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# On (1,2)-absorbing primary ideals and uniformly primary ideals with order $\leq 2$

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**Abstract**

This paper introduces a subset of the set of 1-absorbing primary ideals introduced in [3]. An ideal $I$ of a ring $R$ is (1,2)-absorbing primary if, whenever non-unit elements $\alpha, \beta, \gamma \in R$ with $\alpha\beta\gamma \in I$, then $\alpha\beta \in I$ or $\gamma^2 \in I$. The introduced notion is related to uniformly primary ideals introduced in [5]. The first main objective of this paper is to compare (1,2)-absorbing primary ideals with uniformly primary ideals with order less than or equal 2, as well as to characterize them in many classes of rings. The second part of this paper characterizes, by using (1,2)-absorbing primary ideals, the rings $R$ for which all ideals lie between $\mathrm{N}(R)$ (the nil-radical of $R$) and $\mathrm{N}(R)_2$.

## 1 Introduction

Throughout this paper, $R$ is a commutative with unit ($\neq 0$) and $I$ is a proper ideal of $R$ (that is, $I \neq R$). Let $\sqrt{I}$ , $\mathrm{Id}(R)$, $\mathrm{Id}(R)^*$, $\mathrm{N}(R)$, $\mathrm{Spec}(R)$, $\mathrm{Prim}(R)$, and $\mathrm{Max}(R)$ denote the radical of $I$, the set of proper ideals $R$, the set of nonzero proper ideals $R$, the nil-radical of $R$, the set of all prime ideals of $R$, the set of primary ideals of $R$, and the set of all maximal ideals of $R$, respectively.

Primary ideals are one of the most important tools of commutative algebra and algebraic geometry. In [5], the authors defined uniformly primary ideals.

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$I$ is said to be uniformly primary if there exists an integer $n \geq 0$ such that, whenever $\alpha, \beta \in R$ with $\alpha\beta \in I$, then $\alpha \in I$ or $\beta^n \in I$. The order of $I$ is the smallest positive integer such that the above property holds. Prime ideals are just uniformly primary ideals with order 1. Thus, the order of a uniformly primary ideal $I$ measures how far away $I$ is from being prime. An interesting characterization of uniformly primary ideals is given by [5, Proposition 8]. Recently, in [3], Badawi and Yetkin have introduced an important generalization of primary ideals. $I$ is called 1-absorbing primary (1-AP for short) if, for each non-unit elements $\alpha, \beta, \gamma \in R$, $\alpha\beta\gamma \in I$ implies $\alpha\beta \in I$ or $\gamma \in \sqrt{I}$. It is clear that every primary ideal is 1-AP. Moreover, if the rings is non-local the two notions coincide ([3, Corollary 1]). However, over local rings, the two concepts are different in general ([3, Example 3])

This paper focus on uniformly primary ideals with order $\leq 2$. For the order one, the absorbing version of prime ideals is defined and studied in [12] as follows: $I$ is called 1-absorbing prime if, for each non-unit elements $\alpha, \beta, \gamma \in R$, $\alpha\beta\gamma \in I$ implies $\alpha\beta \in I$ or $\gamma \in I$. So, in this paper, we define the absorbing version of uniformly primary ideal of order $\leq 2$ as follows: $I$ is said to be (1,2)-absorbing primary ((1,2)-AP for short) if, for each non-unit elements $\alpha, \beta, \gamma \in R$, $\alpha\beta\gamma \in I$ implies $\alpha\beta \in I$ or $\gamma^2 \in I$.

A characterization of (1,2)-AP ideals is given at the beginning of section 2 . Indeed, it is proved that $I$ is (1,2)-AP if and only if $I$ is 1-AP with $\sqrt{I} = \{x \in R \mid x^2 \in I\}$, or equivalently $I$ is 1-AP and $\left(\sqrt{I}\right)_2 \subseteq I$ (Theorem 2). We show also that uniformly primary ideals with ord $\leq 2$ are (1,2)-AP and that, over non-local rings, the two notions are the same. Proposition 7 gives us a simple way to construct examples of (1,2)-AP ideals that are not primary. Theorem 9 characterizes local Noetherian rings over which all (1,2)-AP ideals are uniformly primary with order $\leq 2$. It is obvious that every prime ideal is (1,2)-AP. However, Noetherian rings over which every (1,2)-AP ideal is prime must be von Neumann regular (Theorem 10). We describe explicitly in Theorem 11 (resp. Theorem 12) the (1,2)-AP ideals, which are exactly the uniformly primary ideals with order $\leq 2$, in rings whose non-zero prime ideals are maximal (resp. principle ideal rings). It is proved in Theorem 13 that a Noetherian domain that is not a field is a Dedekind domain if and only if the only non-zero (1,2)-AP ideals are $P$ and $P^2$ with $P$ prime if and only if the only non-zero uniformly primary ideal with ord $\leq 2$ are $P$ and $P^2$ with $P$ prime. Proposition 15 uses the product of ideals to characterizes the (1,2)-AP ideals. At the end of Section 2, we study the behavior of (1,2)-AP ideals over certain ring extensions. Namely, quotient ring, localization of a ring, and polynomial ring. Section 3 is devoted to characterize rings over which every (non-zero) proper ideal is (1,2)-AP (resp. uniformly primary with order $\leq 2$).

## 2 (1,2)-absorbing primary ideals vs uniformly primary with order $\leq 2$

Throughout, $R$ is a commutative with unit ($\neq 0$) and $\mathfrak{I}$ is a proper ideal of $R$.

**Definition 1.** *$I$ is said to be (1,2)-absorbing primary ((1,2)-AP for short) if, whenever non-unit elements $\alpha, \beta, \gamma \in R$, $\alpha\beta\gamma \in I$ implies $\alpha\beta \in I$ or $\gamma^2 \in I$. The set of (1,2)-AP ideals of $R$ is denoted $(1,2) - AP(R)$.*

Let $I_2$ denotes the ideal generated by the squares of elements of $I$; $I_2 = \big(a^2 \mid a \in I\big)$ ([2]).

The first result of this section gives a characterization of (1,2)-AP ideals by comparing them with 1-AP ideals.

**Theorem 2.** *The following are equivalent:*

1. *$I$ is (1,2)-AP.*
2. *$I$ is 1-AP and $\left(\sqrt{I}\right)_2 \subseteq I$.*
3. *$I$ is 1-AP and $\sqrt{I} = \big\{x \in R \mid x^2 \in I\big\}$.*

*Consequently, if $I$ is (1,2)-AP then $\sqrt{I}$ is prime.*

*Proof.* (1) $\Rightarrow$ (2) Since every (1,2)-AP is 1-AP, it suffices to prove that $\left(\sqrt{I}\right)_2 \subseteq I$. Consider $x \in \sqrt{I}$ and let $n$ be the smallest integer such that $x^n \in I$. Suppose that $n \geq 3$. Since $I$ is (1,2)-AP and $xx^{n-2}x = x^n \in I$, we obtain that $xx^{n-2} = x^{n-1} \in I$ or $x^2 \in I$, a contradiction. Hence, $n \leq 2$, and so $x^2 \in I$, as desired.
(2) $\Rightarrow$ (1) Let $\alpha, \beta, \gamma \in R - U(R)$ s such that $\alpha\beta\gamma \in I$ and $\alpha\beta \notin I$. Since $I$ is 1-AP, we get that $\gamma \in \sqrt{I}$. Thus, $\gamma^2 \in \left(\sqrt{I}\right)_2 \subseteq I$. Consequently, $I$ is (1,2)-AP.
(2) $\Leftrightarrow$ (3) It suffices to see that $\left(\sqrt{I}\right)_2 \subseteq I$ if and only if $\sqrt{I} = \big\{x \in R \mid x^2 \in I\big\}$.
The last statement follows from [3, Theorem 2]. $\square$

Clearly, every (1,2)-AP is 1-AP. However, a 1-AP ideal $I$ need not satisfy $\left(\sqrt{I}\right)_2 \subseteq I$, and so need not be (1,2)-AP. Indeed, the zero ideal of $\mathbb{Z}/8\mathbb{Z}$ is primary since $\sqrt{(\overline{0})} = (\overline{2})$ is maximal Hence, $(\overline{0})$ is 1-AP. However, $\left(\sqrt{(\overline{0})}\right)_2 = (\overline{4}) \nsubseteq (\overline{0})$.
We recall the following definition.

**Definition 3** ([5]). *$I$ is said to be uniformly primary if there exists an integer $n \geq 0$ such that, whenever $\alpha, \beta \in R$ with $\alpha\beta \in I$, then $\alpha \in I$ or $\beta^n \in I$. The order of $I$ is the smallest positive integer such that the above property holds. Let $\mathrm{U}-\mathrm{Prim}(R)$ denotes the set of uniformly primary ideals of $R$ and, for a given positive integer $n$, let $\mathrm{U}-\mathrm{Prim}(R)_{\leq n}$ denotes the set of uniformly primary ideals of $R$ of order $\leq n$.*

Using [5, Proposition 8] and its proof, we conclude the following characterization of uniformly primary ideals with ord $\leq 2$.

**Proposition 4.** *The following are equivalent:*

1. $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.
2. $I \in \mathrm{Prim}(R)$ *and* $\sqrt{I} = \{x \in R \mid x^2 \in I\}$.
3. $I \in \mathrm{Prim}(R)$ *and* $\left(\sqrt{I}\right)_2 \subseteq I$.

Since primary ideals are 1-AP, comparing Theorem 2 and Proposition 4, we conclude that uniformly primary ideals with ord $\leq 2$ are (1,2)-AP. Moreover, using [3, Corollary 1], we deduce that over a non-local ring, the two notions coincide. Accordingly, we have the following.

**Corollary 5.** *Suppose that $R$ is non-local. Then, the following are equivalent:*

1. $I \in (1,2)-AP(R)$.
2. $I \in \mathrm{Prim}(R)$ *and* $\left(\sqrt{I}\right)_2 \subseteq I$.
3. $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.

Let $S$ and $T$ be two rings. It is known that

$$\mathrm{Prim}(S \times T) = (\mathrm{Prim}(S) \times T) \cup (S \times \mathrm{Prim}(T)).$$

Let $I \in \mathrm{Id}(S)$ and $K \in \mathrm{Id}(T)$. We have, $\left(\sqrt{I \times T}\right)_2 \subseteq I \times T$ $\left(\text{resp. } \left(\sqrt{S \times K}\right)_2 \subseteq S \times K\right)$ if and only if $\left(\sqrt{I}\right)_2 \subseteq I$ $\left(\text{resp. } \left(\sqrt{K}\right)_2 \subseteq K\right)$. Therefore, given Corollary 5, we have the following result.

**Corollary 6.** *Let $S$ and $T$ be two rings and $J$ a proper ideal of $R := S \times T$. Then, the following are equivalent:*

1. $J \in (1,2)-AP(R)$.
2. $J \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.

3. *$J = I \times T$ for some $I \in \mathrm{U} - \mathrm{Prim}(S)_{\leq 2}$ or $J = S \times K$ for some $K \in \mathrm{U} - \mathrm{Prim}(T)_{\leq 2}$.*

**Proposition 7.** *Suppose that $(R, M)$ is local and let $P \in \mathrm{Spec}(R)$ . Then,*

1. *$PM$ is (1,2)-AP. Moreover, $PM$ is primary if and only if $P = M$ or $PM = P$.*
2. *In particular, if $(0) \neq P = (x) \neq M$ for some $x \in R$, then $PM$ is (1,2)-AP that is not primary.*

*Proof.* (1) Following [3, Theorem 7], $PM$ is 1-AP with $\sqrt{PM} = P$. Moreover, $P_2 \subseteq P^2 \subseteq PM$. Using Theorem 2, we conclude that $PM$ is (1,2)-AP.
Suppose now that $PM \in \mathrm{Prim}(R)$ and $P \neq M$. Let $x \in M \backslash P$. For each $p \in P$, we have $px \in PM$. Since $PM$ is primary and $x \notin \sqrt{PM} = P$, we obtain that $p \in PM$. Thus, $P = PM$. Conversely, if $P = M$ then $PM = M^2$ is clearly primary (as a power of the maximal ideal $M$). Also, if $PM = P$ is prime then it is clearly primary.
(2) If $(x) = P = PM$ then $x \in PM$. Thus, $x(1 - a) = 0$ for some $a \in M$. Hence, $x = 0$ since $1 - a$ is a unit, a contradiction. Accordingly, $P \neq PM$. Now, the desired result follows from (1). □

By definitions, it is clear that 1-absorbing prime ideals are (1,2)-AP. The following example shows that these two notions do not coincide.

**Example 8.** *Let $k$ be a field and set $A = \frac{k[x,y]}{(x^2,xy)}$. We have $\mathfrak{M} = (\overline{x}, \overline{y}) \in \mathrm{Max(A)}$. So, set $R = A_{\mathfrak{M}}$. Then, $(R, M)$ is local with $M = \mathfrak{M}R_{\mathfrak{M}} = \left(\frac{\overline{x}}{1}, \frac{\overline{y}}{1}\right)$, and $P = \left(\frac{\overline{x}}{1}\right) \in \mathrm{Spec}(R)$. By Proposition 7, $PM = (0_R)$ is (1,2)-AP. However, it is not 1-absorbing prime. Indeed, $\left(\frac{\overline{y}}{1}\right)^2 . \frac{\overline{x}}{1} = 0_R$ but neither $\left(\frac{\overline{y}}{1}\right)^2 = 0_R$ nor $\frac{\overline{x}}{1} = 0_R$.*

Seen [3, Example 3] and keeping in mind Proposition 7, we observe that, over a local ring, (1,2)-AP ideals do not need be primary.
A ring in which the prime ideals are comparable to all principal ideals is called a divided ring. Note that, over a divided ring (even local), every 1-AP ideal is primary ([3, Example 3]). In particular, every (1,2)-AP is primary. Next, we classify the local Noetherian rings in which the (1,2)-AP ideals are all primary.

**Theorem 9.** *Suppose that $(R, M)$ is local Noetherian. The following are equivalent:*

1. *$(1, 2) - AP(R) \subseteq \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$.*

2. $(1,2)-AP(R) \subseteq \mathrm{U}-\mathrm{Prim}(R)$.

3. $(1,2)-AP(R) \subseteq \mathrm{Prim}(R)$

4. *R is a divided ring.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial since

$$\mathrm{U}-\mathrm{Prim}(R)_{\leq 2} \subseteq \mathrm{U}-\mathrm{Prim}(R) \subseteq \mathrm{Prim}(R).$$

(3) $\Rightarrow$ (4) Let $P \in \mathrm{Spec}(R)$. Using Proposition 7, we get $PM = P$ or $P = M$. If $PM = P$, the Nakayama's lemma implies that $P = \{0\}$. Hence, $P = (0)$ or $P = M$. Thus, $\mathrm{Spec}(R) \subseteq \{(0), M\}$. Thus, prime ideals are comparable to all (principal) ideals. Hence, $R$ is a divided ring as desired.
(4) $\Rightarrow$ (1) Let $I \in (1,2)-AP(R)$. Then, by Theorem 2, $I$ is 1-AP and $\sqrt{I} = \{x \in R \mid x^2 \in I\}$. Hence, by [3, Theorem 9], $I$ is primary and $\sqrt{I} = \{x \in R \mid x^2 \in I\}$. Which means, by using Proposition 4, that $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$, as desired. □

Recall that a ring $R$ is called von Neumann regular ($VNR$ for short) if, for every $a \in R$, there exists $b \in R$ such that $a = a^2 b$ [6, 7]. As a consequence, every ideal in a $VNR$ ring is radical. In particular, if $R$ is VNR then

$$(1,2)-AP(R) = \mathrm{U}-\mathrm{Prim}(R)_{\leq 2} = \mathrm{Prim}(R) = \mathrm{Spec}(R).$$

**Theorem 10.** *Suppose that* $\mathrm{U}-\mathrm{Prim}(R)_{\leq 2} \subseteq \mathrm{Spec}(R)$. *Then,*

(a) $M^2 = M$ *for each* $M \in \mathrm{Max}(R)$, *and*

(b) $\mathfrak{p}^2 R_{\mathfrak{p}} = \mathfrak{p} R_{\mathfrak{p}}$ *for each* $\mathfrak{p} \in \mathrm{Spec}(R)$.

*In particular, if R is Noetherian then, the following are equivalent:*

1. $(1,2)-AP(R) \subseteq \mathrm{Spec}(R)$.

2. $\mathrm{U}-\mathrm{Prim}(R)_{\leq 2} \subseteq \mathrm{Spec}(R)$.

3. *R is* $VNR$.

*Proof.* Let $M \in \mathrm{Max}(R)$. Since $M^2 \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$, we get that $M^2 \in \mathrm{Spec}(R)$. So, $M^2 = \sqrt{M^2} = M$.
Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and set $A = R/\mathfrak{p}^2$, $P = \mathfrak{p}/\mathfrak{p}^2$, and $S = A_P$. We have that $\sqrt{(0_S)} = \mathrm{N}(S) = \mathrm{N}(A)_P = P_P$ is maximal. Thus, $(0_S)$ is a primary ideal of $S$. Then, $(0_S) = I_P$ for some $I \in \mathrm{Prim}(A)$ such that $I \subseteq P$. Hence, $I = I_0/\mathfrak{p}^2$ for some $I_0 \in \mathrm{Prim}(R)$ such that $\mathfrak{p}^2 \subseteq I_0 \subseteq \mathfrak{p}$. By hypothesis, $I_0$ is prime

since $I_0 \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$. Hence, $I_0 = \mathfrak{p}$. Thus, $(0_S) = P_P$ is maximal. Accordingly, $S$ is a field. On the other hand, $S = A_P = \left(R/\mathfrak{p}^2\right)_{\mathfrak{p}/\mathfrak{p}^2} \cong R_{\mathfrak{p}}/\mathfrak{p}^2 R_{\mathfrak{p}}$. Thus, $\mathfrak{p}^2 R_{\mathfrak{p}}$ is the maximal ideal of the local ring $R_{\mathfrak{p}}$. Then, $\mathfrak{p}^2 R_{\mathfrak{p}} = \mathfrak{p} R_{\mathfrak{p}}$.
Suppose now that $R$ is Noetherian.
(1) $\Rightarrow$ (2) Trivial since $\mathrm{U}-\mathrm{Prim}(R)_{\leq 2} \subseteq (1,2) - AP(R)$.
(3) $\Rightarrow$ (1) Since $R$ is $VNR$ then every ideal is radical. Since (1,2)-AP ideals have prime radical, every (1,2)-AP ideal of $R$ becomes prime.
(2) $\Rightarrow$ (3) The local ring $R_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Since $\mathfrak{p}^2 R_{\mathfrak{p}} = \mathfrak{p} R_{\mathfrak{p}}$, by applying the Nakayama's lemma, we obtain that $\mathfrak{p} R_{\mathfrak{p}} = (0)$. Thus, $R_{\mathfrak{p}}$ is a field. Hence, $R$ is a $VNR$ ring. □

We saw that over a $VNR$ ring, the sets $(1,2) - AP(R)$ and $\mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$ are the same and coincide with $\mathrm{Spec}(R)$. The next two theorems give other classes of rings over which, (1,2)-AP ideals coincide with uniformly primary ideals with ord $\leq 2$, and have a simple form.

**Theorem 11.** *Suppose that every non-zero prime ideal is maximal (for example, $R$ is a 0-dimensional ring or a 1-dimensional domain). Then, the following are equivalent:*

1. $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.
2. $I \in (1,2) - AP(R)$.
3. $P_2 \subseteq I \subseteq P$ *for some* $P \in \mathrm{Spec}(R)$.

*Proof.* (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Follows from Theorem 2.
(3) $\Rightarrow$ (1) Let $P \in \mathrm{Spec}(R)$ and $I$ an ideal such that $P_2 \subseteq I \subseteq P$. Then, $P = \sqrt{P_2} \subseteq \sqrt{I} \subseteq P$. Thus, $\sqrt{I} = P$. If $P = (0)$, then $R$ is a domain and $I = (0)$ is prime. Now, suppose that $P \neq (0)$. Then, $P \in \mathrm{Max}(R)$, and so $I$ is primary with $\left(\sqrt{I}\right)_2 \subseteq I$. Thus, by Proposition 4, $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$. □

The ring $R$ (not necessarily a domain) is called a principal ideal ring ($PIR$ for short) if every ideal of $R$ is principal.

**Theorem 12.** *Suppose that $R$ is a $PIR$. Then, the following are equivalent:*

1. $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.
2. $I \in (1,2) - AP(R)$.
3. $I$ *is prime or* $I = M^2$ *for some* $M \in \mathrm{Max}(R)$.

*Proof.* (3) $\Rightarrow$ (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Following the Zariski-Samuel theorem [11, Theorem 33], $R$ is isomorphic to a direct product $\prod_{i=1}^{n} R_i$ of $PIR$, where each $R_i$ is either a domain or $\mathrm{N}(R_i)$ is maximal. Without loss of generality, write $R = \prod_{i=1}^{n} R_i$. For each $i$, the unique possible non-maximal prime ideal of $R_i$ is $(0)$, and in this case $R_i$ must be a domain. Thus, non-maximal prime ideals of $R$ are $\prod_{i=1}^{n} P_i$, where $P_j = (0)$ for some $j$ and $P_i = R_i$ for each $i \neq j$. Thus, non-maximal prime ideals of $R$ are generated by idempotent.
Let $I \in (1,2) - AP(R)$. If $\sqrt{I} = P$ is not maximal then $P^2 = P$. Since $R$ is a $PIR$, $P^2 = P_2 \subseteq I \subseteq P$. Thus, $I = P$. Suppose now that $\sqrt{I} = M$ is maximal. In this case $I$ becomes primary and $M_2 = M^2 = (m^2) \subseteq I = (x) \subseteq M = (m)$ for some $m, x \in R$. Set $m^2 = \alpha x$ and $x = m\beta$ for some $\alpha, \beta \in R$. We have $m\beta \in I$, and so $m \in I$ or $\beta \in \sqrt{I} = M$. Hence, $M = I$ or $x = m\beta \in M^2$. Then, $M = I$ or $I \subseteq M^2$. Consequently, $I = M$ or $I = M^2$. $\square$

A domain $R$ is a Dedekind domain if every ideal of $R$ is a product of prime ideals.

**Theorem 13.** *Suppose that $R$ is a Noetherian domain which is not a field. Then, the following are equivalent:*

1. *The only non-zero (1,2)-AP ideals are $P$ and $P^2$ with $P \in \mathrm{Spec}(R)$.*
2. *The only non-zero uniformly primary ideal with $\mathrm{ord} \leq 2$ are $P$ and $P^2$ with $P \in \mathrm{Spec}(R)$.*
3. *$R$ is a Dedekind domain.*

*Proof.* (3) $\Rightarrow$ (1) Let $I \neq (0)$ be a (1,2)-AP ideal and set $P = \sqrt{I}$. Clearly, $P \neq (0)$, otherwise $I = (0)$. By [10, Theorem 5.2.15], $R$ is a 1-dimensional domain, and so $P$ is maximal. However, $R_P$ is a $PID$ (again by [10, Theorem 5.2.15]). Hence, $P_2 R_P = P^2 R_P$ and for each $P \neq M \in \mathrm{Max}(R)$, we have $P_2 R_M = R_M = P^2 R_M$ since $P_2 \subseteq P^2 \nsubseteq M$. Hence, $P_2 = P^2$. Using Theorem 2, $I$ lies between $P_2 = P^2$ and $P$. Accordingly, by [9, Theorem 6.20], $I = P^2$ or $I = P$.
(1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Let $M \in \mathrm{Max}(R)$. If an idea $I$ satisfies $M^2 \subseteq I \subseteq M$, then $\sqrt{I} = M$, and so $I \in \mathrm{Prim}(R)$. Thus, by Proposition 4, $I \in \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$ since $\left(\sqrt{I}\right)_2 = M_2 \subseteq I$, and so $I$ is equal to $M$ or $M^2$. It follows from [9, Theorem 6.20] that $R$ is a Dedekind domain. $\square$

**Proposition 14.** *The following are equivalent:*

1. *$I[X] \in (1,2) - AP(R[X])$.*

2. $I[X] \in \mathrm{U}-\mathrm{Prim}(R[X])_{\leq 2}$.
3. $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.

*Proof.* (1) $\Leftrightarrow$ (2) is a particular case of Corollary 5 since $R[X]$ is never local.
(2) $\Rightarrow$ (3) Following Proposition 4, $I[X] \in \mathrm{Prim}(R[X])$ and $\left(\sqrt{I[X]}\right)_2 \subseteq I[X]$. Then, $I$ is primary. Moreover, for each $a \in \sqrt{I} \subseteq \sqrt{I}[X] = \sqrt{I[X]}$, and so $a^2 \in \left(\sqrt{I[X]}\right)_2 \subseteq I[X]$. Hence, $a^2 \in I$. Then, $\left(\sqrt{I}\right)_2 \subseteq I$. Therefore, by Proposition 4, $I \in \mathrm{U}-\mathrm{Prim}(R)_{\leq 2}$.
(3) $\Rightarrow$ (2) Following Proposition 4, $I \in \mathrm{Prim}(R)$ and $\left(\sqrt{I}\right)_2 \subseteq I$. Then, $I[X] \in \mathrm{Prim}(R[X])$. Let $f \in \sqrt{I[X]}$. Set $f = \sum_{k=0}^{n} a_k X^k$. We have

$$f^2 = \sum_{k=0}^{n} (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} 2a_i a_j X^{i+j}$$

$$= \sum_{k=0}^{n} (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} \left[(a_i + a_j)^2 - a_i^2 - a_j^2\right] X^{i+j}$$

Since $a_i \in \sqrt{I}$ for each $i$, we get that $(a_i)^2, (a_i + a_j)^2 \in \left(\sqrt{I}\right)_2 \subseteq I$ for each $i$ and $j$. Hence, $f^2 \in I[X]$. Then, $\left(\sqrt{I[X]}\right)_2 \subseteq I[X]$. Thus, by Proposition 4, $I[X] \in \mathrm{U}-\mathrm{Prim}(R[X])_{\leq 2}$. $\square$

Using ideals, the definition of (1,2)-AP ideals can be rephrased as follows:

**Proposition 15.** *The following are equivalent:*

1. $I \in (1,2) - AP(R)$.
2. *For each* $X, Y, Z \in \mathrm{Id}(R)$, $XYZ \subseteq I$ *implies* $XY \subseteq I$ *or* $Z_2 \subseteq I$.

*Proof.* (1) $\Rightarrow$ (2) Let $X, Y, Z \in \mathrm{Id}(R)$. Suppose that $XYZ \subseteq I$ and $XY \nsubseteq I$. Consider $x \in X$ and $y \in Y$ such that $xy \notin I$. For each $z \in Z$, we have $xyz \in I$. Hence, since $I \in (1,2) - AP(R)$, $z^2 \in I$. Thus, $Z_2 \subseteq I$.
(2) $\Rightarrow$ (1) Let $x, y, z \in R - U(R)$ such that $xyz \in I$. Assume that $xy \notin I$ and set $X = (x)$, $Y = (y)$, and $Z = (z)$. Then, $XYZ \subseteq I$ and $XY \nsubseteq I$. Thus, $(z^2) = (z)_2 = Z_2 \subseteq I$. Hence, $z^2 \in I$. Thus, $I \in (1,2) - AP(R)$. $\square$

**Proposition 16.** *Let* $\varphi : R \to S$ *be a ring homomorphism such that* $\varphi^{-1}(\mathrm{U}(S)) \subseteq \mathrm{U}(R)$. *Then, If* $I' \in (1,2) - AP(S)$, *then* $\varphi^{-1}(I') \in (1,2) - AP(R)$.

*Proof.* Let $\alpha, \beta, \gamma \in R - U(R)$. Suppose that $\alpha\beta\gamma \in \varphi^{-1}(I')$ and $\gamma^2 \notin \varphi^{-1}(I')$. Since $\varphi^{-1}(\mathrm{U}(S)) \subseteq \mathrm{U}(R)$, we get that $\varphi(\alpha), \varphi(\beta), \varphi(\gamma) \in S - U(S)$. Hence, since $I' \in (1,2) - AP(S)$, $\varphi(\alpha\beta\gamma) \in I'$, and $\varphi(\gamma^2) \notin I'$, we get $\varphi(\alpha\beta) \in I'$. Thus, $\alpha\beta \in \varphi^{-1}(I')$. So, $\varphi^{-1}(I') \in (1,2) - AP(R)$. □

**Remark 17.** *Let $R$ be a ring admitting a (1,2)-AP ideal $I$ that is not primary (see Example 8). Consider the ring homomorphism $\varphi : R \times R \to R$ defined by $\varphi(x, y) = x$. Then, $\varphi^{-1}(I) = I \times R$. If $I \times R \in (1,2) - AP(R \times R)$ then, by Corollary 6, $I \in \mathrm{Prim}(R)$, a contradiction. Hence, in general, the inverse image of a (1,2)-AP ideal by a ring homomorphism need not be (1,2)-AP.*

**Proposition 18.** *Let $\varphi : R \to S$ be a ring epimorphism. Then, If $\ker(\varphi) \subseteq I \in (1,2) - AP(R)$, then $\varphi(I) \in (1,2) - AP(S)$.*

*Proof.* Let $\alpha, \beta, \gamma \in S - U(S)$. Suppose that $\alpha\beta\gamma \in \varphi(I)$ and $\gamma^2 \notin \varphi(I)$. Set $\alpha = \varphi(a)$, $\beta = \varphi(b)$, and $\gamma = \varphi(c)$ for some $a, b, c \in R$. Clearly, $a, b, c \notin U(R)$. Since $\varphi(abc) = \alpha\beta\gamma \in \varphi(I)$ and $\ker(\varphi) \subseteq I$, we get $abc \in I$. Moreover, $\varphi(c^2) = \gamma^2 \notin \varphi(I)$ implies that $c^2 \notin I$. Hence, since $I \in (1,2) - AP(R)$, we get $ab \in I$. Thus, $\alpha\beta = \varphi(ab) \in \varphi(I)$. Consequently, $\varphi(I) \in (1,2) - AP(S)$. □

**Proposition 19.** *Let $S$ be a multiplicative subset of $R$ (with $0 \notin S$).*

1. *If $I \in (1,2) - AP(R)$ such that $\emptyset = S \cap I$, then $S^{-1}I \in (1,2) - AP(S^{-1}R)$.*
2. *Suppose that $a/1$ is non-unit in $S^{-1}R$ for every $a \notin U(R)$. If $J \in (1,2) - AP(S^{-1}R)$, then $J^c = \{x \in R \mid x/1 \in J\} \in (1,2) - AP(R)$.*

*Proof.* (1) First, note that $S^{-1}I \neq S^{-1}R$ since $I \cap S = \emptyset$. Now, let $\frac{a}{s_1}, \frac{b}{s_2}, \frac{c}{s_3} \in S^{-1}R - U(S^{-1}R)$ such that $\frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$ and $\frac{a}{s_1}\frac{b}{s_2} \notin S^{-1}I$. Thus, $abcs \in I$ for some $s \in S$ and $abw \notin I$ for all $w \in S$. In particular, $abs \notin I$. So, $c^2 \in I$. Then, $\left(\frac{c}{s_3}\right)^2 \in S^{-1}I$. This shows that $S^{-1}I \in (1,2) - AP(S^{-1}R)$.
(2) Consider the natural ring homomorphism $\varphi : R \to S^{-1}R$; $\varphi(a) = a/1$. Since $a/1$ is non-unit in $S^{-1}R$ for every $a \notin U(R)$, we get that $\varphi^{-1}\left(\mathrm{U}(S^{-1}R)\right) \subseteq \mathrm{U}(R)$. Hence, by Proposition 16, $J^c = \varphi^{-1}(J) \in (1,2) - AP(R)$. □

## 3 Rings over which every (non-zero) proper ideal is (1,2)-absorbing primary

This section is devoted to characterize rings satisfying one the of inclusion/ equality: $\mathrm{Id}(R)^* \subseteq \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$, $\mathrm{Id}(R)^* \subseteq (1,2) - AP(R)$, $\mathrm{Id}(R)^* = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$, $\mathrm{Id}(R)^* = (1,2) - AP(R)$, $\mathrm{Id}(R) = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$, and

$\mathrm{Id}(R) = (1,2) - AP(R)$.

Recall that a nonzero ideal of a ring is said to be minimal if it is minimal by inclusion over nonzero proper ideals. A minimal ideal need not be unique. Let $L$ be the intersection of all non-zero ideals of $R$. If $L \neq (0)$, then $R$ is called subdirectly irreducible and $L$ is called the little ideal of $R$ ([8]). Thus, a little ideal $L$ (if it exists) is a non-zero ideal included in all other non-zero ideals of $R$. It is clear that $L$ is unique and it is the unique minimal ideal of $R$. Similarly, we define the notion of the little sub-ideal of an ideal as follows:

**Definition 20.** *Let $I \neq (0)$ be an ideal of a ring $R$. Set $L$ the intersection of all non-zero sub-ideals of $I$. If $L \neq (0)$ then $L$ is called the little sub-ideal of $I$ (note that $L$ must be unique).*

We need the following lemmas.

**Lemma 21.** *Suppose that $\mathrm{N}(R) \subseteq I$. Then, $I$ is a 1-AP ideal of $R$ if and only if $I/\mathrm{N}(R)$ is a 1-AP ideal of $R/\mathrm{N}(R)$.*

*Proof.* Set $S = R/\mathrm{N}(R)$ and $\overline{I} = I/\mathrm{N}(R)$.
Suppose that $I$ is a 1-AP ideal of $R$. Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \in S - U(S)$ such that $\overline{\alpha\beta\gamma} \in \overline{I}$ and $\overline{\alpha\beta} \notin \overline{I}$. Clearly $\alpha, \beta, \gamma \notin U(R)$ and $\alpha\beta\gamma \in I$ since $\mathrm{N}(R) \subseteq I$. Thus, $\gamma \in \sqrt{I}$, and so $\overline{\gamma} \in \sqrt{I}/\mathrm{N}(R) = \sqrt{\overline{I}}$. Accordingly, $\overline{I}$ is a 1-AP ideal of $S$.
Conversely, assume that $\overline{I}$ is a 1-AP ideal of $S$. Let $\alpha, \beta, \gamma \in R - U(R)$ such that $\alpha\beta\gamma \in I$ and $\alpha\beta \notin I$. If $\overline{\alpha} \in U(S)$ then $\alpha t - 1 \in \mathrm{N}(R)$ for some $t \in R$. Thus, $\alpha t \in \mathrm{N}(R) + 1 \subseteq U(R)$. So, $\alpha \in U(R)$, a contradiction. Thus, $\overline{\alpha} \notin U(S)$. Similarly $\overline{\beta}, \overline{\gamma} \notin U(S)$. Moreover, $\overline{\alpha\beta} \notin \overline{I}$. Thus, since $\overline{\alpha\beta\gamma} \in \overline{I}$ , we get that $\overline{\gamma} \in \sqrt{\overline{I}} = \sqrt{I}/\mathrm{N}(R)$. Hence, $\gamma \in \sqrt{I}$ since $\mathrm{N}(R) \subseteq \sqrt{I}$. Consequently, $I$ is a 1-AP ideal of $R$. $\square$

$R$ is said to be a $UN$-ring if $\mathrm{N}(R)$ is a maximal ideal of $R$ ([4]).

**Lemma 22.** *Every non-zero proper ideal of $R$ is 1-AP if and only if*

*(a) $R \cong k_1 \times k_2$, where $k_1$ and $k_2$ are fields, or*

*(b) $R$ is a $UN$-ring, or*

*(c) $(R, M)$ is local with $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ such that $\mathrm{N}(R)M$ is zero or the little sub-ideal of $\mathrm{N}(R)$.*

*Proof.* ($\Rightarrow$) Assume that $R$ is not local and let $M_1 \neq M_2$ be two maximal ideals. If $M_1 \cap M_2 \neq (0)$, then $M_1 \cap M_2$ is 1-AP, and so $M_1 \cap M_2 = \sqrt{M_1 \cap M_2}$ is prime. Hence, $M_1$ and $M_2$ are comparable, a contradiction. Then, $M_1 \cap M_2 = (0)$. Therefore, $R \cong R/M_1 \times R/M_2$, and so $(a)$ holds.

Suppose now that $(R, M)$ is local. If $R$ is a domain then, by [1, Corollary 2.14], $R$ is a field or $\mathrm{Spec}(R) = \{0, M\}$. So, $R$ satisfies either $(b)$ or $(c)$.
Next, assume that $R$ is not a domain. We claim that prime ideals are comparable. Consider two incomparable prime ideals $P_1$ and $P_2$. If $P_1 \cap P_2 \neq 0$ then $P_1 \cap P_2$ is 1-AP and so prime since it is a radical ideal. Hence, $P_1$ and $P_2$ are comparable, a contradiction. Then, $P_1 \cap P_2 = (0)$. Therefore, $\mathrm{N}(R) = \sqrt{(0)} = \sqrt{P_1 \cap P_2} = P_1 \cap P_2 = (0)$. Hence, $R$ is reduced. Consider $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$. Thus, $x + y \notin P_1$. Let $p_1 \in P_1 \backslash \{0\}$ (such $p_1$ exists since $P_1 \neq (0)$). Since $R$ is reduced, $p_1^2 \neq 0$. We have $p_1^2 y \in P_1 \cap P_2 = (0)$. If $p_1^2 x = 0 \in P_2$ then $p_1^2 \in P_1 \cap P_2 = (0)$, a contradiction. Then, $(x + y)p_1^2 = xp_1^2 \neq 0$. Thus, $\big((x + y)p_1^2\big)$ is 1-AP. Since $(x + y)p_1^2 \in \big((x + y)p_1^2\big)$ and $x + y \notin \sqrt{((x + y)p_1^2)} \subseteq P_1$, we obtain that $p_1^2 \in \big((x + y)p_1^2\big)$. So, $p_1^2 = (x + y)p_1^2 \alpha = xp_1^2 \alpha$ for some $\alpha \in R$. Thus, $p_1^2(1 - x\alpha) = 0$. Since $R$ is local, $1 - x\alpha$ is unit. Then, $p_1^2 = 0$, a contradiction. Consequently, prime ideals are comparable. Thus, $(0) \neq \mathrm{N}(R)$ is prime.
Consider the domain $R' = R/\mathrm{N}(R)$. By Lemma 21, every proper ideal of $R'$ is 1-AP. By [1, Corollary 2.14], $R'$ is a field or $\mathrm{Spec}(R') = \{(0_{R'}), M/\mathrm{N}(R)\}$. Hence, either $R$ is a $UN$-ring or $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$. Assume that $R$ is not a $UN$-ring. Let $I$ be a non-zero sub-ideal of $\mathrm{N}(R)$. Let $a \in \mathrm{N}(R)$, $m \in M$, and $m' \in M \backslash \mathrm{N}(R)$. Then, $I + (amm')$ is a non-zero ideal of $R$ with $\sqrt{I + (amm')} = \mathrm{N}(R)$. Hence, $I + (amm')$ is 1-AP. Since $amm' \in I + (amm')$ and $m' \notin \mathrm{N}(R)$, we get $am \in I + (amm')$. Hence, for some $r \in R$, we have $am(1 - rm') \in I$. Note that $1 - rm' \notin \mathrm{N}(R) = \sqrt{I}$, otherwise $1 \in M$. Then, since $I$ is 1-AP, we obtain that $am \in I$. Then, $\mathrm{N}(R)M \subseteq I$ for each non zero sub-ideal $I$ of $\mathrm{N}(R)$. Hence, either $\mathrm{N}(R)M = (0)$ or $\mathrm{N}(R)M$ is the little sub-ideal of $\mathrm{N}(R)$.
$(\Leftarrow)$ Let $k_1$ and $k_2$ be two field. non-zero proper ideals of $k_1 \times k_2$ are $(0) \times k_2$ and $k_1 \times (0)$ which are maximal and so 1-AP.
If $R$ is a $UN$-ring then every (non-zero) ideal is primary, and so 1-AP.
Suppose now that $R$ is local such that $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ and either $\mathrm{N}(R)M = (0)$ or $\mathrm{N}(R)M$ is the little sub-ideal of $\mathrm{N}(R)$. Let $I$ be a non-zero ideal. If $\sqrt{I} = M$ then $I$ is primary, and so 1-AP. If $\sqrt{I} = \mathrm{N}(R)$ then $I$ is a non-zero sub-ideal of $\mathrm{N}(R)$. Hence, $\mathrm{N}(R)M \subseteq I$ (in the both cases). Let $x, y, z \in R - U(R)$ with $xyz \in I$ and $z \notin \sqrt{I} = \mathrm{N}(R)$. Then, either $x \in \mathrm{N}(R)$ or $y \in \mathrm{N}(R)$. In the both cases, $xy \in \mathrm{N}(R)M \subseteq I$. Thus, $I$ is 1-AP. $\square$

Comparing Lemma 22 with [1, Theorem 2.11], we conclude that $\mathrm{Id}(R)^*$ coincide with the set of 1-AP ideals if and only if $R \cong k_1 \times k_2$, where $k_1$ and $k_2$ are fields, or $(R, M)$ is local with $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ such that $\mathrm{N}(R)M$ is the little sub-ideal of $\mathrm{N}(R)$.

Rings $R$ such that $\mathrm{Id}(R) = \mathrm{Prim}(R)$ are treated in [1, Corollary 2.15]. Next we characterizes rings that satisfy the inclusion $\mathrm{Id}(R)^* \subseteq \mathrm{Prim}(R)$.

**Corollary 23.** $\mathrm{Id}(R)^* \subseteq \mathrm{Prim}(R)$ *if and only if*

*(a)* $R \cong k_1 \times k_2$*, where* $k_1$ *and* $k_2$ *are fields, or*

*(b)* $R$ *is a* $UN$*-ring, or*

*(c)* $(R, M)$ *is local and* $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ *such that* $\mathrm{N}(R)$ *is zero or a minimal ideal of* $R$*.*

*Proof.* ($\Rightarrow$) Suppose that $R$ satisfies neither (a) nor (b). By Lemma 22, $R$ is local and $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ with $\mathrm{N}(R)M = (0)$ or $\mathrm{N}(R)M$ is the little sub-ideal of $\mathrm{N}(R)$. If $\mathrm{N}(R)M \neq (0)$ then it is primary. Since $\mathrm{N}(R) \neq M$, using Proposition 7, we get $\mathrm{N}(R)M = \mathrm{N}(R)$. Thus, $\mathrm{N}(R)$ is the little sub-ideal of $\mathrm{N}(R)$. That is, $\mathrm{N}(R)$ is a minimal ideal of $R$.
Now suppose that $\mathrm{N}(R)M = (0)$ but $\mathrm{N}(R) \neq (0)$. Let $I \neq (0)$ be a sub-ideal of $\mathrm{N}(R)$ and $m \in M \backslash \mathrm{N}(R)$. For each $x \in \mathrm{N}(R)$, we have $xm = 0 \in I$ and $m \notin \sqrt{I} = \mathrm{N}(R)$. Thus, $x \in I$. Hence, $\mathrm{N}(R) = I$. Thus, $\mathrm{N}(R)$ is again minimal.
($\Leftarrow$) If $R$ satisfies (a) or (b), the desired result follows easily. So, assume that ($c$) holds. Let $I \neq (0)$ be an ideal. If $\sqrt{I} = M$ then $I$ is primary. So, assume that $\sqrt{I} = \mathrm{N}(R)$. Thus, $\mathrm{N}(R)$ must be minimal since $I \neq (0)$. Hence, $I = \mathrm{N}(R)$, and so is prime. □

Comparing [1, Corollary 2.15] with Corollary 23, we conclude that $\mathrm{Id}(R)^* = \mathrm{Prim}(R)$ if and only if $R \cong k_1 \times k_2$, where $k_1$ and $k_2$ are fields or $(R, M)$ is local such that $\mathrm{Spec}(R) = \{\mathrm{N}(R), M\}$ and $\mathrm{N}(R)$ is a minimal ideal.

The main result of this section is the following.

**Theorem 24.** *The following are equivalent:*

1. $\mathrm{Id}(R)^* \subseteq \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$.
2. $\mathrm{Id}(R)^* \subseteq (1, 2) - AP(R)$.
3. *One of the following holds:*
   *(a)* $R \cong k_1 \times k_2$*, where* $k_1$ *and* $k_2$ *are fields.*
   *(b)* $R$ *is a* $UN$*-ring such that* $\mathrm{N}(R)_2$ *is zero or the little ideal of* $R$*.*

*Proof.* (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Following Lemma 22, either $R \cong k_1 \times k_2$, where $k_1$ and $k_2$ are fields or $R$ is local and $\mathrm{N}(R)$ is prime. Suppose that $R$ is local. Let $x \in R - U(R)$. Suppose that $x^3 \neq 0$. Since $(x^3)$ is (1,2)-AP and $x \in \sqrt{(x^3)}$, we get that $x^2 \in (x^3)$. Thus, $x^2(1 - xt) = 0$ for some $t \in R$. Since $R$ is local, $1 - xt$ is a unit element of $R$. Hence, $x^2 = 0$, a contradiction since $x^3 \neq 0$. Consequently, for each $x \in R - U(R)$, we have $x^3 = 0$, and then $R$ is a $UN$-ring. Suppose that $\mathrm{N}(R)_2 \neq (0)$ and let $I \in \mathrm{Id}(R)^*$. Then, $I$ is (1,2)-AP with $\sqrt{I} = \mathrm{N}(R)$. Hence, $\mathrm{N}(R)_2 \subseteq I$. Thus, $\mathrm{N}(R)_2$ is the little ideal of $R$.
(3) $\Rightarrow$ (1) Let $k_1$ and $k_2$ be two field. non-zero proper ideals of $k_1 \times k_2$ are $k_1 \times (0)$ and $(0) \times k_2$ which are maximal, and so uniformly primary with ord $\leq 2$.
Suppose now that $R$ is a $UN$-ring such that $\mathrm{N}(R)_2 = (0)$ or $\mathrm{N}(R)_2$ is the little ideal of $R$. Let $I \in \mathrm{Id}(R)^*$. Clearly $\sqrt{I} = \mathrm{N}(R)$ (hence, primary) and $\mathrm{N}(R)_2 \subseteq I$. Thus, $I \in \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$. □

**Example 25.** *1. $\mathbb{Z}/4\mathbb{Z}$ is a non-reduced $UN$-ring such $\mathrm{N}(\mathbb{Z}/4\mathbb{Z})_2 = (0)$.*

*2. $\mathbb{Z}/8\mathbb{Z}$ is a $UN$-ring such $\mathrm{N}(\mathbb{Z}/8\mathbb{Z})_2 = (\overline{4})$ is the little ideal of $\mathbb{Z}/8\mathbb{Z}$.*

**Remark 26.** *If $R$ is non-local, we can conclude easily from Theorem 24 that:*

$$\mathrm{Id}(R)^* = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2} \Leftrightarrow \mathrm{Id}(R)^* = (1,2) - AP(R) \Leftrightarrow R \cong k_1 \times k_2,$$

*where $k_1$ and $k_2$ are fields.*

*In the local context, the equality $\mathrm{Id}(R)^* = (1,2) - AP(R)$ means that $(0)$ is not $(1,2) - AP$. But that means also that $R$ is $UN$, and so $(0)$ is primary. Thus, by Theorem 2, we must have $\left(\sqrt{(0)}\right)_2 \nsubseteq (0)$. That is, $\mathrm{N}(R)_2 \neq (0)$. So, by Theorem 24, $\mathrm{N}(R)_2$ is the little ideal of $R$. Conversely, we can show that if $R$ is $UN$ with little ideal $\mathrm{N}(R)_2$, then $\mathrm{Id}(R)^* = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$. Accordingly, in the local context, we have*

$$\mathrm{Id}(R)^* = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2} \Leftrightarrow \mathrm{Id}(R)^* = (1,2) - AP(R) \Leftrightarrow$$

$$\Leftrightarrow R \text{ is } UN \text{ with little ideal } \mathrm{N}(R)_2.$$

**Corollary 27.** *Suppose that $R$ is $PIR$. Then, the following are equivalent:*

1. $\mathrm{Id}(R)^* \subseteq \left\{M, M^2 \mid M \in \mathrm{Max}(R)\right\}$.
2. $\mathrm{Id}(R)^* \subseteq \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$.
3. $\mathrm{Id}(R)^* \subseteq (1,2) - AP(R)$.

4. *$R \cong k_1 \times k_2$, where $k_1$ and $k_2$ are fields or $R$ is a $UN$-ring such that $\mathrm{N}(R)^3 = (0)$.*

5. *$R$ has at most four ideals.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (4) From Theorem 24, if $R$ is not isomorphic to a product of two fields, then $R$ is $UN$ such that $\mathrm{N}(R)_2 = (0)$ or $\mathrm{N}(R)_2$ is the little ideal of $R$. Set $\mathrm{N}(R) = (x)$. Suppose that $x^3 \neq 0$. Then, $\mathrm{N}(R)_2 = (x^2) \neq (0)$, and so $\mathrm{N}(R)_2 = (x^2)$ is the little ideal of $R$. Thus, $(x^2) \subseteq (x^3)$ Hence, $x^2(1 - xt) = 0$ for some $t \in R$. Thus, since $R$ is local, $x^2 = 0$, a contradiction. Hence, $x^3 = 0$.
(4) $\Rightarrow$ (5) It suffices to show that if $R$ is $UN$ such that $\mathrm{N}(R)^3 = (0)$ then $R$ has at most four ideals. Set $\mathrm{N}(R) = (x)$. Let $I = (y)$ be a non-zero proper ideal of $R$. Then, $y \in (x)$. Write $y = x\alpha$ for some $\alpha \in R$. If $\alpha$ is unit then $I = (x)$. Now, suppose that $\alpha \notin U(R)$. Hence, $\alpha = x\beta$ for some $\beta \in R$. If $\beta \notin U(R)$ then $\beta = x\gamma$ for some $\gamma \in R$, and so $y = x^3\gamma = 0$, a contradiction. Then, $\beta \in U(R)$ and $y = x^2\beta$. Thus, $I = \mathrm{N}(R)^2$. Hence, the only possible non-zero proper ideals of $R$ are $\mathrm{N}(R)$ and $\mathrm{N}(R)^2$.
(5) $\Rightarrow$ (1) If $R$ is a field the result is trivial. If $R$ admits three ideals $(0)$, $M$ and $R$. Then, $\mathrm{Id}(R)^* = \{M\} = \mathrm{Max}(R)$. Suppose now that $R$ admits exactly four ideals $(0)$, $I$, $J$ and $R$. If $I$ and $J$ are incomparable then they are maximal and then $\mathrm{Id}(R)^* = \{I, J\} = \mathrm{Max}(R)$. If $I \subseteq J$ then $R$ is local with maximal ideal $J$. Let $x \in I - \{0\}$ and $y \in J - I$. Then, $I = (x)$ and $J = (y)$. If $J^2 = J$ then $y(1 - yt) = 0$ for some $t \in R$. But $R$ is local, and so $y = 0$, a contradiction. If $J^2 = (0)$, write $x = yt$. If $t \notin U(R)$ then $t = yw$ and then $x = y^2w = 0$, a contradiction. Thus, $t \in U(R)$ and $I = (x) = (y) = J$, a contradiction. Then, $J^2 = I$, and so $I$ is a square of a maximal ideal. □

**Corollary 28.** *Suppose that $R$ is a $PID$. Let $p$ be a non-zero prime element of $R$, and $n \geq 1$ be an integer. Set $A_n = R/(p^n)$. Then, $\mathrm{Id}(R)^* \subseteq (1,2) - AP(R)$ if and only if $n \leq 3$.*

*Proof.* The ring $A_n$ is a principal $UN$-ring with $\mathrm{N}(A_n) = pA_n$. Then,

$$\mathrm{Id}(R)^* \subseteq (1,2) - AP(R) \iff p^3 A_n = (0) \iff n \leq 3.$$

□

**Corollary 29.** *The following are equivalent:*

1. *$\mathrm{Id}(R) = \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$.*

2. *$\mathrm{Id}(R) = (1,2) - AP(R)$.*

3. *$R$ is a $UN$-ring such that $\mathrm{N}(R)_2 = (0)$.*

*Proof.* (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Since (0) is (1,2)-AP, $\mathrm{N}(R) = \sqrt{0} \in \mathrm{Spec}(R)$ and $\mathrm{N}(R)_2 \subseteq (0)$. Hence, $R$ cannot be isomorphic to a product of fields and $\mathrm{N}(R)_2 = (0)$. Moreover, by Theorem 24, $R$ is a $UN$-ring.
(3) $\Rightarrow$ (1) Let $I$ be a proper ideal of $R$. Then, $I \in \mathrm{Prim}(R)$ since $\sqrt{I} = \mathrm{N}(R) \in \mathrm{Max}(R)$. Thus, since $\mathrm{N}(R)_2 = (0) \subseteq I$, $I \in \mathrm{U} - \mathrm{Prim}(R)_{\leq 2}$. □

**Corollary 30.** *Suppose that R is reduced. Then,*

1. $\mathrm{Id}(R)^* \subseteq (1,2) - AP(R)$ *if and only if R is a field or R is isomorphic to a product of two fields.*

2. $\mathrm{Id}(R) = (1,2) - AP(R)$ *if and only if R is a field.*

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