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Green-Lindsay thermoelasticity for double porous materials

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Abstract

The main purpose of this paper is to obtain new results in the thermoelasticity for double porous materials, starting from the classical theory of Green-Lindsay's elasticity. The novelty of the proposed method consists in proving a reciprocal theorem and obtaining the energy equation in the context of Green-Lindsay's thermoelasticity for double porous materials. The added value in this article is the result regarding the uniqueness of the solution of the problem with mixed data for bodies with double porosity.

1 Introduction

In the last years many researchers have shown their interest into the theory of Green and Lindsay due to the fact that it takes into consideration also the temperature rate as a constitutive variable and it permits the propagation of the waves at finite speeds. This type of theory was approached from the point of view of classical thermoelasticity [Green(1972)], thermoviscoelasticity [Aouadi(2019)], thermoelastic solid [Nieto(2018)], thermoelasticity of dipolar bodies [Marin and Craciun(2020)]. Our study uses this theory in the context of double porous thermoelastic materials.

The concept of double porosity was introduced for the first time by Barenblatt in [Barenblatt(1960)], [Barenblatt(1963)]. The materials with double porosity structure have a large number of applications in many fields as:

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biomechanics [Svanadze(2013)],[Scarpetta(2014)], seismology, geomagnetism and geodesy, [Wilson(1982)], energy production [Kumar(2016)] and also petroleum engineering [Bai(1994)], [Masters(2000)].

In the last years materials with double porosity structure were approached using various theories. Some studies have been published for the backward in time problems of these type of materials [Florea(2019)]. The asymptotic behavior was studied in [Bazarra(2019)], the theory of Moore-Gibson-Thompson was approached in the context of bodies with double porosity in [Florea(2021)], vibrations problems in the case of thermoelasticity for double porous bodies ware published in [Florea(2019)], [Svanadze(2020)]. The uniqueness theories for the thermoelastic bodies with porosity were studied in the last decays by many scholars i.e. [Svanadze(2014)].

The present study is structured as follows. In Section 2 the behavior of a body with double porosity structure using Green-Lindsay theory is described. The main results of the present study are highlighted in Section 3. Here is proved a Betti type result that establishes a reciprocity relation between two systems of external loadings. This reciprocity relation is useful in order to obtain the uniqueness results regarding the considered mixed with initial and boundary data problem in the context of double porous materials using the Green Lindsay function and the Biot's energy function.

2 Basic equations

The behavior of a body with double porosity structure using Green-Lindsay thermoelasticity is described by the following variables: $u_i(t,x)$ the displacement components, $\varphi(t,x)$, $\psi(t,x)$ the fractional volume fields corresponding to the pores and cracks, respectively and $\theta(t,x)$ the temperature.

The internal energy denoted by Ψ depends on the deformation tensors. By adding the temperature and its derivative as an independent variable, is obtained the Helmholtz energy or the so-called free energy noted by ω, which depends on the internal energy Ψ, the temperature θ and the entropy η: $\omega = \Psi - \theta\eta$.

Based on the Green-Lindsay theory the heat flux components are expressed by:

$$Q_i = -\theta_0(b_i\dot{\theta} + K_{ij}\theta_{,j}). \tag{1}$$

Knowing that in the case of linear theory the temperature difference from some basic temperature is very small, θ_0 is a constant temperature.

A material with two porosities is governed by the following equations of motion:

$$\rho \ddot{u}_i = t_{ji,j} + \rho f_i, \tag{2}$$

the balances of the equilibrated forces:

$$\begin{aligned} K_1\ddot{\varphi} &= \sigma_{j,j} + p + \rho G, \\ K_2\ddot{\psi} &= \tau_{j,j} + r + \rho L, \end{aligned} \tag{3}$$

and by the energy equation:

$$\rho\theta_0\dot{\eta} = Q_{j,j} + \rho h. \tag{4}$$

The constitutive equations are functions of the strain tensors and some constants of materials:

$$\begin{aligned} &t_{ij} = C_{ijkl}u_{k,l} + B_{ij}\varphi + D_{ij}\psi - \beta_{ij}(\theta + \alpha\dot{\theta}) = \tfrac{\partial\omega}{\partial u_{i,j}}, \\ &\sigma_i = a_{ij}\varphi_{,j} + b_{ij}\psi_{,j} = \tfrac{\partial\omega}{\partial\varphi_{,i}}, \\ &\sigma_i = b_{ij}\varphi_{,j} + \delta_{ij}\psi_{,j} = \tfrac{\partial\omega}{\partial\psi_{,i}}, \\ &p = -B_{ij}u_{i,j} - \alpha_1\varphi - \alpha_3\psi + \gamma_1(\theta + \alpha\dot{\theta}) = -\tfrac{\partial\omega}{\partial\varphi}, \\ &r = -D_{ij}u_{i,j} - \alpha_3\varphi - \alpha_2\psi + \gamma_2(\theta + \alpha\dot{\theta}) = \tfrac{\partial\omega}{\partial\psi}, \\ &\eta = \beta_{ij}u_{i,j} + \gamma_1\varphi + \gamma_2\psi + c(\theta + \alpha\dot{\theta}) = -\tfrac{\partial\omega}{\partial(\theta+\alpha\dot{\theta})}, \\ &Q_i = -\theta_0(b_i\dot{\theta} + K_{ij}\theta_{,j}) = \tfrac{\partial\omega}{\partial\theta_{,i}}. \end{aligned} \tag{5}$$

Based on the above relations (5) we will obtain the quadratic form of the Helmholtz energy:

$$\begin{aligned} \omega = \tfrac{1}{2}C_{ijkl}u_{k,l}u_{i,j} + B_{ij}\varphi u_{i,j} + D_{ij}\psi u_{i,j} - \beta_{ij}u_{i,j}(\theta + \alpha\dot{\theta}) + \\ + \tfrac{1}{2}a_{ij}\varphi_{,i}\varphi_{,j} + b_{ij}\varphi_{,i}\psi_{,j} + \tfrac{1}{2}\delta_{ij}\psi_{,i}\psi_{,j} + \tfrac{1}{2}\alpha_1\varphi^2 + \tfrac{1}{2}\alpha_2\psi^2 + \\ + \alpha_3\varphi\psi - \gamma_1\varphi(\theta + \alpha\dot{\theta}) - \gamma_2\psi(\theta + \alpha\dot{\theta}) + \\ + \tfrac{1}{2}c(\theta + \alpha\dot{\theta})^2 + b_i\theta_0\dot{\theta}\theta_{,i} + \tfrac{1}{2}\theta_0K_{ij}\theta_{,i}\theta_{,j}. \end{aligned} \tag{6}$$

If we take into account (1) and $(5)_6$ then the energy equation becomes:

$$\rho\beta_{ij}\dot{u_{i,j}} + \rho\gamma_1\dot{\varphi} + \rho\gamma_2\dot{\psi} + \rho a\dot{\theta} + \rho c\alpha\ddot{\theta} + b_i\dot{\theta_{,i}} + K_{ij}\theta_{,ij} - \frac{\rho h}{\theta_0} = 0. \tag{7}$$

We consider a three dimensional space $B \subset \mathbb{R}^3$ that is filled up by the double porous body. The boundary of the considered domain is denoted by ∂B and the normal at the domain surface has the components n_i. Therefore we may define the expression of the surface couple (α, β), the force traction (t_i) and also the flow (Q).

$$t_i = t_{ij}n_j, \quad \alpha = \sigma_i n_i, \quad \beta = \tau_i n_i, \quad Q = Q_i n_i. \tag{8}$$

The boundary conditions are:

$$\begin{array}{llll} u_i = u_i^b & \text{on} \quad \partial B_1 \times [0,\infty), & t_i = t_i^b & \text{on} \quad \partial B_1^c \times [0,\infty), \\ \varphi = \varphi^b & \text{on} \quad \partial B_2 \times [0,\infty), & \alpha = \alpha^b & \text{on} \quad \partial B_2^c \times [0,\infty), \\ \psi = \psi^b & \text{on} \quad \partial B_3 \times [0,\infty), & \beta = \beta^b & \text{on} \quad \partial B_3^c \times [0,\infty), \\ \theta = \theta^b & \text{on} \quad \partial B_4 \times [0,\infty), & Q = Q^b & \text{on} \quad \partial B_4^c \times [0,\infty), \end{array} \tag{9}$$

where u_i^b, φ^b, ψ^b, θ^b, t_i^b, α^b, β^b, Q^b are known functions. The boundary of the considered domain ∂B is divided into four subsurfaces noted by $B_i, i = \overline{1,4}$ with their complements $B_i^c, i = \overline{1,4}$ that fulfill the following conditions:

$$\partial B_i \cup B_i^c = \partial B, \quad \partial B_i \cap B_i^c = \Phi, \quad i = \overline{1,4}. \tag{10}$$

We consider that at the initial moment t_0 we have the initial conditions:

$$\begin{array}{c} u_i(0,x) = \dot{u}_i(0,x) = 0, \quad \varphi(0,x) = \dot{\varphi}(0,x) = 0, \\ \psi(0,x) = \dot{\psi}(0,x) = 0, \quad \theta(0,x) = \dot{\theta}(0,x) = 0. \end{array} \tag{11}$$

The mixed data problem for bodies with double porosity consists of equations (2), (3), (4), (5), with the initial conditions (11) and the boundary conditions (9).

The solution $(u_i, \varphi, \psi, \theta)$ of the problem with mixed data for bodies with double porosity is represented by the response of a system at the external actions. This system of external actions is defined by:

$$H = (f_i, G, L, h, u_i^b, \varphi^b, \psi^b, \theta^b, t_i^b, \alpha^b, \beta^b, Q^b).$$

and it generates a thermoelastic state defined by:

$$S = (u_i, \varphi, \psi, \theta, \varepsilon_{ij}, \chi_{ij}, t_{ij}, \sigma_i, \tau_i, p, r, Q_i, \eta).$$

The thermoelastic states corresponding to the two systems are:

$$S^{(a)} = (u_i^{(a)}, \varphi^{(a)}, \psi^{(a)}, \theta^{(a)}, \varepsilon_{ij}^{(a)}, \chi_{ij}^{(a)}, t_{ij}^{(a)}, \sigma_i^{(a)}, \tau_i^{(a)}, p^{(a)}, r^{(a)}, Q_i^{(a)}, \eta^{(a)}),$$
$$a = 1, 2.$$

Let us consider two function m and n. The convolution product is defined as follows:

$$(m * n)(t,x) = \int_0^t m(t-\zeta,x)\; n(\zeta,x)\; d\zeta,$$

respectively,

$$(m * \hat{n})(t,x) = \int_0^t m(t-\zeta,x)\; \frac{\partial n}{\partial \zeta}(\zeta,x)\; d\zeta.$$

3 Main results

The following section contains the main results of the present study. Therefore the first theorem is a result of Betti type that establishes a reciprocity relation between the two systems of external loadings. This theorem is very useful in order to obtain the next uniqueness results regarding the mixed problem for the double porous materials using the Green-Lindsay function and also the energy function of Biot.

Theorem 1. *The following reciprocal relationship occurs:*

$$\begin{aligned}
&-\int_B \rho(f_i^{(1)} * \hat{u}_i^{(2)} - f_i^{(2)} * \hat{u}_i^{(1)})\; dV+\\
&+\int_B \rho(G^{(1)} * \hat{\varphi}^{(2)} - G^{(2)} * \hat{\varphi}^{(1)})\; dV + \int_B \rho(L^{(1)} * \hat{\psi}^{(2)} - L^{(2)} * \hat{\psi}^{(1)})\; dV+\\
&+\int_{\partial B_1} (t_{ij}^{(1)} * \hat{u}_i^{b(2)} - t_{ji}^{(2)} * \hat{u}_i^{b(1)})n_j\; dA + \int_{\partial B_1^c} (t_i^{b(1)} * \hat{u}_i^{(2)} - t_i^{b(2)} * \hat{u}_i^{(1)})\; dA+\\
&+\int_{\partial B_2} (\sigma_j^{(1)} * \hat{\varphi}^{b(2)} - \sigma_j^{(2)} * \hat{\varphi}^{b(1)})n_j\; dA + \int_{\partial B_2^c} (\alpha^{b(1)} * \hat{\varphi}^{(2)} - \alpha^{b(2)} * \hat{\varphi}^{(1)})\; dA+\\
&+\int_{\partial B_3} (\tau_j^{(1)} * \hat{\psi}^{b(2)} - \tau_j^{(2)} * \hat{\psi}^{b(1)})n_j\; dA + \int_{\partial B_3^c} (\beta^{b(1)} * \hat{\psi}^{(2)} - \beta^{b(2)} * \hat{\psi}^{(1)})\; dA =\\
&= \frac{1}{\theta_0}\int_B \rho(h^{(1)} * \theta^{(2)} - h^{(2)} * \theta^{(1)})\; dV + \frac{\alpha}{\theta_0}\int_B \rho(h^{(1)} * \hat{\theta}^{(2)} - h^{(2)} * \hat{\theta}^{(1)})\; dV+\\
&+\frac{1}{\theta_0}\int_B (Q_i^{(2)} * \theta_{,i}^{(1)} - Q_i^{(1)} * \theta_{,i}^{(2)})\; dV + \frac{\alpha}{\theta_0}\int_B (Q_i^{(2)} * \hat{\theta}_{,i}^{(1)} - Q_i^{(1)} * \hat{\theta}_{,i}^{(2)})\; dV-\\
&-\frac{1}{\theta_0}\int_{\partial B_4} (Q_i^{(2)} * \theta^{b(1)} - Q_i^{(1)} * \theta^{b(2)})n_i\; dA - \frac{\alpha}{\theta_0}\int_{\partial B_4} (Q_i^{(2)} * \hat{\theta}^{b(1)} - Q_i^{(1)} * \hat{\theta}^{b(2)})n_i\; dA-\\
&-\frac{1}{\theta_0}\int_{\partial B_4^c} (Q^{b(2)} * \theta^{(1)} - Q^{b(1)} * \theta^{(2)})\; dA - \frac{\alpha}{\theta_0}\int_{\partial B_4^c} (Q^{b(2)} * \hat{\theta}^{(1)} - Q^{b(1)} * \hat{\theta}^{(2)})\; dA.
\end{aligned}$$

Proof. We apply the Laplace transform to the equations (1) - (4) and (7):

$$L[f(t,x)](s) = \tilde{f}(s,x) \int_0^\infty f(t,x)\; e^{-st}\; dt.$$

We will use the derivation property of the original and take into account the null initial conditions (11):

$$\begin{aligned}
&L[\ddot{u}_i(t,x)](s) = s^2\tilde{u}_i(s,x) - s\cdot u_i(0,x) - \dot{u}_i(0,x) = s^2\tilde{u}_i(s,x),\\
&L[\ddot{\varphi}(t,x)](s) = s^2\tilde{\varphi}(s,x) - s\cdot \varphi(0,x) - \dot{\varphi}(0,x) = s^2\tilde{\varphi}(s,x),\\
&L[\ddot{\psi}(t,x)](s) = s^2\tilde{\psi}(s,x) - s\cdot \psi(0,x) - \dot{\psi}(0,x) = s^2\tilde{\psi}(s,x).
\end{aligned}$$

Thus the equations that govern the body with double porosity structure (2), (3) can be written:

$$\begin{aligned}
&\rho s^2\tilde{u}_i^{(a)} = \tilde{t}_{ji,j}^{(a)} + \rho\tilde{f}_i^{(a)},\\
&K_1 s^2\tilde{\varphi}^{(a)} = \tilde{\sigma}_{j,j}^{(a)} + \tilde{p}^{(a)} + \rho\tilde{G}^{(a)}, \qquad a=1,2.\\
&K_2 s^2\tilde{\psi}^{(a)} = \tilde{\tau}_{j,j}^{(a)} + \tilde{r}^{(a)} + \rho\tilde{L}^{(a)}.
\end{aligned} \tag{12}$$

The energy equation (7) through the Laplace transform will have the following form:

$$\rho\beta_{ij}s\tilde{u}_{i,j}^{(a)} + \rho\gamma_1 s\tilde{\varphi}^{(a)} + \rho\gamma_2 s\tilde{\psi}^{(a)} + \rho c\alpha s^2\tilde{\theta}^{(a)} + b_i s\tilde{\theta}_{,i}^{(a)} + K_{ij}\tilde{\theta}_{,ij}^{(a)} - \frac{\rho\tilde{h}^{(a)}}{\theta_0} = 0. \tag{13}$$

The constitutive equations (5) through the Laplace transform become:

$$\begin{aligned}
&\tilde{t}_{ij}^{(a)} = C_{ijkl}\tilde{u}_{k,l}^{(a)} + B_{ij}\tilde{\varphi}^{(a)} + D_{ij}\tilde{\psi}^{(a)} - \beta_{ij}(\tilde{\theta}^{(a)} + \alpha s\tilde{\theta}^{(a)}),\\
&\tilde{\sigma}_i^{(a)} = a_{ij}\tilde{\varphi}_{,j}^{(a)} + b_{ij}\tilde{\psi}_{,j}^{(a)},\\
&\tilde{\tau}_i^{(a)} = b_{ij}\tilde{\varphi}_{,j}^{(a)} + \delta_{ij}\tilde{\psi}_{,j}^{(a)},\\
&\tilde{p}^{(a)} = -B_{ij}\tilde{u}_{i,j}^{(a)} - \alpha_1\tilde{\varphi}^{(a)} - \alpha_3\tilde{\psi}^{(a)} + \gamma_1(\tilde{\theta}^{(a)} + \alpha s\tilde{\theta}^{(a)}),\\
&\tilde{r}^{(a)} = -D_{ij}\tilde{u}_{i,j}^{(a)} - \alpha_3\tilde{\varphi}^{(a)} - \alpha_2\tilde{\psi}^{(a)} + \gamma_2(\tilde{\theta}^{(a)} + \alpha s\tilde{\theta}^{(a)}),\\
&\tilde{\eta}^{(a)} = \beta_{ij}\tilde{u}_{i,j}^{(a)} + \gamma_1\tilde{\varphi}^{(a)} + \gamma_2\tilde{\psi}^{(a)} + c(\tilde{\theta}^{(a)} + \alpha s\tilde{\theta}^{(a)}).
\end{aligned} \tag{14}$$

The image of the heat flux (1) through the Laplace transform will be:

$$\tilde{Q}_i^{(a)} = -\theta_0(b_i s\tilde{Q}^{(a)} + K_{ij}\tilde{Q}_{,j}^{(a)}), \quad a=1,2. \tag{15}$$

The image of the boundary conditions (9) through the Laplace transform will be:

$$\begin{aligned}
&\tilde{u}_i^{(a)} = \tilde{u}_i^{b(a)} \quad \text{on} \quad \partial B_1\times[0,\infty), \quad \tilde{t}_i = \tilde{t}_i^{b(a)} \quad \text{on} \quad \partial B_1^c\times[0,\infty),\\
&\tilde{\varphi}^{(a)} = \tilde{\varphi}^{b(a)} \quad \text{on} \quad \partial B_2\times[0,\infty), \quad \tilde{\alpha} = \tilde{\alpha}^{b(a)} \quad \text{on} \quad \partial B_2^c\times[0,\infty),\\
&\tilde{\psi}^{(a)} = \tilde{\psi}^{b(a)} \quad \text{on} \quad \partial B_3\times[0,\infty), \quad \tilde{\beta} = \tilde{\beta}^{b(a)} \quad \text{on} \quad \partial B_3^c\times[0,\infty),\\
&\tilde{\theta}^{(a)} = \tilde{\theta}^{b(a)} \quad \text{on} \quad \partial B_4\times[0,\infty), \quad \tilde{Q} = \tilde{Q}^{b(a)} \quad \text{on} \quad \partial B_4^c\times[0,\infty), \qquad a=1,2,
\end{aligned} \tag{16}$$

where with b higher index was noted the value on the boundary. The relation $(12)_1$ for the two systems of loads is written as follows:

$$\begin{aligned}\rho s^2 \tilde{u}_i^{(1)} \tilde{u}_i^{(2)} &= \tilde{t}_{ji,j}^{(1)} \tilde{u}_i^{(2)} + \rho \tilde{f}_i^{(1)} \tilde{u}_i^{(2)},\\ \rho s^2 \tilde{u}_i^{(2)} \tilde{u}_i^{(1)} &= \tilde{t}_{ji,j}^{(2)} \tilde{u}_i^{(1)} + \rho \tilde{f}_i^{(2)} \tilde{u}_i^{(1)}.\end{aligned}$$

Performing the subtraction between the last two relations and integrating on B we have:

$$\int_B \rho(\tilde{f}_i^{(1)} \tilde{u}_i^{(2)} - \tilde{f}_i^{(2)} \tilde{u}_i^{(1)})\, dV = \int_B (\tilde{t}_{ji}^{(1)} \tilde{u}_i^{(2)} - \tilde{t}_{ji}^{(2)} \tilde{u}_i^{(1)})_{,j}\, dV + \int_B (\tilde{t}_{ji}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{t}_{ji}^{(1)} \tilde{u}_{i,j}^{(2)})\, dV. \tag{17}$$

For the first integral of (17) we apply the divergence theorem and for the second integral of (17) we take into account the constitutive equations (14). We obtain:

$$\begin{aligned}&\int_B (\tilde{t}_{ji}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{t}_{ji}^{(1)} \tilde{u}_{i,j}^{(2)})\, dV =\\ &\quad= \int_B \Big[C_{ijkl} \tilde{u}_{k,l}^{(2)} + B_{ij} \tilde{\varphi}^{(2)} + D_{ij} \tilde{\psi}^{(2)} + \beta_{ij}(\tilde{\theta}^{(2)} + \alpha s \tilde{\theta}^{(2)})\Big] \cdot \tilde{u}_{i,j}^{(1)} -\\ &\quad- \Big[C_{ijkl} \tilde{u}_{k,l}^{(1)} + B_{ij} \tilde{\varphi}^{(1)} + D_{ij} \tilde{\psi}^{(1)} + \beta_{ij}(\tilde{\theta}^{(1)} + \alpha s \tilde{\theta}^{(1)})\Big] \cdot \tilde{u}_{i,j}^{(2)}\, dV =\\ &\quad= \int_B [B_{ij}(\tilde{\varphi}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{\varphi}^{(1)} \tilde{u}_{i,j}^{(2)}) + D_{i,j}(\tilde{\psi}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{\psi}^{(1)} \tilde{u}_{i,j}^{(2)}) -\\ &\quad- \beta_{ij}(\tilde{\theta}^{(2)} + \alpha s \tilde{\theta}^{(2)}) \tilde{u}_{i,j}^{(1)} + \beta_{ij}(\tilde{\theta}^{(1)} + \alpha s \tilde{\theta}^{(1)}) \tilde{u}_{i,j}^{(2)}]\, dV.\end{aligned} \tag{18}$$

Therefore, the relationship (17) becomes:

$$\begin{aligned}&\int_B \rho(\tilde{f}_i^{(1)} \tilde{u}_i^{(2)} - \tilde{f}_i^{(2)} \tilde{u}_i^{(1)})\, dV =\\ &\quad= \int_{\partial B_1} (\tilde{t}_{ji}^{(1)} \tilde{u}_i^{b(2)} - \tilde{t}_{ji}^{(2)} \tilde{u}_i^{b(1)}) n_j\, dA + \int_{\partial B_1^c} (\tilde{t}_i^{b(1)} \tilde{u}_i^{(2)} - \tilde{t}_i^{b(2)} \tilde{u}_i^{(1)})\, dA +\\ &\quad+ \int_B [B_{ij}(\tilde{\varphi}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{\varphi}^{(1)} \tilde{u}_{i,j}^{(2)}) + D_{ij}(\tilde{\psi}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{\psi}^{(1)} \tilde{u}_{i,j}^{(2)}) -\\ &\qquad- \beta_{ij}(1 + \alpha s)(\tilde{\theta}^{(2)} \tilde{u}_{i,j}^{(1)} - \tilde{\theta}^{(1)} \tilde{u}_{i,j}^{(2)})]\, dV.\end{aligned} \tag{19}$$

We will proceed in an analogous way for the relation $(12)_2$ for the two systems of loads. Performing the subtraction between the two obtained relations and integrating on B we have:

$$\begin{aligned}&\int_B \rho(\tilde{G}^{(1)}\tilde{\varphi}^{(2)} - \tilde{G}^{(2)}\tilde{\varphi}^{(1)})\, dV = \\ &\qquad = \int_B (\tilde{\sigma}_j^{(2)}\tilde{\varphi}^{(1)} - \tilde{\sigma}_j^{(1)}\tilde{\varphi}^{(2)})_{,j}\, dV + \int_B (\tilde{\sigma}_j^{(1)}\tilde{\varphi}_{,j}^{(2)} - \tilde{\sigma}_j^{(2)}\tilde{\varphi}_{,j}^{(1)})\, dV + \\ &\qquad + \int_B (\tilde{p}^{(2)}\tilde{\varphi}^{(1)} - \tilde{p}^{(1)}\tilde{\varphi}^{(2)})\, dV.\end{aligned} \tag{20}$$

In the integrals from the relation (20) we apply the divergence theorem and take into account the constitutive equations. Therefore the relation (20) will have the following form:

$$\begin{aligned}&\int_B \rho(\tilde{G}^{(1)}\tilde{\varphi}^{(2)} - \tilde{G}^{(2)}\tilde{\varphi}^{(1)})\, dV = \\ &\qquad = \int_{\partial B_2^c} (\tilde{\alpha}^{b(2)}\tilde{\varphi}^{(1)} - \tilde{\alpha}^{b(1)}\tilde{\varphi}^{(2)})\, dA + \int_{\partial B_2} (\tilde{\sigma}_j^{(2)}\tilde{\varphi}^{b(1)} - \tilde{\sigma}_j^{(1)}\tilde{\varphi}^{b(2)}) n_j\, dA + \\ &\qquad + \int_B b_{ij}(\tilde{\psi}_{,j}^{(1)}\tilde{\varphi}_{,j}^{(2)} - \tilde{\psi}_{,j}^{(2)}\tilde{\varphi}_{,j}^{(1)})\, dV + \int_B [-B_{ij}(\tilde{u}_{i,j}^{(2)}\tilde{\varphi}^{(1)} - \tilde{u}_{i,j}^{(1)}\tilde{\varphi}^{(2)}) - \\ &\qquad - \alpha_3(\tilde{\psi}^{(2)}\tilde{\varphi}^{(1)} - \tilde{\psi}^{(1)}\tilde{\varphi}^{(2)}) + \gamma_1(1+\alpha s)(\tilde{\varphi}^{(1)}\tilde{\theta}^{(2)} - \tilde{\varphi}^{(2)}\tilde{\theta}^{(1)})]\, dV.\end{aligned} \tag{21}$$

For the third equation from (12) we perform the subtraction between the relations of the two loading systems and by integration on B we obtain:

$$\begin{aligned}&\int_B \rho(\tilde{L}^{(1)}\tilde{\psi}^{(2)} - \tilde{L}^{(2)}\tilde{\psi}^{(1)})\, dV = \\ &\qquad = \int_B (\tilde{\tau}_j^{(2)}\tilde{\psi}^{(1)} - \tilde{\tau}_j^{(1)}\tilde{\psi}^{(2)})_{,j}\, dV + \int_B (\tilde{\tau}_j^{(1)}\tilde{\psi}_{,j}^{(2)} - \tilde{\tau}_j^{(2)}\tilde{\psi}_{,j}^{(1)})\, dV + \\ &\qquad + \int_B (\tilde{r}^{(2)}\tilde{\psi}^{(1)} - \tilde{r}^{(1)}\tilde{\psi}^{(2)})\, dV.\end{aligned} \tag{22}$$

Using the divergence theorem and considering the constitutive equations the relation (22) becomes:

$$\begin{aligned}&\int_B \rho(\tilde{L}^{(1)}\tilde{\psi}^{(2)} - \tilde{L}^{(2)}\tilde{\psi}^{(1)})\, dV = \\ &\qquad = \int_{\partial B_3^c} (\tilde{\beta}^{b(2)}\tilde{\psi}^{(1)} - \tilde{\beta}^{b(1)}\tilde{\psi}^{(2)})\, dA + \int_{\partial B_3} (\tilde{\tau}_j^{(2)}\tilde{\psi}^{b(1)} - \tilde{\tau}_j^{(1)}\tilde{\psi}^{b(2)}) n_j\, dA + \\ &\qquad + \int_B b_{ij}(\tilde{\varphi}_{,j}^{(1)}\tilde{\psi}_{,j}^{(2)} - \tilde{\varphi}_{,j}^{(2)}\tilde{\psi}_{,j}^{(1)})\, dV + \int_B D_{ij}(\tilde{u}_{i,j}^{(1)}\tilde{\psi}^{(2)} - \tilde{u}_{i,j}^{(2)}\tilde{\psi}^{(1)}) + \\ &\qquad + \alpha_3(\tilde{\varphi}^{(1)}\tilde{\psi}^{(2)} - \tilde{\varphi}^{(2)}\tilde{\psi}^{(1)}) + \gamma_2(1+\alpha s)(\tilde{\psi}^{(1)}\tilde{\theta}^{(2)} - \tilde{\psi}^{(2)}\tilde{\theta}^{(1)})\, dV.\end{aligned} \tag{23}$$

The last step into obtaining the results of the theorem 1 is to write the energy equation (13) for the two loads and to repeat the same procedure as above using the subtraction of the two relations written for both loading systems and by integration on B we obtain:

$$\begin{aligned}\int_B \rho\beta_{ij}s\left(\tilde{u}_{i,j}^{(1)}\tilde{\theta}^{(2)}-\tilde{u}_{i,j}^{(2)}\tilde{\theta}^{(1)}\right)+\rho\gamma_2 s\left(\tilde{\psi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\psi}^{(2)}\tilde{\theta}^{(1)}\right)+\\ +\rho\gamma_1 s\left(\tilde{\varphi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\varphi}^{(2)}\tilde{\theta}^{(1)}\right)+b_i s\left(\tilde{\theta}_{,i}^{(1)}\tilde{\theta}^{(2)}-\tilde{\theta}_{,i}^{(2)}\tilde{\theta}^{(1)}\right)+\\ +K_{ij}\left(\tilde{\theta}_{,ij}^{(1)}\tilde{\theta}^{(2)}-\tilde{\theta}_{,ij}^{(2)}\tilde{\theta}^{(1)}\right)-\tfrac{\rho}{\theta_0}\left(\tilde{h}^{(1)}\tilde{\theta}^{(2)}-\tilde{h}^{(2)}\tilde{\theta}^{(1)}\right)\,dV=0.\end{aligned} \tag{24}$$

We apply the divergence theorem and take into account the boundary conditions (16) and the equation (15). The relation (24) becomes:

$$\begin{aligned}\int_B \rho\beta_{ij}s\left(\tilde{u}_{i,j}^{(1)}\tilde{\theta}^{(2)}-\tilde{u}_{i,j}^{(2)}\tilde{\theta}^{(1)}\right)+\rho\gamma_1 s\left(\tilde{\varphi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\varphi}^{(2)}\tilde{\theta}^{(1)}\right)-\\ -\tfrac{\rho}{\theta_0}\left(\tilde{h}^{(1)}\tilde{\theta}^{(2)}-\tilde{h}^{(2)}\tilde{\theta}^{(1)}\right)+\rho\gamma_2 s\left(\tilde{\psi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\psi}^{(2)}\tilde{\theta}^{(1)}\right)\,dV+\\ +\tfrac{1}{\theta_0}\int_{\partial B_4^c}\left(\tilde{\theta}^{(1)}\tilde{Q}^{b(2)}-\tilde{\theta}^{(2)}\tilde{Q}^{b(1)}\right)\,dA+\tfrac{1}{\theta_0}\int_{\partial B_4}\left(\tilde{\theta}^{b(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}^{b(2)}\tilde{Q}_i^{(1)}\right)n_i\,dA-\\ -\tfrac{1}{\theta_0}\int_B\left(\tilde{\theta}_{,i}^{(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}_{,i}^{(2)}\tilde{Q}_i^{(1)}\right)\,dV=0.\end{aligned} \tag{25}$$

We notice that in (25) there are terms from (19), (21) and (23).
Thus, from (19) we obtain:

$$\begin{aligned}\int_B \rho\beta_{ij}s\left(\tilde{\theta}^{(2)}\tilde{u}_{i,j}^{(1)}-\tilde{\theta}^{(1)}\tilde{u}_{i,j}^{(2)}\right)\,dV=\\ [\int_{\partial B_1}\left(\tilde{t}_{ji}^{(1)}\tilde{u}_i^{b(2)}-\tilde{t}_{ji}^{(2)}\tilde{u}_i^{b(1)}\right)n_j\,dA+\int_{\partial B_1^c}\left(\tilde{t}_i^{b(1)}\tilde{u}_i^{(2)}-\tilde{t}_i^{b(2)}\tilde{u}_i^{(1)}\right)\,dA+\\ +\int_B\left[B_{ij}\left(\tilde{\varphi}^{(2)}\tilde{u}_{i,j}^{(1)}-\tilde{\varphi}^{(1)}\tilde{u}_{i,j}^{(2)}\right)+D_{ij}\left(\tilde{\psi}^{(2)}\tilde{u}_{i,j}^{(1)}-\tilde{\psi}^{(1)}\tilde{u}_{i,j}^{(2)}\right)\right]\,dV-\\ -\int_B\rho\left(\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}-\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}\right)\,dV]\cdot\tfrac{\rho s}{1+\alpha s}.\end{aligned} \tag{26}$$

From (21) we obtain:

$$\begin{aligned}\int_B \rho\gamma_1 s\left(\tilde{\varphi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\varphi}^{(2)}\tilde{\theta}^{(1)}\right)\,dV=\\ =[\int_B\rho\left(\tilde{G}^{(1)}\tilde{\varphi}^{(2)}-\tilde{G}^{(2)}\tilde{\varphi}^{(1)}\right)\,dV-\int_{\partial B_2^c}\left(\tilde{\alpha}^{b(2)}\tilde{\varphi}^{(1)}-\tilde{\alpha}^{b(1)}\tilde{\varphi}^{(2)}\right)\,dA-\\ -\int_{\partial B_2}\left(\tilde{\sigma}_j^{(2)}\tilde{\varphi}^{b(1)}-\tilde{\sigma}_j^{(1)}\tilde{\varphi}^{b(2)}\right)n_j\,dA-\int_B b_{ij}\left(\tilde{\psi}_{,j}^{(1)}\tilde{\varphi}_{,j}^{(2)}-\tilde{\psi}_{,j}^{(2)}\tilde{\varphi}_{,j}^{(1)}\right)\,dV+\\ +\int_B\left[B_{ij}\left(\tilde{u}_{i,j}^{(2)}\tilde{\varphi}^{(1)}-\tilde{u}_{i,j}^{(1)}\tilde{\varphi}^{(2)}\right)+\alpha_3\left(\tilde{\psi}^{(2)}\tilde{\varphi}^{(1)}-\tilde{\psi}^{(1)}\tilde{\varphi}^{(2)}\right)\right]\,dV]\cdot\tfrac{\rho s}{1+\alpha s}.\end{aligned} \tag{27}$$

From (23) we obtain:

$$
\begin{aligned}
&\int_B \rho\gamma_2 s\left(\tilde{\psi}^{(1)}\tilde{\theta}^{(2)}-\tilde{\psi}^{(2)}\tilde{\theta}^{(1)}\right)\,dV=\\
&=[\int_B \rho\left(\tilde{L}^{(1)}\tilde{\psi}^{(2)}-\tilde{L}^{(2)}\tilde{\psi}^{(1)}\right)\,dV-\int_{\partial B_3^c}\left(\tilde{\beta}^{b(2)}\tilde{\psi}^{(1)}-\tilde{\beta}^{b(1)}\tilde{\psi}^{(2)}\right)\,dA-\\
&-\int_{\partial B_3}\left(\tilde{\tau}_j^{(2)}\tilde{\psi}^{b(1)}-\tilde{\tau}_j^{(1)}\tilde{\psi}^{b(2)}\right)n_j\,dA-\int_B b_{ij}\left(\tilde{\varphi}_{,j}^{(1)}\tilde{\psi}_{,j}^{(2)}-\tilde{\varphi}_{,j}^{(2)}\tilde{\psi}_{,j}^{(1)}\right)\,dV-\\
&-\int_B\left[D_{ij}\left(\tilde{u}_{i,j}^{(1)}\tilde{\psi}^{(2)}-\tilde{u}_{i,j}^{(2)}\tilde{\psi}^{(1)}\right)+\alpha_3\left(\tilde{\varphi}^{(1)}\tilde{\psi}^{(2)}-\tilde{\varphi}^{(2)}\tilde{\psi}^{(1)}\right)\right]\,dV]\cdot\frac{\rho s}{1+\alpha s}.
\end{aligned}
\tag{28}
$$

By replacing the relations (26)-(28) in the relation (25), we obtain:

$$
\begin{aligned}
&\frac{\rho s}{1+\alpha s}\cdot[\int_{\partial B_1}\left(\tilde{t}_{ij}^{(1)}\tilde{u}_i^{b(2)}-\tilde{t}_{ji}^{(2)}\tilde{u}_i^{b(1)}\right)n_j\,dA+\int_{\partial B_1^c}\left(\tilde{t}_i^{b(1)}\tilde{u}_i^{(2)}-\tilde{t}_i^{b(2)}\tilde{u}_i^{(1)}\right)\,dA+\\
&+\int_B\left[B_{ij}\left(\tilde{\varphi}^{(2)}\tilde{u}_{i,j}^{(1)}-\tilde{\varphi}^{(1)}\tilde{u}_{i,j}^{(2)}\right)+D_{ij}\left(\tilde{\psi}^{(2)}\tilde{u}_{i,j}^{(1)}-\tilde{\psi}^{(1)}\tilde{u}_{i,j}^{(2)}\right)-\right.\\
&\left.-\rho\left(\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}-\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}\right)\right]\,dV+\\
&+\int_B\rho\left(\tilde{G}^{(1)}\tilde{\varphi}^{(2)}-\tilde{G}^{(2)}\tilde{\varphi}^{(1)}\right)\,dV-\int_{\partial B_2^c}\left(\tilde{\alpha}^{b(2)}\tilde{\varphi}^{(1)}-\tilde{\alpha}^{b(1)}\tilde{\varphi}^{(2)}\right)\,dA-\\
&-\int_{\partial B_2}\left(\tilde{\sigma}_j^{(2)}\tilde{\varphi}^{b(1)}-\tilde{\sigma}_j^{(1)}\tilde{\varphi}^{b(2)}\right)n_j\,dA-\int_B b_{ij}\left(\tilde{\psi}_{,j}^{(1)}\tilde{\varphi}_{,j}^{(2)}-\tilde{\psi}_{,j}^{(2)}\tilde{\varphi}_{,j}^{(1)}\right)\,dV+\\
&+\int_B\left[B_{ij}\left(\tilde{u}_{i,j}^{(2)}\tilde{\varphi}^{(1)}-\tilde{u}_{i,j}^{(1)}\tilde{\varphi}^{(2)}\right)+\alpha_3\left(\tilde{\psi}^{(2)}\tilde{\varphi}^{(1)}-\tilde{\psi}^{(1)}\tilde{\varphi}^{(2)}\right)\right]\,dV+\\
&+\int_B\rho\left(\tilde{L}^{(1)}\tilde{\psi}^{(2)}-\tilde{L}^{(2)}\tilde{\psi}^{(1)}\right)\,dV-\int_{\partial B_3^c}\left(\tilde{\beta}^{b(2)}\tilde{\psi}^{(1)}-\tilde{\beta}^{b(1)}\tilde{\psi}^{(2)}\right)\,dA-\\
&-\int_{\partial B_3}\left(\tilde{\tau}_j^{(2)}\tilde{\psi}^{b(1)}-\tilde{\tau}_j^{(1)}\tilde{\psi}^{b(2)}\right)n_j\,dA-\int_B b_{ij}\left(\tilde{\varphi}_{,j}^{(1)}\tilde{\psi}_{,j}^{(2)}-\tilde{\varphi}_{,j}^{(2)}\tilde{\psi}_{,j}^{(1)}\right)\,dV-\\
&-\int_B\left[D_{ij}\left(\tilde{u}_{i,j}^{(1)}\tilde{\psi}^{(2)}-\tilde{u}_{i,j}^{(2)}\tilde{\psi}^{(1)}\right)+\alpha_3\left(\tilde{\varphi}^{(1)}\tilde{\psi}^{(2)}-\tilde{\varphi}^{(2)}\tilde{\psi}^{(1)}\right)\right]\,dV]=\\
&=\int_B\frac{\rho}{\theta_0}\left(\tilde{h}^{(1)}\tilde{\theta}^{(2)}-\tilde{h}^{(2)}\tilde{\theta}^{(1)}\right)\,dV-\frac{1}{\theta_0}\int_{\partial B_4^c}\left(\tilde{\theta}^{(1)}\tilde{Q}^{b(2)}-\tilde{\theta}^{(2)}\tilde{Q}^{b(1)}\right)\,dA-\\
&-\frac{1}{\theta_0}\int_{\partial B_4}\left(\tilde{\theta}^{b(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}^{b(2)}\tilde{Q}_i^{(1)}\right)n_i\,dA+\frac{1}{\theta_0}\int_B\left(\tilde{\theta}_{,i}^{(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}_{,i}^{(2)}\tilde{Q}_i^{(1)}\right)\,dV,
\end{aligned}
$$

which leads to:

$$\begin{aligned}
\rho s\cdot [-\int_B \rho\left(\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}-\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}\right)\,dV+\int_B \rho\left(\tilde{G}^{(1)}\tilde{\varphi}^{(2)}-\tilde{G}^{(2)}\tilde{\varphi}^{(1)}\right)\,dV+\\
+\int_B \rho\left(\tilde{L}^{(1)}\tilde{\psi}^{(2)}-\tilde{L}^{(2)}\tilde{\psi}^{(1)}\right)\,dV+\\
+\int_{\partial B_1}\left(\tilde{t}_{ij}^{(1)}\tilde{u}_i^{b(2)}-\tilde{t}_{ji}^{(2)}\tilde{u}_i^{b(1)}\right)n_j\,dA+\int_{\partial B_1^c}\left(\tilde{t}_i^{b(1)}\tilde{u}_i^{(2)}-\tilde{t}_i^{b(2)}\tilde{u}_i^{(1)}\right)\,dA+\\
+\int_{\partial B_2}\left(\tilde{\sigma}_j^{(1)}\tilde{\varphi}^{b(2)}-\tilde{\sigma}_j^{(2)}\tilde{\varphi}^{b(1)}\right)n_j\,dA+\int_{\partial B_2^c}\left(\tilde{\alpha}^{b(1)}\tilde{\varphi}^{(2)}-\tilde{\alpha}^{b(2)}\tilde{\varphi}^{(1)}\right)\,dA+\\
+\int_{\partial B_3}\left(\tilde{\tau}_j^{(1)}\tilde{\psi}^{b(2)}-\tilde{\tau}_j^{(2)}\tilde{\psi}^{b(1)}\right)n_j\,dA+\int_{\partial B_3^c}\left(\tilde{\beta}^{b(1)}\tilde{\psi}^{(2)}-\tilde{\beta}^{b(2)}\tilde{\psi}^{(1)}\right)\,dA]=\\
=(1+\alpha s)\cdot\frac{1}{\theta_0}[\int_B \rho\left(\tilde{h}^{(1)}\tilde{\theta}^{(2)}-\tilde{h}^{(2)}\tilde{\theta}^{(1)}\right)\,dV+\int_B\left(\tilde{\theta}_{,i}^{(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}_{,i}^{(2)}\tilde{Q}_i^{(1)}\right)\,dV-\\
-\int_{\partial B_4}\left(\tilde{\theta}^{b(1)}\tilde{Q}_i^{(2)}-\tilde{\theta}^{b(2)}\tilde{Q}_i^{(1)}\right)n_i\,dA-\int_{\partial B_4^c}\left(\tilde{\theta}^{(1)}\tilde{Q}^{b(2)}-\tilde{\theta}^{(2)}\tilde{Q}^{b(1)}\right)\,dA].
\end{aligned} \tag{29}$$

We will apply the inverse Laplace transform to the relation (29) and we will take into account the convolution product. We will also take into account:

$$s\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}=\tilde{f}_i^{(1)}\cdot s\cdot\tilde{u}_i^{(2)}=L\left[f_i^{(1)}\right]\cdot L\left[\frac{\partial u_i^{(2)}}{\partial t}\right]=f_i^{(1)}*\hat{u}_i^{(2)},$$

where $\hat{u}_i^{(2)}=\frac{\partial u_i^{(2)}}{\partial t}$. The result of Theorem 1 is obtained immediately. □

In order to obtain a uniqueness result for the mixed problem with initial values and boundary values considered above, we will use the generalized free energy function, ω, proposed by Biot as follows:

$$\omega=\Psi-\eta\theta_0, \tag{30}$$

which leads to the fact that the internal energy, Ψ, has the following expression:

$$\Psi=\omega+\eta\theta_0.$$

Green and Lindsay introduce the scalar function:

$$\phi=\theta_0+\theta+\alpha\dot{\theta}+\beta\theta\dot{\theta}+\frac{1}{2}\gamma\dot{\theta}^2,$$

and the expression of the energy function is:

$$\xi=\Psi-\eta\phi=\omega+\eta\theta_0-\eta\theta_0-\eta\theta-\eta\alpha\dot{\theta}-\eta\beta\theta\dot{\theta}-\frac{1}{2}\eta\gamma\dot{\theta}^2. \tag{31}$$

Taking into account the constitutive equations the energy function can be written in the following form:

$$\begin{aligned}\xi \; &= \tfrac{1}{2}C_{ijkl}u_{k,l}u_{i,j} + B_{ij}\varphi u_{i,j} + D_{ij}\psi u_{i,j} + \tfrac{1}{2}a_{ij}\varphi_{,i}\varphi_{,j} + b_{ij}\varphi_{,i}\psi_{,j} + \\ &+ \tfrac{1}{2}\delta_{ij}\psi_{,i}\psi_{,j} + \tfrac{1}{2}\alpha_1\varphi^2 + \tfrac{1}{2}\alpha_2\psi^2 + \alpha_3\varphi\psi + b_i\theta_0\dot{\theta}\theta_{,i} + \tfrac{1}{2}\theta_0 K_{ij}\theta_{,i}\theta_{,j} - \\ &- a(\theta + \alpha\dot{\theta}) - \tfrac{c}{2}\theta^2 - e\theta\dot{\theta} - f\dot{\theta}^2 - c\beta\theta^2\dot{\theta} - c\alpha\beta\theta\dot{\theta}^2 - \tfrac{1}{2}c\gamma\theta\dot{\theta}^2 - \tfrac{1}{2}c\alpha\gamma\dot{\theta}^3,\end{aligned} \tag{32}$$

where:

$$\begin{aligned}&a = 2(\beta_{ij}u_{i,j} + \gamma_1\varphi + \gamma_2\psi),\\ &e = (\beta_{ij}u_{i,j} + \gamma_1\varphi + \gamma_2\psi)\beta + c\alpha = \tfrac{a\beta}{2} + c\alpha,\\ &f = (\beta_{ij}u_{i,j} + \gamma_1\varphi + \gamma_2\psi)\tfrac{\gamma}{2} + \tfrac{c\alpha^2}{2} = \tfrac{a\gamma}{4} + \tfrac{c\alpha^2}{2}.\end{aligned}$$

Taking into account the initial conditions, the quadratic form of the energy function is obtained according to Biot:

$$\begin{aligned}\omega = \tfrac{1}{2}C_{ijkl}u_{k,l}u_{i,j} + B_{ij}\varphi u_{i,j} + D_{ij}\psi u_{i,j} + \tfrac{1}{2}a_{ij}\varphi_{,i}\varphi_{,j} + b_{ij}\varphi_{,i}\psi_{,j} + \\ + \tfrac{1}{2}\delta_{ij}\psi_{,i}\psi_{,j} + \tfrac{1}{2}\alpha_1\varphi^2 + \tfrac{1}{2}\alpha_2\psi^2 + \alpha_3\varphi\psi + b_i\theta_0\dot{\theta}\theta_{,i} + \tfrac{1}{2}\theta_0 K_{ij}\theta_{,i}\theta_{,j} + \\ + \tfrac{1}{2}c\theta^2 + c\alpha\theta\dot{\theta} + \tfrac{\alpha^2 c}{2}\dot{\theta}^2.\end{aligned} \tag{33}$$

Theorem 2. *The energy equation in the context of Green-Lindsay thermoelasticity for double porosity bodies has the following form:*

$$\begin{aligned}\tfrac{d}{dt}\int_B (\xi_c + \omega)\ dV = \\ = \rho\int_B \left(f_i\dot{u}_i + G\dot{\varphi} + L\dot{\psi} + \tfrac{h}{\theta_0}\left(\theta + \alpha\dot{\theta}\right)\right)\ dV + \\ + \int_{\partial B}\left(t_{ji}\dot{u}_i + \sigma_j\dot{\varphi} + \tau_j\dot{\psi}\right)n_i\ dA + \\ + \tfrac{1}{\rho}\int_{\partial B}\left[\tfrac{Q_i}{\theta_0}\left(\theta + \alpha\dot{\theta}\right) + Q_i\dot{\theta}\right]n_i\ dA + \int_{\partial B} b_i\theta_0\theta\ddot{\theta}n_i\ dA.\end{aligned} \tag{34}$$

Proof. We consider the kinetic energy of the body with double porosity:

$$\xi_c = \frac{1}{2}\rho\left[\dot{u}_i(t)\right]^2 + \frac{1}{2}K_1\left[\dot{\varphi}(t)\right]^2 + \frac{1}{2}K_2\left[\dot{\psi}(t)\right]^2, \tag{35}$$

whose derivative in relation to time is:

$$\frac{d\xi_c}{dt} = \rho\dot{u}_i(t)\ddot{u}_i(t) + K_1\dot{\varphi}(t)\ddot{\varphi}(t) + K_2\dot{\psi}(t)\ddot{\psi}(t). \tag{36}$$

Multiplying by $\dot{u}_i(t), \dot{\phi}(t)$ and $\dot{\psi}(t)$ the equations (2) , $(3)_1$ and $(3)_2$, respectively, we have:

$$\begin{aligned}
&\rho\ddot{u}_i\dot{u}_i = t_{ji,j}\dot{u}_i + \rho f_i\dot{u}_i,\\
&K_1\ddot{\varphi}\dot{\varphi} = \sigma_{j,j}\dot{\varphi} + p\dot{\varphi} + \rho G\dot{\varphi},\\
&K_2\ddot{\psi}\dot{\psi} = \tau_{j,j}\dot{\psi} + r\dot{\psi} + \rho L\dot{\psi}.
\end{aligned} \tag{37}$$

We replace (36) in (37) and we have:

$$\begin{aligned}
\tfrac{d\xi_c}{dt} &= t_{ji,j}\dot{u}_i + \rho f_i\dot{u}_i + \sigma_{j,j}\dot{\varphi} + p\dot{\varphi} + \rho G\dot{\varphi} + \tau_{j,j}\dot{\psi} + r\dot{\psi} + \rho L\dot{\psi} =\\
&= \rho f_i\dot{u}_i + \rho G\dot{\varphi} + \rho L\dot{\psi} + p\dot{\varphi} + r\dot{\psi} + (t_{ji}\dot{u}_i)_{,j} - t_{ji}\dot{u}_{i,j} +\\
&+ (\sigma_j\dot{\varphi})_{,j} - \sigma_j\dot{\varphi}_{,j} + (\tau_j\dot{\psi})_{,j} - \tau_j\dot{\psi}_{,j}.
\end{aligned} \tag{38}$$

We will integrate (38) on the domain B and apply the divergence theorem:

$$\begin{aligned}
\tfrac{d}{dt}\int_B \xi_c(t)\, dV &= \int_B \left(\rho f_i\dot{u}_i + \rho G\dot{\varphi} + \rho L\dot{\psi} + p\dot{\varphi} + r\dot{\psi}\right)\, dV -\\
&- \int_B \left(t_{ji}\dot{u}_{i,j} + \sigma_j\dot{\varphi}_{,j} + \tau_j\dot{\psi}_{,j}\right)\, dV +\\
&+ \int_{\partial B} t_{ji}\dot{u}_i n_i\, dA + \int_{\partial B} \sigma_j\dot{\varphi}n_i\, dA + \int_{\partial B} \tau_j\dot{\psi}n_i\, dA =\\
&= \int_B \left(\rho f_i\dot{u}_i + \rho G\dot{\varphi} + \rho L\dot{\psi}\right)\, dV +\\
&+ \int_B [-B_{ij}u_{i,j}\dot{\varphi} - \alpha_1\varphi\dot{\varphi} - \alpha_3\psi\dot{\varphi} + \gamma_1\left(\theta + \alpha\dot{\theta}\right)\dot{\varphi} -\\
&- D_{ij}u_{i,j}\dot{\psi} - \alpha_3\varphi\dot{\psi} - \alpha_2\psi\dot{\psi} + \gamma_2\left(\theta + \alpha\dot{\theta}\right)\dot{\psi} -\\
&- C_{ijkl}u_{k,l}\dot{u}_{i,j} - B_{ij}\varphi\dot{u}_{i,j} - D_{ij}\psi\dot{u}_{i,j} + \beta_{ij}\left(\theta + \alpha\dot{\theta}\right)\dot{u}_{i,j} -\\
&- a_{ij}\varphi_{,j}\dot{\varphi}_{,j} - b_{ij}\psi_{,j}\dot{\varphi}_{,j} - b_{ij}\varphi_{,j}\dot{\psi}_{,j} - \delta_{ij}\psi_{,j}\dot{\varphi}_{,j}]\, dV +\\
&+ \int_{\partial B} t_{ji}\dot{u}_i n_i\, dA + \int_{\partial B} \sigma_j\dot{\varphi}n_i\, dA + \int_{\partial B} \tau_j\dot{\psi}n_i\, dA.
\end{aligned} \tag{39}$$

We derive (33) in relation to time:

$$\begin{aligned}
\tfrac{d}{dt}\omega &= C_{ijkl}u_{k,l}\dot{u}_{i,j} + B_{ij}\varphi\dot{u}_{i,j} + B_{ij}\dot{\varphi}u_{i,j} + D_{ij}\dot{\psi}u_{i,j} + D_{ij}\psi\dot{u}_{i,j} +\\
&+ a_{ij}\varphi_{,i}\dot{\varphi}_{,j} + b_{ij}\dot{\varphi}_{,i}\psi_{,j} + \delta_{ij}\psi_{,i}\dot{\psi}_{,j} + \alpha_1\varphi\dot{\varphi} + \alpha_2\psi\dot{\psi} + \alpha_3\varphi\dot{\psi} +\\
&+ b_i\theta_0\ddot{\theta}\theta_{,i} + b_i\theta_0\dot{\theta}\dot{\theta}_{,i} + \theta_0 K_{ij}\theta_{,i}\dot{\theta}_{,j} + c\theta\dot{\theta} + m\dot{\theta}^2 + m\theta\ddot{\theta} + m\alpha\dot{\theta}\ddot{\theta},
\end{aligned} \tag{40}$$

where: $m = c\alpha$.

We will add (39) to (40) and we obtain:

$$\begin{aligned}\frac{d}{dt}\int_B (\xi_c+\omega)\ dV = \int_B \left(\rho f_i \dot{u}_i + \rho G\dot{\varphi} + \rho L\dot{\psi}\right)\ dV + \\ +\int_B \left(\gamma_1\dot{\varphi} + \gamma_2\dot{\psi} + \beta_{ij}\dot{u}_{i,j}\right)\left(\theta + \alpha\dot{\theta}\right)\ dV + \\ +\int_B b_i\theta_0\ddot{\theta}\theta_{,i} + b_i\theta_0\dot{\theta}\dot{\theta}_{,i} + \theta_0 K_{ij}\theta_{,i}\dot{\theta}_{,j} + c\theta\dot{\theta} + c\alpha\dot{\theta}^2 + c\alpha\theta\ddot{\theta} + c\alpha^2\dot{\theta}\ddot{\theta}\ dV + \\ +\int_{\partial B} t_{ji}\dot{u}_i n_i\ dA + \int_{\partial B} \sigma_j\dot{\varphi}n_i\ dA + \int_{\partial B} \tau_j\dot{\psi}n_i\ dA.\end{aligned} \tag{41}$$

We take into account (1) and we obtain:

$$\theta_0\left(b_i\dot{\theta} + K_{ij}\theta_{,i}\right)\dot{\theta}_{,j} = -Q_i\dot{\theta}_{,j}. \tag{42}$$

From (7) we obtain:

$$\beta_{ij}\dot{u}_{i,j} + \gamma_1\dot{\varphi} + \gamma_2\dot{\psi} = \frac{h}{\theta_0} - c\dot{\theta} - c\alpha\ddot{\theta} - \frac{1}{\rho}b_i\dot{\theta}_{,i} - \frac{1}{\rho}K_{ij}\theta_{,ij}. \tag{43}$$

From the above relations (41)-(43) we have:

$$\begin{aligned}\frac{d}{dt}\int_B (\xi_c+\omega)\ dV = \rho\int_B \left(f_i\dot{u}_i + G\dot{\varphi} + L\dot{\psi} + \frac{h}{\theta_0}\left(\theta + \alpha\dot{\theta}\right)\right)\ dV + \\ +\int_{\partial B}\left(t_{ji}\dot{u}_i + \sigma_j\dot{\varphi} + \tau_j\dot{\psi}\right)n_i\ dA + \frac{1}{\rho}\int_{\partial B}\frac{Q_i}{\theta_0}\left(\theta + \alpha\dot{\theta}\right)n_i\ dA + \\ +\int_{\partial B} b_i\theta_0\ddot{\theta}\theta n_i\ dA + \int_{\partial B} Q_i\dot{\theta}n_i\ dA.\end{aligned} \tag{44}$$

□

Theorem 3. *The mixed problem for double porous bodies admits a unique solution if the energy function of Biot, ω, from (2.5) is positively defined.*

Proof. We consider that the mixed problem admits two solutions: $S_1 = (u_i^{(1)}, \varphi^{(1)}, \psi^{(1)}, \theta^{(1)})$ and $S_2 = (u_i^{(2)}, \varphi^{(2)}, \psi^{(2)}, \theta^{(2)})$.

The difference between the two solutions is also a solution of the mixed problem for double porous bodies: $S_1 = (u_i^{(1)} - u_i^{(2)}, \varphi^{(1)} - \varphi^{(2)}, \psi^{(1)} - \psi^{(2)}, \theta^{(1)} - \theta^{(2)})$.

We notice that this difference leads to zero loads: $f_i \equiv 0$, $G \equiv 0$, $L \equiv 0$.

If we take into account the zero conditions on the boundary then the energy equation (2.15) is reduced to:

$$\frac{d}{dt}\int_B (\xi_c+\omega)\ dV = \int_{\partial B}\left(\frac{Q_i}{\rho\theta_0}\left(\theta + \alpha\dot{\theta}\right) + \frac{Q_i}{\rho}\dot{\theta} + b_i\theta_0\theta\ddot{\theta}n_i\right)\ dA.$$

We take into account (1) and we have: $\frac{d}{dt}\int\limits_B (\xi_c + \omega)\ dV \leq 0$ for $\forall t \geq 0$.

If we take into account the initial conditions (11) at the moment $t = 0$ we have: $\xi_c = \omega = 0$.

But as ξ_c and ω are positively defined it follows that $\xi_c = \omega = 0, \forall t \geq 0$.

From this we deduce that the difference S is zero, so the problem admits a unique solution.

□

4 Conclusions

The first main result of this paper is a reciprocity relation between two systems of external loadings. This reciprocal theorem is a very important Betty-type result. Another useful result is the energy equation in the context of Green-Lindsay thermoelasticity for double porous materials. Both the reciprocal relation and the energy equation are used, along with the Green-Lindsay function and Biots energy function, to prove the uniqueness of the solution of the considered mixed with initial and boundary values problem in the context of materials with double porosity structure.

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Declarations

No potential conflict of interest was reported by the authors.

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