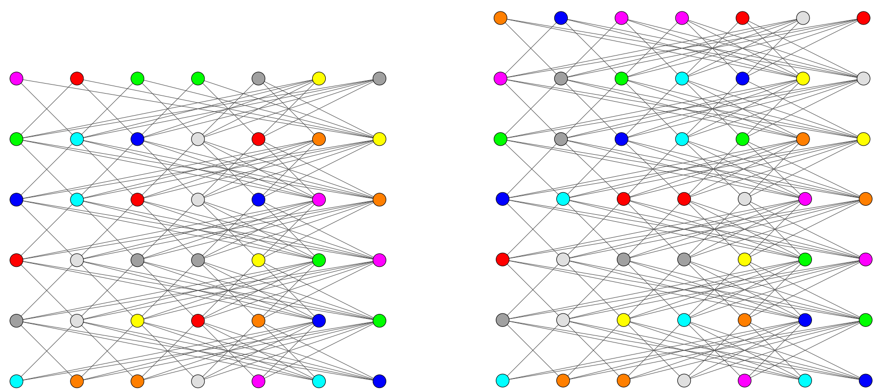

Figure 5: 7-dynamic proper 9-coloring of $P_l \times F_{2,5}$, for $l \in \{6, 7\}$.

- **Case** $(n, r) = (6, 7)$.

  From Lemma 7, $10 \leq \chi_7(P_l \times F_{2,6})$. This lower bound is reached, because of the 7-dynamic proper 10-coloring of $P_l \times F_{2,6}$ such that, for all $i < l$, $j < 2$ and $k < 6$, we have that $c(u_i v_j) = 3j + 4 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right)$ and

$$c(u_i w_k) = \begin{cases} 2k + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right), & \text{if } k \in \{0, 1\}, \\ 3k - 2 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } k \in \{2, 3\}, \\ 2k - 8 + \left(\left(1 + \left\lfloor \frac{i}{2} \right\rfloor\right) \bmod 2\right), & \text{if } k \in \{4, 5\}, \end{cases}$$

  (Figure 6 illustrates the case $l = 8$.)

$\square$

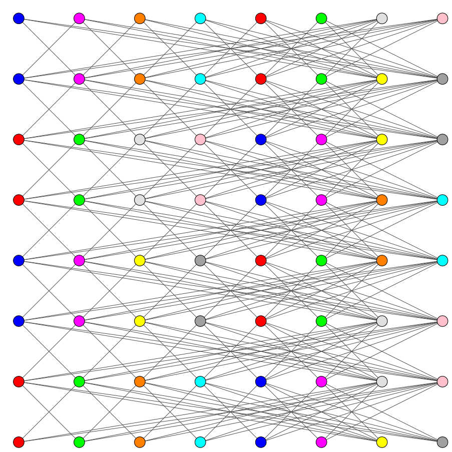

Figure 6: 7-dynamic proper 10-coloring of $P_8 \times F_{2,6}$.

The next result shows that an extra color is however required for $(m, n, r) = (2, 4, 7)$.

Proposition 3. *It is verified that* $\chi_7(P_l \times F_{2,4}) = 9$.

Proof. Let c be a 7-dynamic proper $\chi_7(P_l \times F_{2,4})$-coloring of $P_l \times F_{2,4}$. From Lemma 7, this map c requires, at least, eight distinct colors. If this lower bound were tight, then Condition (1) implies that both sets $\{c(u_1w_0), c(u_1w_1), c(u_1w_2), c(u_1w_3)\}$ and $\{c(u_0w_1), c(u_0w_2), c(u_2w_1), c(u_2w_2)\}$ would be formed by the same four pairwise distinct colors. None of these colors coincides with one of the four distinct colors of the set $\{c(u_0v_0), c(u_0v_1), c(u_2v_0), c(u_2v_1)\}$. As a consequence, two distinct colors of this last set would be required for coloring the vertices u_1v_0 and u_1v_1. But then, $|c(N(u_1v_0)| \leq 6$, which contradicts Condition (1). Hence, $9 \leq \chi_7(P_l \times F_{2,4})$. In order to prove that this lower bound is tight, it is enough to define the map c so that, for all $i < l$, $j < 2$ and $k < 4$, we have that $c(u_iv_j) = 3 + j + 2\left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right)$ and

$$c(u_iw_k) = \begin{cases} 3 + (i \bmod 6), & \text{if } k = 0, \\ \left\lfloor \frac{i}{2} \right\rfloor \bmod 3, & \text{if } k = 1, \\ \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } k = 2, \\ 4 + (i \bmod 6), & \text{if } i \text{ is even and } k = 3, \\ 2 + (i \bmod 6), & \text{if } i \text{ is odd and } k = 3. \end{cases}$$

(Figure 7 illustrates the case $l = 8$.) $\square$

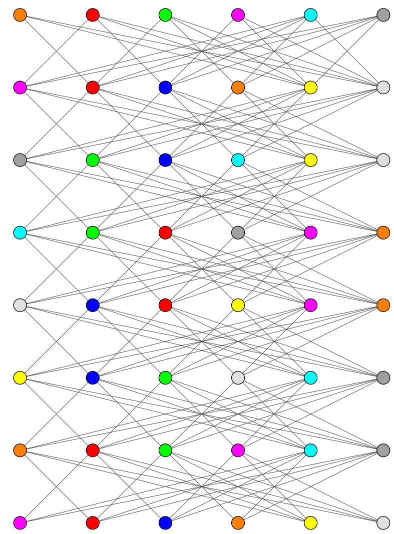

Figure 7: 7-dynamic proper 9-coloring of $P_8 \times F_{2,4}$.

Let us focus now on the case $n = 3$ and $r \in \{2m + 2,\, 2m + 3\}$.

Proposition 4. *Let* $r \in \{2m + 2,\, 2m + 3\}$*. Then,*

$$\chi_r(P_l \times F_{m,3}) = \begin{cases} 2m + 4, & \textit{if } \begin{cases} m = 1, \\ m > 1 \textit{ and } r = 2m + 2, \end{cases} \\ 2m + 5, & \textit{otherwise.} \end{cases}$$

Proof. Let c be an r-dynamic proper $\chi_r(P_l \times F_{m,3})$-coloring of $P_l \times F_{m,3}$. From the proof of Lemma 7, it requires $2m + 2$ colors for the vertices in $N(u_1 w_0)$. Moreover, adjacency implies that $u_1 w_0$ and $u_1 w_2$ require two extra colors. Hence, $2m + 4 \leq \chi_r(P_l \times F_{m,3})$. The following study of cases arises.

- **Case** $m = 1$.

  The described lower bound is reached in this case. To prove it, it is enough to consider the map c so that, for each pair of non-negative integers $i < l$ and $k < 3$, we have that $c(u_i v_0) = \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3$ and

$$c(u_i w_k) = \begin{cases} 3k + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } k \in \{0, 1\}, \\ 3 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } k = 2. \end{cases}$$

(Figure 8 (left) illustrates the case $l = 7$.)

- **Case** $m > 1$ and $r = 2m + 2$.

  Again, the lower bound is tight. To prove it, it is enough to define the map c so that, for all $i < l$, $j < m$ and $k < 3$, we have that

$$c(u_i v_j) = \begin{cases} 3j + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } j \in \{0,1\}, \\ 4 + 2j + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right), & \text{otherwise.} \end{cases}$$

  and

$$c(u_i w_k) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor \bmod 3, & \text{if } k = 0, \\ 6 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right), & \text{if } k = 1, \\ 3 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } k = 2. \end{cases}$$

  (Figure 8 (center) illustrates the case $(l, m) = (7, 3)$.)

- **Case** $m > 1$ and $r = 2m + 3$.

  Since $r > 6$, Condition (1) applied to $u_1 v_0$ implies that the set $\{c(u_0 w_0)$, $c(u_0 w_1)$, $c(u_0 w_2)$, $c(u_2 w_0)$, $c(u_2 w_1)$, $c(u_2 w_2)\}$ is formed by six distinct colors. That condition applied to the vertices $u_1 w_k$, with $k \in \{0, 1, 2\}$, also implies that at most one of these six colors belongs to the set $\{c(u_0 v_0), \ldots, c(u_0 v_{m-1}), c(u_2 v_0), \ldots, c(u_2 v_{m-1})\}$. Hence, it is $2m + 5 \leq \chi_{2m+3}(P_l \times F_{m,3})$, whenever $m > 1$. This lower bound is reached, because of the map c that is described so that, for all $i < l$, $j < m$ and $k < n$, we have that

$$c(u_i v_j) = \begin{cases} \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } j = 0, \\ 5 + 2j + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right), & \text{otherwise.} \end{cases}$$

  and

$$c(u_i w_k) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor \bmod 3, & \text{if } k = 0, \\ 1 + 2k + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right), & \text{if } k \in \{1, 2\}. \end{cases}$$

  (Figure 8 (right) illustrates the case $(l, m) = (7, 3)$.)

$\square$

Let us finish our study on the case $n < r \in \{2m + 2,\, 2m + 3\}$ by dealing with $(m, n, r) = (1, 4, 5)$.

Proposition 5. *It is verified that* $\chi_5(P_l \times F_{1,4}) = \begin{cases} 6, & \textit{if } l \neq 5, \\ 7, & \textit{if } l = 5. \end{cases}$

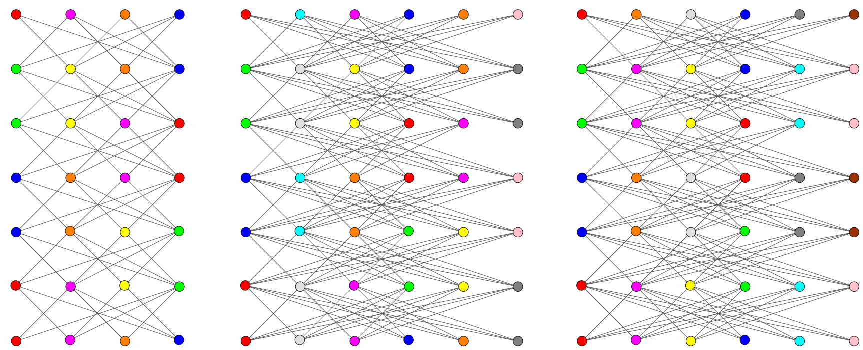

Figure 8: r-dynamic proper 6-coloring of $P_7 \times F_{1,3}$, for $r \in \{4, 5\}$ (left); 8-dynamic proper 10-coloring of $P_7 \times F_{3,3}$ (center); and 9-dynamic proper 11-coloring of $P_7 \times F_{3,3}$ (right).

Proof. Let c be a 5-dynamic proper $\chi_5(P_l \times F_{1,4})$-coloring of the direct product $P_l \times F_{1,4}$. From Lemma 7, this map c requires at least six distinct colors. Figure 9 (left and center) illustrates that the mentioned lower bound is tight whenever $l \in \{3, 4\}$. Further, a simple study by brute force enables one to ensure that no 5-dynamic proper 6-coloring exists for $l = 5$. So, a seventh color is required in that case. In fact, Figure 9 (right) illustrates that $\chi_r(P_5 \times F_{1,4}) = 7$.

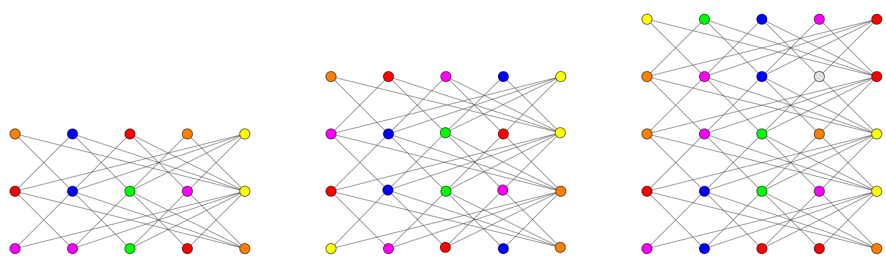

Figure 9: 5-dynamic proper ℓ-coloring of the direct product $P_l \times F_{1,4}$, for $(l, \ell) \in \{(3, 6), (4, 6), (5, 7)\}$.

In order to prove that the lower bound or six colors is also reached for all $l > 5$, it is enough to define the map c so that, for each pair of non-negative integers $i < l$ and $k < 4$,

$$c(u_iv_0) = \begin{cases} 3, & \text{if } i \bmod 6 \in \{3,4\}, \\ 4, & \text{if } i \bmod 6 \in \{0,5\}, \\ 5, & \text{otherwise.} \end{cases}$$

and

$$c(u_iw_k) =$$

$$\begin{cases} 1, & \text{if } (i,k) \in \{(0,0),\ (1,3)\}, \\ 3, & \text{if } (i,k) \in \{(0,2),\ (0,3),\ (1,0)\}, \\ c(u_{l-3}w_0), & \text{if } l \text{ is odd and } (i,k) \in \{(l-1,1),\ (l-2,3)\} \\ c(u_{l-3}w_1), & \text{if } l \text{ is even and } (i,k) \in \{(l-1,0),\ (l-2,3)\}, \\ c(u_{l-3}w_2), & \text{if } l \text{ is odd and } (i,k) \in \{(l-1,3),\ (l-2,0)\}, \\ c(u_{l-3}v_0), & \text{if } l \text{ is odd and } (i,k) = (l-1,0), \\ c(u_{l-4}v_0), & \text{if } l \text{ is even and } (i,k) \in \{(l-1,2),(l-1,3),(l-2,0)\}, \\ 3 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } 1 < i < l-2 \text{ and } k \in \{0,3\}, \\ \left\lfloor \frac{i}{2} \right\rfloor \bmod 3, & \text{if } k = 1 \text{ and } \begin{cases} l \text{ is even}, \\ l \text{ is odd and } i \neq l-1, \end{cases} \\ \left(2 + \left\lfloor \frac{i-1}{2} \right\rfloor\right) \bmod 3, & \text{if } k = 2 \text{ and } i > 0 \text{ and } \begin{cases} l \text{ is odd}, \\ l \text{ is even and } i \neq l-1, \end{cases} \end{cases}$$

(Figure 10 illustrates the case $l \in \{6,7,8\}$.) □

3.4 Case $r \leq 2m+1$

Lemma 8. *If* $r \leq 2m+1$, *then* $[\chi_r(P_l \times F_{m,n}) \leq \max\{\min\{r,\, 2n\} + \min\{r,\, m+1\} - 1,\ r + \min\{r,\, n\} - 2\}$.

Proof. Let c be an r-dynamic proper coloring of the direct product $P_l \times F_{m,n}$. From Condition (1), this map c requires at least

- $\min\{r,\, 2n\}$ distinct colors for $N(u_1v_0) = \{u_0w_0, \ldots, u_0w_{n-1},\ u_2w_0, \ldots, u_2w_{n-1}\}$, which must be pairwise different of the, at least, $\min\{r,\, m+1\} - 1$ distinct colors that are necessary for the set $N(u_0w_0) \setminus \{u_1w_1\} = \{u_1v_0, \ldots, u_1v_{m-1}\}$; and
- $r-2$ distinct colors for coloring the set $N(u_1w_0) \setminus \{u_0w_1,\ u_2w_1\} = \{u_0v_0, \ldots, u_0v_{m-1},\ u_2v_0, \ldots, u_2v_{m-1}\}$, which must be pairwise different of the, at least, $\min\{r,\, n\}$ distinct colors that are necessary for coloring the set $N(u_0v_0) = \{u_1w_0, \ldots, u_1w_{n-1}\}$.

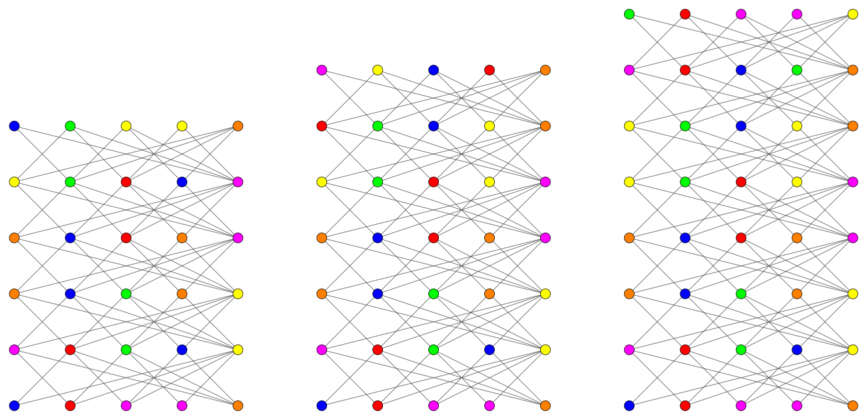

Figure 10: 5-dynamic proper 6-coloring of $P_l \times F_{1,4}$, for $l \in \{6, 7, 8\}$.

Hence, the number of required distinct colors is, at least, the maximum of these two values. $\square$

In the following results, we show how the lower bound described in Lemma 8 enables one to deal with the dynamic chromatic problem under consideration.

Proposition 6. *If $r \leq \delta(P_l \times F_{m,n})$, then*

$$\chi_r(P_l \times F_{m,n}) = \begin{cases} 2, & \text{if } r = 1, \\ 2r - 1, & \text{otherwise.} \end{cases}$$

Proof. The case $r = 1$ follows from Lemmas 2, 3 and 5 once it is noticed that $\delta(P_l) = 1$. Now, if $1 < r \leq \delta(P_l \times F_{m,n}) = \delta(F_{m,n})$, then Lemmas 4 and 5 imply that $\chi_r(P_l \times F_{m,n}) \leq 2r - 1$. This lower bound is tight because of Lemma 8. $\square$

Proposition 7. *If $\delta(P_l \times F_{m,n}) < r \leq \min\{2m + 1, n\}$, then*

$$\chi_r(P_l \times F_{m,n}) = \begin{cases} 5, & \text{if } m = 1, \\ 2r - 2, & \text{otherwise.} \end{cases}$$

Proof. If $m = 1$, then it must be $r = 3$. From Lemma 2, we have that $4 \leq \chi_3(P_l \times F_{1,n})$. Let us prove that the assumption of the existence of a

3-dynamic proper 4-coloring c of the direct product $P_l \times F_{1,n}$ gives rise to a contradiction.

Condition (1) applied to the vertex u_0v_0 implies that the set $\{c(u_1w_0), \ldots, c(u_1w_{n-1})\}$ is formed by three distinct colors and hence, $c(u_2v_0) = c(u_0v_0)$. The same condition applied to the vertex u_1w_0 implies that the set $\{c(u_1w_0), c(u_0w_1), c(u_2w_1), c(u_0v_0)\}$ is formed by the four distinct colors under consideration. Then, since c is a proper coloring, it must be $c(u_1w_2) = c(u_1w_0)$.

If $n = 3$, then it contradicts that the set $\{c(u_1w_0), c(u_1w_1), c(u_1w_2)\}$ is formed by three distinct colors. In any case, whatever the positive integer n is, it would be $|c(N(u_0w_1))| \leq 2 < 3 = d(u_0w_1)$, which contradicts Condition (1) applied to the vertex u_0w_1. Hence, $5 \leq \chi_r(P_l \times F_{1,n})$. In order to prove that this lower bound is reached, it is enough to consider the 3-dynamic proper 5-coloring c of $P_l \times F_{1,n}$ such that, for each pair of non-negative integers $i < l$ and $k < n$, we have that $c(u_iv_0) = 3 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right)$ and $c(u_iw_k) = k \bmod 3$. (Figure 11 (left) illustrates the case $(l, n, r) = (4, 4, 3)$.)

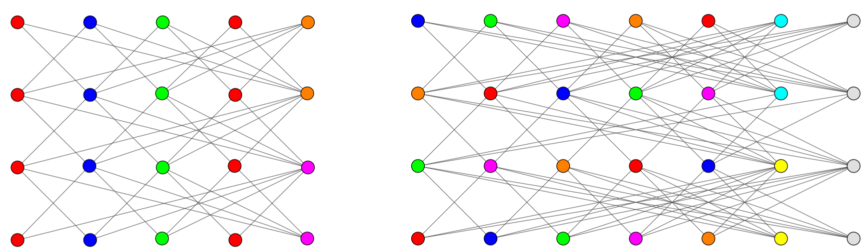

Figure 11: r-dynamic proper ℓ-coloring of the direct product $P_4 \times F_{m,n}$, for $(\ell, m, n, r) = (5, 1, 4, 3)$ and $(\ell, m, n, r) = (8, 2, 5, 5)$.

Let us focus now on the case $m > 1$. It must be $r > m + 1$ and hence, Lemma 8 implies that $2r - 2 \leq \chi_r(P_l \times F_{m,n})$. This lower bound is tight, because of the r-dynamic proper $(2r - 2)$-coloring c of the direct product $P_l \times F_{m,n}$ that is described so that, for all $i < l$, $j < m$ and $k < n$, we have that $c(u_iw_k) = (2i + k) \bmod r$ and

$$c(u_iv_j) = \begin{cases} r + m + j, & \text{if } i \bmod 4 \in \{2, 3\} \text{ and } j < r - m - 2, \\ r + j, & \text{otherwise.} \end{cases}$$

(Figure 11 (right) illustrates the case $(l, m, n, r) = (4, 2, 5, 5)$.) □

Proposition 8. *If* $n < r \leq \min\{2m+1,\, 2n\}$*, then*

$$\chi_r(P_l \times F_{m,n}) = \begin{cases} 2r-1, & \textit{if } r \leq m+1, \\ r+m, & \textit{if } n < m+1 < r, \\ r+n-2, & \textit{if } m+1 \leq n < r. \end{cases}$$

Proof. Notice that $m > 1$. Let us prove each case separately.

- **Case** $r \leq m+1$.

  From Lemma 8, we have that $\chi_r(P_l \times F_{m,n}) \geq 2r-1$. In order to prove that this lower bound is tight, it is enough to consider the r-dynamic proper $(2r-1)$-coloring c of the direct product $P_l \times F_{m,n}$ such that, for all $i < l$, $j < m$ and $k < n$, we have that $c(u_iv_j) = r + (j \bmod (r-1))$ and

  $$c(u_iw_k) = \begin{cases} n+k, & \text{if } i \bmod 4 \in \{1,2\} \text{ and } k < r-n, \\ k, & \text{otherwise.} \end{cases}$$

  (Figure 12 illustrates the case $(l,m,n,r) = (4,3,3,4)$.)

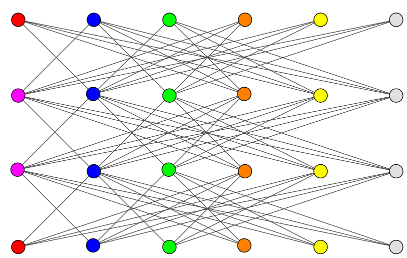

Figure 12: 4-dynamic proper 7-coloring of $P_4 \times F_{3,3}$.

- **Case** $n < m+1 < r$.

  Here, $m > 2$ and $n+2 \leq r \leq \min\{2m+1,\, 2n\} = 2n$. From Lemma 8, we have that $r + m \leq \chi_r(P_l \times F_{m,n})$. To prove that this lower bound is reached, we define an r-dynamic proper $(r+m)$-coloring c of $P_l \times F_{m,n}$. More specifically, for each pair of non-negative integers $i < l$ and $j < m$, we define

  $$c(u_iv_j) = \begin{cases} 3j + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } j < r-n, \\ r+j, & \text{otherwise.} \end{cases}$$

In addition, the following two subcases arise, for each pair of non-negative integers $i < l$ and $k < n$.

– **Subcase** $m = n = r - 2$.

$$c(u_iw_k) = \begin{cases} 6, & \text{if } k = 0 \text{ and } n > 3,\\ \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } k = 1,\\ 5 + k, & \text{if } 2 \leq k < n - 2,\\ 3 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } \begin{cases} (k, n) = (0, 3),\\ k = n - 2 > 1, \end{cases}\\ n + 3, & \text{if } k = n - 1. \end{cases}$$

(Figure 13 illustrates the case $m \in \{3, 5\}$, for $l = 5$.)

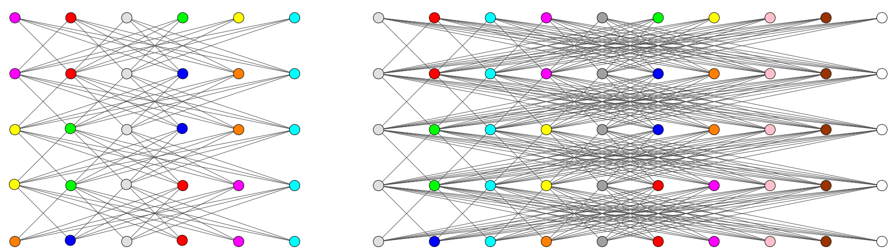

Figure 13: $(m + 2)$-dynamic proper $(2m + 2)$-coloring of the direct product $P_5 \times F_{m,m}$, for $m \in \{3, 5\}$.

– **Subcase** $r > n + 2$.

$$c(u_iw_k) =$$

$$\begin{cases} 6 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } k = 0 \text{ and } n > 3,\\ \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } k = 1,\\ 3(k+1) + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } 1 < k < \min\{n - 2, r - n - 1\},\\ 2(r - n) + k + 1, & \text{if } r - n - 1 \leq k < n - 2,\\ 3 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } \begin{cases} (k, n) = (0, 3),\\ k = n - 2 > 1, \end{cases}\\ 3(n-1) + \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } (k, r) = (n - 1, 2n),\\ 2r - n - 1, & \text{if } k = n - 1 \text{ and } r < 2n. \end{cases}$$

(Figure 14 illustrates the case $(m, n, r) \in \{(4, 3, 6), (5, 5, 8)\}$, for $l = 5$.)

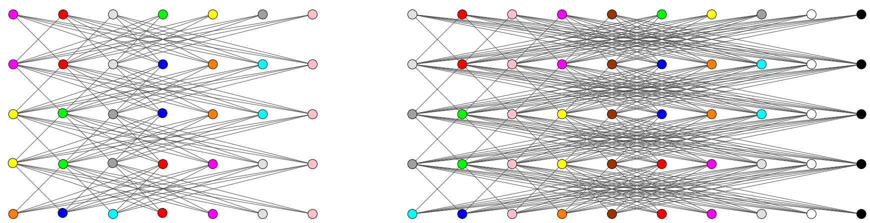

Figure 14: r-dynamic proper $(r+m)$-coloring of the direct product $P_5 \times F_{m,n}$, for $(m,n,r) \in \{(4,3,6),\ (5,5,8)\}$.

- **Case** $m+1 \leq n < r$.

  Here, $m+2 \leq r \leq \min\{2m+1,\, 2n\} = 2m+1$. From Lemma 8, $r+n-2 \leq \chi_r(P_l \times F_{m,n})$. To prove that this lower bound is reached, we define an appropriate r-dynamic proper $(r+n-2)$-coloring c of $P_l \times F_{m,n}$.

  - **Subcase** $m = 2$.

    Here, $(n,r) \in \{(3,4),\ (3,5),\ (4,5)\}$. For all $i < l$, $j < m$ and $k < n$, it is enough to define $c(u_iv_j) = \left\lfloor \frac{i}{2} \right\rfloor \bmod 3$ and

$$c(u_iw_k) = \begin{cases} 3 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2\right)(r-4), & \text{if } (k,n) = (0,3),\\ 3 + \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 3\right), & \text{if } (k,n) = (0,4),\\ \left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3, & \text{if } k = 1,\\ 3 + \left(\left(1 + \left\lfloor \frac{i+1}{2} \right\rfloor\right) \bmod 3\right), & \text{if } (k,n) = (2,4),\\ r+n-3, & \text{if } k = n-1. \end{cases}$$

    (Figure 15 illustrates the case $l = 5$.)

  - **Subcase** $m > 2$.

    Here, $r+n-2 \geq m+n \geq 6$. The map c is such that, for all $i < l$, $j < m$ and $k < n$, we have that $c(u_iw_k) = \left(2\left\lfloor \frac{i}{2} \right\rfloor + k\right) \bmod n$ and

$$c(u_iv_j) =$$

$$\begin{cases} n+j, & \text{if } \begin{cases} j < 2m-r+2,\\ j \geq 2m-r+2 \text{ and } i \bmod 4 \in \{0,1\}, \end{cases}\\ n+r-2-m+j, & \text{if } j \geq 2m-r+2 \text{ and } i \bmod 4 \in \{2,3\}. \end{cases}$$

    (Figure 16 illustrates the case $(l,m,n,r) = (5,3,4,6)$.)

$\square$

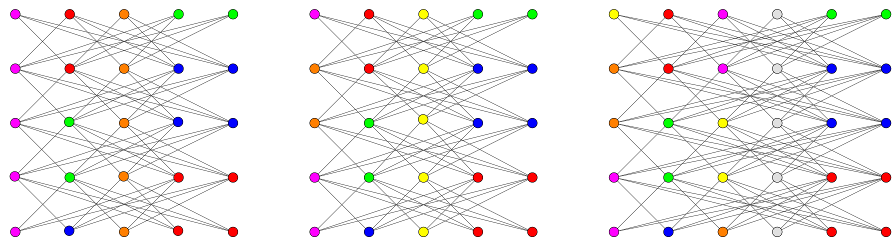

Figure 15: r-dynamic proper $(r+n-2)$-coloring of $P_5 \times F_{2,n}$, for $(n,r) \in \{(3,4),\ (3,5),\ (4,5)\}$.

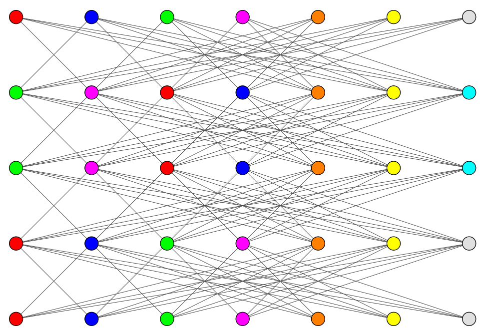

Figure 16: 6-dynamic proper 8-coloring of $P_5 \times F_{3,4}$.

Proposition 9. *If $2n < r \leq 2m+1$, then*

$$\chi_r(P_l \times F_{m,n}) = \begin{cases} 2n+r-1, & \text{if } r \leq m+1, \\ m+2n, & \text{if } m+2 \leq r < m+n+2, \\ n+r-2, & \text{if } m+n+2 \leq r \text{ and } (l,n) \in \{(3,4),\ (4,4)\}, \\ n+r-1, & \text{otherwise.} \end{cases} .$$

Proof. Here, $n \leq m$ and hence, $m \geq 3$. Let c be an r-dynamic proper $\chi_r(P_l \times F_{m,n})$-coloring of the direct product $P_l \times F_{m,n}$.

- **Case** $r \leq m+1$.

  From Lemma 8, $2n+r-1 \leq \chi_r(P_l \times F_{m,n})$. This lower bound is reached, because of the map c such that, for all $i < l$, $j < m$ and $k < n$, it is $c(u_i v_j) = 2n + (j \bmod (r-1))$ and

$$c(u_i w_k) = \begin{cases} k, & \text{if } i \bmod 4 \in \{0,1\}, \\ n+k, & \text{otherwise.} \end{cases}$$

(Figure 17 illustrates the case $(l, m, n, r) = (4, 7, 3, 7)$.)

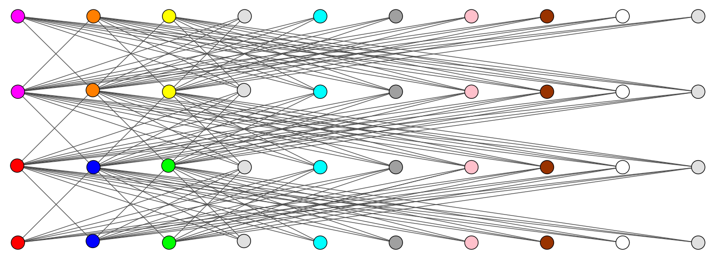

Figure 17: 7-dynamic proper 12-coloring of $P_4 \times F_{7,3}$.

- **Case** $m + 2 \leq r < m + n + 2$.

  From Lemma 8, $m + 2n \leq \chi_r(P_l \times F_{m,n})$. This lower bound is reached, because of the map c such that, for all $i < l$, $j < m$ and $k < n$, we have that

$$c(u_i v_j) = \begin{cases} n+j, & \text{if } i \bmod 6 \in \{0,5\} \text{ and } j < n, \\ j, & \text{if } i \bmod 6 \in \{3,4\} \text{ and } j < n, \\ 2n+j, & \text{otherwise.} \end{cases}$$

  and

$$c(u_i w_k) = \begin{cases} k, & \text{if } i \bmod 6 \in \{0,1\}, \\ n+k, & \text{if } i \bmod 6 \in \{2,3\}, \\ 2n+k, & \text{otherwise.} \end{cases}$$

  (Figure 18 illustrates the case $(l, m, n) = (6, 4, 3)$, for $r \in \{7, 8\}$.)

- **Case** $m + n + 2 \leq r$.

  Since $r \leq 2m + 1$ and $n \leq m$, it must be $n < m$. From Lemma 8, $n + r - 2 \leq \chi_r(P_l \times F_{m,n})$. In order to prove that this lower bound is reached for $(l, n) \in \{(3, 4), (4, 4)\}$, it is enough to define the map c so that, for all $i < l$, $j < m$ and $k < 4$,

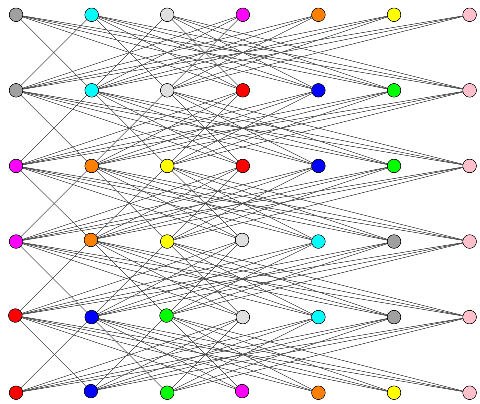

Figure 18: r-dynamic proper 11-coloring of $P_6 \times F_{4,3}$, for $r \in \{7, 8\}$.

$$c(u_iv_j) = \begin{cases} 0, & \text{if } (i,j) = (3,0), \\ 3, & \text{if } (i,j) = (3,1), \\ m+n+j, & \text{if } \begin{cases} i = 0 \text{ and } j < r-m-2, \\ i = 3 \text{ and } 1 < j < r-m-2, \end{cases} \\ 4+j, & \text{otherwise.} \end{cases}$$

$$c(u_iw_k) = \begin{cases} 0, & \text{if } (i,k) \in \{(0,2),\ (1,0)\}, \\ 1, & \text{if } (i,k) \in \{(1,1),\ (2,1)\}, \\ 2, & \text{if } (i,k) \in \{(1,2),\ (2,2)\}, \\ 3, & \text{if } (i,k) \in \{(0,1),\ (1,3)\}, \\ m+n, & \text{if } (i,k) \in \{(2,0),\ (3,2)\}, \\ m+n+1, & \text{if } (i,k) \in \{(2,3),\ (3,1)\}, \\ m+n+2, & \text{if } (i,k) \in \{(0,0),\ (3,0)\}, \\ m+n+3, & \text{if } (i,k) \in \{(0,3),\ (3,3)\}. \end{cases}$$

(Figure 19 illustrates the case $(l, m, n, r) = (4, 6, 4, 12)$.)

Now, if $(l, n) \notin \{(3, 4),\ (4, 4)\}$, then let us consider the pair of sets $S = \{c(u_1w_0), \ldots, c(u_1w_{n-1})\}$ and $S' = \{c(u_0w_0), \ldots, c(u_0w_{n-1}),\ c(u_2w_0), \ldots, c(u_2w_{n-1})\}$. If $\chi_r(P_l \times F_{m,n}) = n+r-2$, then Condition (1) applied to the vertex u_0v_0 implies that the set S is formed by n distinct colors.

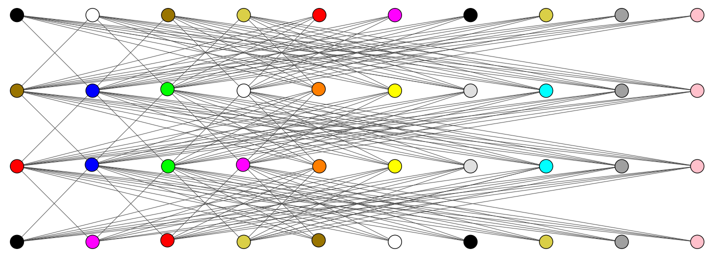

Figure 19: 12-dynamic proper 14-coloring of $P_4 \times F_{6,4}$.

The same condition applied to the vertex u_1w_0 implies that the set $\{c(u_0v_0), \ldots, c(u_0v_{m-1}),\ c(u_2v_0), \ldots, c(u_2v_{m-1})\}$ is formed by the remaining $r-2$ colors. As a consequence, for each non-negative integer $i < n$, the set $c(N(u_1w_i)) \cap S'$ must contain two distinct colors of the set S.

Furthermore, since $r > 2n$, Condition (1) applied to the vertex u_1v_0 implies that all the colors in the set S' are pairwise distinct. Hence, the map c would require at least one extra color whenever $n \neq 4$. From a similar reasoning, if $n = 4 < l$, then it must be $\{c(u_2w_1),\ c(u_2w_2)\} \in S \cap \{c(u_3w_0), \ldots, c(u_3w_{n-1})\}$. But this last intersection is empty, because of Condition (1) applied to vertex u_2v_0 and the fact that $r > 2n$. So, again, an extra color is required. In order to prove that this extra $(n+r-1)^{\text{th}}$ color is sufficient in any of the described cases, it is enough to define the map c so that, for all $i < l$, $j < m$ and $k < n$,

$$c(u_iv_j) = \begin{cases} m+j, & \text{if } \begin{cases} i \bmod 6 \in \{2,3\} \text{ and } j \leq r-m-2, \\ i \bmod 8 \in \{6,7\} \text{ and } n \leq j \leq r-m-2, \end{cases} \\ r-1+j, & \text{if } i \bmod 6 \in \{4,5\} \text{ and } j < n, \\ j, & \text{otherwise.} \end{cases}$$

and

$$c(u_iw_k) = \begin{cases} k, & \text{if } i \bmod 6 \in \{3,4\}, \\ m+k, & \text{if } i \bmod 6 \in \{0,5\}, \\ r-2+k, & \text{if } i \bmod 6 \in \{1,2\}. \end{cases}$$

(Figure 20 illustrates the case $(l,m,n,r) = (8,6,4,12)$.)

□

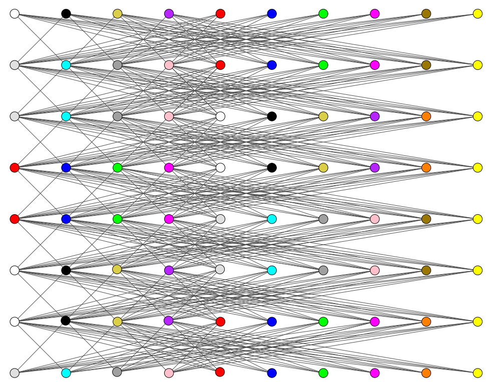

Figure 20: 12-dynamic proper 15-coloring of $P_8 \times F_{6,4}$.

4 Some exact solutions

Based on the results described in the previous section, let us finish our study by deducing some exact solutions of the r-dynamic coloring problem of direct products of paths with fan graphs. Firstly, we determine the case $r \leq 7$.

Theorem 1. *Let $r \in \{1, 2, 3\}$. Then,*

$$\chi_r(P_l \times F_{m,n}) = \begin{cases} r+1, & \text{if } r \in \{1,2\}, \\ 5, & \text{if } r = 3. \end{cases}$$

Proof. The result follows readily from Propositions 6 and 7. □

Theorem 2. *It is verified that*

$$\chi_4(P_l \times F_{m,n}) = \begin{cases} 5, & \text{if } (m,n) = (2,3), \\ 6, & \text{if } \begin{cases} m = 1, \\ m = 2 \text{ and } n \geq 4, \end{cases} \\ 7, & \text{if } \begin{cases} 4 \leq \min\{m+1,\, n\}, \\ m \geq n = 3. \end{cases} \end{cases}$$

Proof. The result holds from the following study of cases: (a) Proposition 1 proves the case $m = 1$ and $n \geq 4$; (b) Proposition 4 proves the case $(m,n) = (1,3)$; (c) Proposition 6 proves the case $4 \leq \min\{m+1,\, n\}$; (d) Proposition 7 proves the case $m = 2$ and $n \geq 4$; and (e) Proposition 8 proves the case $n = 3$ and $m > 1$. □

Theorem 3. *It is verified that*

$$\chi_5(P_l \times F_{m,n}) = \begin{cases} 6, & \textit{if} \begin{cases} (m,n) \in \{(1,3),\, (2,3)\}, \\ (m,n) = (1,4) \ \textit{and} \ l \neq 5, \end{cases} \\ 7, & \textit{if} \begin{cases} m = 1 \ \textit{and} \ n \geq 5, \\ (l,m,n) = (5,1,4), \\ (m,n) \in \{(2,4),\, (3,4)\}, \end{cases} \\ 8, & \textit{if} \begin{cases} \min\{m+1,n\} < 5 \leq \min\{2m+1,n\}, \\ m = n = 3, \end{cases} \\ 9, & \textit{if} \begin{cases} 5 \leq \min\{m+1,n\}, \\ n < 5 \leq m+1, \end{cases} \end{cases}$$

Proof. The result holds from the following study of cases: (a) Proposition 1 proves the case $m = 1$ and $n \geq 5$; (b) Proposition 4 proves the case $(m,n) = (1,3)$; (c) Proposition 5 proves the case $(m,n) = (1,4)$; (d) Proposition 6 proves the case $5 \leq \min\{m+1,n\}$; (e) Proposition 7 proves the case $\min\{m+1,n\} < 5 \leq \min\{2m+1,n\}$; and (f) Proposition 8 proves the cases $n < 5 \leq m+1$ and $(m,n) \in \{(2,3),\, (2,4),\, (3,3),\, (3,4)\}$. □

Theorem 4. *It is verified that*

$$\chi_6(P_l \times F_{m,n}) = \begin{cases} 8, & \textit{if} \begin{cases} m = 1, \\ (m,n) \in \{(2,3),\, (2,4),\, (3,4)\}, \end{cases} \\ 9, & \textit{if } (m,n) \in \{(2,5),\, (3,3),\, (3,5),\, (4,5)\}, \\ 10, & \textit{if} \begin{cases} m = 2 \ \textit{and} \ n \geq 6, \\ \min\{m+1,n\} < 6 \leq \min\{2m+1,n\}, \\ (m,n) \in \{(4,3),\, (4,4)\}, \end{cases} \\ 11, & \textit{if} \begin{cases} 6 \leq \min\{m+1,n\}, \\ n < 6 \leq m+1, \end{cases} \end{cases}$$

Proof. The result holds from the following study of cases: (a) Lemma 6 proves the case $m = 1$; (b) Proposition 1 proves the case $m = 2$ and $n \geq 6$; (c) Proposition 2 proves the case $(m,n) \in \{(2,4),\, (2,5)\}$; (d) Proposition 4 proves the case $(m,n) = (2,3)$; (e) Proposition 6 proves the case $6 \leq \min\{m+1,n\}$; (f) Proposition 7 proves the case $\min\{m+1,n\} < 6 \leq \min\{2m+1,n\}$; and (g) Proposition 8 proves the cases $n < 6 \leq m+1$ and $(m,n) \in \{(3,3),\, (3,4),\, (3,5),\, (4,3),\, (4,4),\, (4,5)\}$. □

Theorem 5. *It is verified that*

$$\chi_7(P_l \times F_{m,n}) = \begin{cases} 8, & \text{if } (m,n) = (1,3), \\ 9, & \text{if } \begin{cases} m = 1 \text{ and } n > 3, \\ (m,n) \in \{(2,3),\, (2,4),\, (2,5),\, (3,3),\, (3,4)\}, \end{cases} \\ 10, & \text{if } (m,n) \in \{(2,6),\, (4,3),\, (3,5),\, (4,5)\}, \\ 11, & \text{if } \begin{cases} m = 2 \text{ and } n \geq 7, \\ (m,n) \in \{(3,6),\, (4,4),\, (4,6),\, (5,3),\, (5,6)\}, \end{cases} \\ 12, & \text{if } \begin{cases} \min\{m+1, n\} < 7 \leq \min\{2m+1, n\}, \\ (m,n) \in \{(5,4),\, (5,5)\}, \\ n = 3 \text{ and } 7 \leq m+1, \end{cases} \\ 13, & \text{if } \begin{cases} 7 \leq \min\{m+1, n\}, \\ n < 7 \leq m+1. \end{cases} \end{cases}$$

Proof. The result holds from the following study of cases: (a) Lemma 6 proves the case $m = 1$; (b) Proposition 1 proves the case $m = 2$ and $n \geq 7$; (c) Proposition 2 proves the case $(m,n) \in \{(2,5),\, (2,6)\}$; (d) Proposition 3 proves the case $(m,n) = (2,4)$; (e) Proposition 4 proves the case $(m,n) = (2,3)$; (f) Proposition 6 proves the case $7 \leq \min\{m+1, n\}$; (g) Proposition 7 proves the case $\min\{m+1, n\} < 7 \leq \min\{2m+1, n\}$; (h) Proposition 8 proves the cases $n < 7 \leq m+1$ and $(m,n) \in \{(3,4),\, (3,5),\, (3,6),\, (4,4),\, (4,5),\, (4,6),\, (5,4),\, (5,5),\, (5,6)\}$; and (i) Proposition 9 proves the case $n = 3 \leq m \leq 5$. $\square$

Finally, we solve the dynamic chromatic problem for the direct product of a path with either a simple fan graph or a double fan graph.

Theorem 6. *Let $l > 2$, $n > 2$ and r be three positive integers. Then,*

$$\chi_r(P_l \times F_{1,n}) = \begin{cases} r+1, & \text{if } \begin{cases} r \in \{1,2\}, \\ (n,r) = (3,5), \\ (n,r) = (4,5) \text{ and } l \neq 5, \end{cases} \\ r+2, & \text{if } \begin{cases} r = 3, \\ (n,r) = (3,4), \\ (l,n,r) = (5,4,5), \\ n \geq r \in \{4,5\}, \\ 6 \leq r \leq 2n, \end{cases} \\ 2n+2, & \text{if } r > 2n. \end{cases}$$

Proof. The result holds from the following study of cases: (a) Lemma 6 proves the case $r \geq 6$; (b) Proposition 1 proves the case $n \geq r \in \{4,5\}$; (c) Proposition 4 proves the case $(n,r) \in \{(3,4),\ (3,5)\}$; (d) Proposition 5 proves the case $(n,r) = (4,5)$; and (e) Theorem 1 proves the case $r \in \{1,2,3\}$. □

Theorem 7. *Let $l > 2$, $n > 2$ and r be three positive integers. Then,*

$$\chi_r(P_l \times F_{2,n}) = \begin{cases} r+1, & \text{if } \begin{cases} r \in \{1,2\}, \\ (n,r) \in \{(3,4),\ (3,5)\} \end{cases} \\ r+2, & \text{if } \begin{cases} r = 3, \\ (n,r) \in \{(3,6),(3,7),(4,5),(4,6),(4,7),(5,7)\}, \end{cases} \\ r+3, & \text{if } (n,r) \in \{(5,6),\ (6,7)\}, \\ r+4, & \text{if } \begin{cases} n \geq r \in \{6,7\}, \\ 8 \leq r \leq 2n, \end{cases} \\ 2r-2, & \text{if } 3 < r \leq \min\{n,5\}, \\ 2n+4. & \text{if } r > 2n. \end{cases}$$

Proof. The result holds from the following study of cases: (a) Lemma 6 proves the case $r \geq 8$; (b) Proposition 1 proves the case $n \geq r \in \{6,7\}$; (c) Proposition 2 proves the case $(n,r) \in \{(4,6),\ (5,6),\ (5,7),\ (6,7)\}$; (d) Proposition 3 proves the case $(n,r) \in \{(4,7)\}$; (e) Proposition 4 proves the case $(n,r) \in \{(3,6),\ (3,7)\}$; (f) Proposition 7 proves the case $3 < r \leq \min\{n,5\}$; (g) Proposition 8 proves the case $(n,r) \in \{(3,5),\ (4,5)\}$; (h) Theorem 1 proves the case $r \in \{1,2,3\}$; and (i) Theorem 2 proves the case $(n,r) = (3,4)$. □

5 Conclusion and further work

In this paper, we have solved the r-dynamic chromatic problem of the direct product of a path P_l with a fan graph $F_{m,n}$, except for the case $n < r \in \{2m+2,\ 2m+3\}$, which, due to its difficulty, it has been solved under certain assumptions. The obtained results have been used to determine in particular the r-dynamic chromatic number of the direct product $P_l \times F_{m,n}$, for all $r \leq 7$, and also, for all $m \in \{1,2\}$. In order to deal with higher orders, the study concerning the remaining cases satisfying that $n < r \in \{2m+2,\ 2m+3\}$ is established as further work.

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References

[1] H. Agustin, D Dafik and A. Y. Harsya, *On r-dynamic coloring of some graph operation*, Int. J. Comb. **1** (2016), 22–30.

[2] A. Ahadi, S. Akbari, A. Dehghana and M. Ghanbari, *On the difference between chromatic number and dynamic chromatic number of graphs*, Discrete Math. **312** (2012), 2579–2583.

[3] S. Akbari, M. Ghanbari and S. Jahanbekam, On the dynamic chromatic number of graphs, in *Combinatorics and graphs*, Contemp. Math. **531** (2010), 11–18.

[4] S. Akbari, M. Ghanbari and S. Jahanbekam, *On the dynamic coloring of Cartesian product graphs*, Ars Comb. **114** (2014), 161–168.

[5] M. Alishahi, *On the dynamic coloring of graphs*, Discrete Appl. Math. **159** (2011), 152–156.

[6] M. Alishahi, *Dynamic chromatic number of regular graphs*, Discrete Appl. Math. **160** (2012), 2098–2103.

[7] T. Deepa, R. M. Falcón and M. Venkatachalam, *On the r-dynamic coloring of the direct product of a path with either a complete graph or a wheel graph*, AIMS Math. **6** (2021), 1470–1496.

[8] T. Deepa and M. Venkatachalam, *On r-dynamic coloring of the total graphs of gear graphs*, Appl. Math. E-Notes **18** (2018), 69–81.

[9] T. Deepa, M. Venkatachalam and R. M. Falcón, *On the r-dynamic coloring of the direct product of a path with either a path or a cycle*, AIMS Math. **5** (2020), 6496–6520.

[10] R. M. Falcón, M. Venkatachalam, S. Gowri and G. Nandini, *On the r-dynamic coloring of some fan graph families*, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat. **29** (2021), 151–181.

[11] R. M. Falcón, M. Venkatachalam, S. Gowri and G. Nandini, *On the r-dynamic coloring of the direct product of a path and a k-subdivision of a star graph*, Discrete Math. Algorithms Appl., **14** (2022), paper 2150121, 19 pages.

[12] S. Gowri, M. Venkatachalam, V. N. Mishra and L. N. Mishra, *On r-dynamic coloring of double star graph families*, Palest. J. Math. **10** (2021), 53–62.

[13] F. Harary, *Graph Theory*, Reading, Massachusetts, Addison Wesley, 1969.

[14] H. J. Lai, B. Montgomery and H. Poon, *Upper bounds of dynamic chromatic number*, Ars Comb. **68** (2003), 193–201.

[15] H. J. Lai, B. Montgomery and Z. Tao, *Conditional colorings of graphs*, Discrete Math. **306** (2006), 1997–2004.

[16] K. Kaliraj, H. N. Kumar and J. V. Vivin, *On dynamic colouring of Cartesian product of complete graph with some graphs*, J. Taibah Univ. Sci. **14** (2020), 168–171.

[17] A. I. Kristiana, M. I. Utoyo, R. Alfarisi and Dafik, *r-dynamic coloring of the corona product of graphs*, Discrete Math. Algorithms Appl. **12** (2020), 2050019.

[18] N. Mohanapriya, J. V. Vivin and M. Venkatachalam, *On dynamic coloring of fan graphs*, Int. J. Pure Appl. Math. **106** (2016), 169–174.

[19] B. Montgomery, *Dynamic coloring of graphs*, PhD Thesis, West Virginia University, ProQuest LLC, Ann Arbor, MI, 2001.

[20] G. Nandini, M. Venkatachalam and S. Gowri, *On r-dynamic coloring of the family of bistar graphs*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2019), 923–928.

[21] G. Nandini, M. Venkatachalam and R. M. Falcón, *On the r-dynamic coloring of subdivision-edge coronas of a path*, AIMS Math. **5** (2020), 4546–4562.

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