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On approximation properties of some non-positive Bernstein-Durrmeyer type operators

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Abstract

In this paper we shall introduce a new type of Bernstein Durrmeyer operators which are not positive on the entire interval $[0,1]$. For these operators we will study the uniform convergence on all continuous functions on $[0,1]$ as well as a result given in terms of modulus of continuity $\omega(f,\delta)$. A Voronovskaja type theorem will be proved as well.

1 Introduction

In order to prove Weierstrass's approximation theorem [15], S. N. Bernstein proposed, in paper [3], a sequence of linear and positive operators, defined as follows: for $f \in C[0,1]$,

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ x \in [0,1], \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $0 \le k \le n$.

These operators proved to be very useful in various domains such as mathematics, engineering and many more. Due to the large fields of applicability,

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Bernstein operators have been extensively studied over the years. More details about their approximation properties can be found in papers [4, 9, 10, 13].

Another modification of Bernstein operators that proved to be very useful was introduced by Durrmeyer in paper [7], which are now known as Bernstein-Durrmeyer operators and they are defined as:

$$D_n\left(f,x\right) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,\ x\in[0,1], \tag{2}$$

for $f \in C[0,1]$ and $p_{n,k}$ defined as before.

These operators are positive and linear.

As well as Bernstein operators, the approximation properties of Bernstein-Durrmeyer operators proved to be of great interest for many authors. Some of the results concerning these operators and their generalizations can be found in papers [1, 2, 6, 11, 12, 14] and many more.

In paper [8], D. A. Meleşteu proposed the following modification of the Bernstein operators: $S_n^\alpha : C[0,1] \to C\left[0, \frac{n}{n+t}\right]$, $\alpha > 0$, $n \in \mathbb{N}$:

$$S_n^\alpha(f,x) = \sum_{k=0}^{n} p_{n,k}^\alpha(x) f\left(\frac{k}{n}\right),\ x\in[0,1], \tag{3}$$

where $p_{n,k}^\alpha(x) = \left(\frac{n+\alpha}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+\alpha} - x\right)^{n-k}$, $0 \le k \le n$. For these modified Bernstein type operators were proved some approximation results, a Voronovskaja type theorem and also a simultaneous approximation result for $x \in [0, 1-\varepsilon]$.

Having in mind the operators introduced by Meleşteu [8] and Deo et al.[5], we propose the following Bernstein-Durrmeyer type operators:

Let $\alpha \ge 0$. For every $f \in C\left[0,1\right]$, we define:

$$D_n^\alpha\left(f,x\right) = (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^\alpha\left(x\right)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha\left(t\right) f\left(t\right)dt,\ x\in\left[0,1\right], \tag{4}$$

where $p_{n,k}^\alpha\left(x\right) = \left(\frac{n+\alpha}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+\alpha} - x\right)^{n-k}$, $n,k\in\mathbb{N},\ k\le n$.

The functions $p_{n,k}^\alpha$ satisfy $\sum_{k=0}^{n} p_{n,k}^\alpha\left(x\right) = 1$.

We make the convention that $p_{n,k}^\alpha\left(x\right) = 0$ if $k > n$.

Remark 1. *For $\alpha = 0$ we obtain the classical Bernstein-Durrmeyer operators, and for $\alpha = 1$ we get the operators studied by Deo N. et al. in paper [5].*

Remark 2. $D_n^\alpha\left(f,x\right)$ *defined in* (4) *is a linear operator which is positive for* $x \in \left[0, \frac{n}{n+\alpha}\right]$ *and non-positive on* $\left(\frac{n}{n+\alpha}, 1\right]$.

Throughout the paper we will need the following result which was proved in the paper [8].

Lemma 3. *We have the following recurrence relation:*

$$x\left(\frac{n}{n+\alpha}-x\right)\left(p_{n,k}^{\alpha}\left(x\right)\right)'=n\left(\frac{k}{n+\alpha}-x\right)p_{n,k}^{\alpha}\left(x\right),\;\; x\in[0,1] \tag{5}$$

2 Preliminaries

Lemma 4. *We have the following:*

$$\int_0^{\frac{n}{n+\alpha}} t^{k+s}\left(\frac{n}{n+\alpha}-t\right)^{n-k}dt=\left(\frac{n}{n+\alpha}\right)^{n+s+1}\beta\left(k+s+1,n-k+1\right), \tag{6}$$

where $\beta\left(a,b\right)$ is the well known Euler's Beta function.

Proof. Is immediate by changhing the variable $u=\frac{n+\alpha}{n}t$. □

Proposition 5. *Operators D_n^{α} satisfy the following relations:*
i) $D_n^{\alpha}\left(e_0,x\right)=1$;
ii)$D_n^{\alpha}\left(e_1,x\right)=\frac{n}{n+2}x+\frac{n}{(n+\alpha)(n+2)}$;
iii)$D_n^{\alpha}\left(e_2,x\right)=\frac{n(n-1)}{(n+2)(n+3)}x^2+\frac{4n^2}{(n+2)(n+3)(n+\alpha)}x+\frac{2n^2}{(n+2)(n+3)(n+\alpha)^2}$,
where $x\in[0,1]$.

Proof. i)

$$D_n^{\alpha}\left(e_0,x\right)=(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}\left(t\right)dt \tag{7}$$

$$=(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)\left(\frac{n+\alpha}{n}\right)^{n}\binom{n}{k}\int_0^{\frac{n}{n+\alpha}}t^{k}\left(\frac{n}{n+\alpha}-t\right)^{n-k}dt.$$

Now, applying the formula (6) for $s=0$, we get that the integral in the expression above is equal to $\frac{1}{n+1}\frac{1}{\binom{n}{k}}\left(\frac{n}{n+\alpha}\right)^{n+1}$, which leads to

$$D_n^{\alpha}\left(e_0,x\right)=\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)=1.$$

ii) For the second expression, we will use again formula (6) for $s=1$, and we will get:

$$D_n^{\alpha}(e_1,x) = (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}} t\cdot p_{n,k}^{\alpha}(t)\,dt$$

$$= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\left(\frac{n+\alpha}{n}\right)^{n}\binom{n}{k}\int_0^{\frac{n}{n+\alpha}} t^{k+1}\left(\frac{n}{n+\alpha}-t\right)^{n-k} dt$$

$$= (n+1)\left(\frac{n+\alpha}{n}\right)^{n+1}\left(\frac{n}{n+\alpha}\right)^{n+2}\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\binom{n}{k}\beta(k+2,n-k+1)$$

$$= (n+1)\left(\frac{n}{n+\alpha}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\frac{n!}{k!\,(n-k)!}\frac{(k+1)!(n-k)!}{(n+2)!}$$

$$= (n+1)\left(\frac{n}{n+\alpha}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\frac{k+1}{(n+1)(n+2)}.$$

We have:

$$D_n^{\alpha}(e_1,x) = \frac{1}{n+2}\left(\frac{n}{n+\alpha}\right)\left[\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\,k+1\right]$$

$$= \frac{1}{n+2}\left(\frac{n+\alpha}{n}\right)^{n-1}\sum_{k=0}^{n} k\frac{n!}{k!\,(n-k)!}x^k\left(\frac{n}{n+\alpha}-x\right)^{n-k} + \frac{n}{(n+2)(n+\alpha)}$$

$$= \frac{1}{n+2}\left(\frac{n+\alpha}{n}\right)^{n-1} xn\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}x^{k-1}\left(\frac{n}{n+\alpha}-x\right)^{n-k} + $$
$$+\frac{n}{(n+2)(n+\alpha)}.$$

Now, changing the summation index we get

$$D_n^{\alpha}(e_1,x) =$$

$$= \frac{1}{n+2}\left(\frac{n+\alpha}{n}\right)^{n-1} xn\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\left(\frac{n}{n+\alpha}-x\right)^{n-1-k} + \frac{n}{(n+2)(n+\alpha)}$$

$$= \frac{1}{n+2}\left(\frac{n+\alpha}{n}\right)^{n-1} xn\left(\frac{n}{n+\alpha}\right)^{n-1} + \frac{n}{(n+2)(n+\alpha)}$$

$$= \frac{n}{n+2}x + \frac{n}{(n+2)(n+\alpha)}.$$

iii) For proving this result we will proceed as before. After some calculations we get:

$$D_n^\alpha(e_2,x) = \frac{1}{(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2\sum_{k=0}^{n}p_{n,k}^\alpha(x)\left(k^2+3k+2\right).$$

Writing $k^2 = k(k-1)+k$ and changing the summation index twice, as shown before, we get:

$$\begin{aligned}
&D_n^\alpha(e_2,x) = \\
&= \frac{1}{(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2\left[\sum_{k=0}^{n}p_{n,k}^\alpha(x)\,k(k-1)+4\sum_{k=0}^{n}p_{n,k}^\alpha(x)\,k+2\right]\\
&= \frac{1}{(n+2)(n+3)}\left(\frac{n}{n+\alpha}\right)^2\left[\left(\frac{n+\alpha}{n}\right)^2 n(n-1)x^2+4x\frac{n+\alpha}{n}n+2\right]\\
&= \frac{n(n-1)}{(n+2)(n+3)}x^2+\frac{4n^2}{(n+2)(n+3)(n+\alpha)}x+\frac{2n^2}{(n+2)(n+3)(n+\alpha)^2}.
\end{aligned}$$

□

Now, we denote by $\mu_{n,m}(x)$ the m-th order moments for the operator D_n^α, which has the following expression:

$$\begin{aligned}
\mu_{n,m}(x) &= D_n^\alpha\left((t-x)^m,x\right) \qquad (8)\\
&= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}}(t-x)^m\cdot p_{n,k}^\alpha(t)\,dt.
\end{aligned}$$

Theorem 6. *The following recurrence relation holds:*

$$\begin{aligned}
(m+n+2)\,\mu_{n,m+1}(x) &= x\left(\frac{n}{n+\alpha}-x\right)\left[2m\mu_{n,m-1}(x)+\mu'_{n,m}(x)\right] \qquad (9)\\
&\quad+(m+1)\left(\frac{n}{n+\alpha}-2x\right)\mu_{n,m}(x).
\end{aligned}$$

Proof.

$$\begin{aligned}
\mu'_{n,m}(x) &= (n+1)\left(\frac{n+\alpha}{n}\right)\Big[\sum_{k=0}^{n}\left(p_{n,k}^\alpha(x)\right)'\int_0^{\frac{n}{n+\alpha}}(t-x)^m\cdot p_{n,k}^\alpha(t)\,dt\\
&\quad-\sum_{k=0}^{n}mp_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}}(t-x)^{m-1}\cdot p_{n,k}^\alpha(t)\,dt\Big].
\end{aligned}$$

So, we get:

$$\mu'_{n,m}(x) =$$
$$= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left(p_{n,k}^{\alpha}(x)\right)'\int_0^{\frac{n}{n+\alpha}}(t-x)^m \cdot p_{n,k}^{\alpha}(t)\,dt - m\mu_{n,m-1}(x).$$

After rearranging the terms, we proceed with calculations using (5):

$$x\left(\frac{n}{n+\alpha}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right] =$$
$$= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} n\left(\frac{k}{n+\alpha}-x\right)p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}}(t-x)^m \cdot p_{n,k}^{\alpha}(t)\,dt$$
$$= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}} n\left(\frac{k}{n+\alpha}-x\right)p_{n,k}^{\alpha}(t)\,(t-x)^m\,dt.$$

We can write the expression above as:

$$x\left(\frac{n}{n+\alpha}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right] = (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\times$$
$$\times\left[\int_0^{\frac{n}{n+\alpha}} n\left(\frac{k}{n+\alpha}-t\right)p_{n,k}^{\alpha}(t)\,(t-x)^m\,dt + n\int_0^{\frac{n}{n+\alpha}}(t-x)^{m+1}\,p_{n,k}^{\alpha}(t)\,dt\right],$$

which, by using relation (5), leads to:

$$x\left(\frac{n}{n+\alpha}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right] =$$
$$= (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}} t\left(\frac{n}{n+\alpha}-t\right)\left(p_{n,k}^{\alpha}(t)\right)'(t-x)^m\,dt +$$
$$+n\mu_{n,m+1}(x)$$

Now, we write:

$$x\left(\frac{n}{n+\alpha}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right] =$$
$$= n\mu_{n,m+1}(x) + (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\times$$
$$\times\int_0^{\frac{n}{n+\alpha}}\left[x\left(\frac{n}{n+\alpha}-x\right)+\left(\frac{n}{n+\alpha}-2x\right)(t-x)-(t-x)^2\right]\left(p_{n,k}^{\alpha}(t)\right)'(t-x)^m\,dt.$$

Using integration by parts formula, we get:

$$x\left(\frac{n}{n+\alpha}-x\right)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right]=$$
$$n\mu_{n,m+1}(x)-mx\left(\frac{n}{n+\alpha}-x\right)\mu_{n,m-1}(x)-(m+1)\left(\frac{n}{n+\alpha}-2x\right)\mu_{n,m}(x)$$
$$+(m+2)\mu_{n,m+1}(x).$$

Finally, rearranging the terms we get the stated result. □

Theorem 7. *We have:*
i) $\mu_{n,0}(x)=1$;
ii)$\mu_{n,1}(x)=-\frac{2}{n+2}x+\frac{n}{(n+\alpha)(n+2)}$;
iii)$\mu_{n,2}(x)=\frac{2}{(n+2)(n+3)}\left[(n-3)x\left(\frac{n}{n+\alpha}-x\right)+\left(\frac{n}{n+\alpha}\right)^2\right]$;
iv)$\mu_{n,3}(x)=\frac{6}{(n+2)(n+3)(n+4)}\left(\frac{n}{n+\alpha}-2x\right)\left[(2n-2)x\left(\frac{n}{n+\alpha}-x\right)+\left(\frac{n}{n+\alpha}\right)^2\right]$;
v)

$$\mu_{n,4}(x)=\frac{12}{(n+2)(n+3)(n+4)(n+5)}\left[x^2\left(\frac{n}{n+\alpha}-x\right)^2\left(n^2-21n+10\right)+\right.$$
$$\left.+(6n-10)x\left(\frac{n}{n+\alpha}-x\right)\left(\frac{n}{n+\alpha}\right)^2+2\left(\frac{n}{n+\alpha}\right)^4\right].$$

Proof. i) $\mu_{n,0}(x)=D_n^\alpha(e_0,x)=1$.
ii) $\mu_{n,1}(x)=D_n^\alpha((t-x),x)=D_n^\alpha(e_1,x)-xD_n^\alpha(e_0,x)=\frac{n}{n+\alpha}x+\frac{n}{(n+2)(n+\alpha)}-x=-\frac{2}{n+2}x+\frac{n}{(n+\alpha)(n+2)}$.
iii)-v) Using the recurrence formula for the moments (9) and arranging the terms in a convenient way, we get the stated expressions for $\mu_{n,2}(x)$, $\mu_{n,3}(x)$ and $\mu_{n,4}(x)$. □

3 Convergence properties of D_n^α

Having the expressions of D_n^α for the test functions $e_0,\ e_1$ and e_2 found above, we can prove the following by applying Korovkin's theorem.

Theorem 8. *For all* $\alpha\geq 0,\ f\in C[0,1]$, *and for all* $\varepsilon\in(0,1)$, *the following holds:*

$$\lim_{n\to\infty}D_n^\alpha(f,x)=f(x),\ \textit{uniformly on}\ [0,1-\varepsilon]. \tag{10}$$

Proof. There exists $\tilde{n} \in \mathbb{N}$ such that for all $n \geq \tilde{n}$, we have $\frac{n}{n+\alpha} > 1 - \varepsilon$.

Now, using the fact that $\lim_{n\to\infty} D_n^\alpha(e_i, x) = e_i$, for $i = 0, 1, 2$, uniformly on $[0, 1-\varepsilon]$, we apply Korovkin theorem for the sequence D_n^α and get the uniform approximation of f by D_n^α on $[0, 1-\varepsilon]$. □

As it can be seen in the result above, the uniform convergence is proved only on the interval where the operators are positive. The aim of the following results is to prove that the convergence is uniform for all continous functions on the interval $[0, 1]$ even though the operators are not positive on the entire interval.

First, we will prove that $\lim_{n\to\infty} D_n^\alpha(e_l, x) = e_l$ uniformly on $[0, 1]$, where $e_l = t^l$, hence we will need the following result:

Proposition 9. *For $l \geq 0$ we have:*

$$D_n^\alpha(e_l, x) = (n+1)\frac{(n!)^2}{(n+l+1)!}\sum_{i=0}^{\min\{n,l\}}\binom{l}{i}\frac{l!}{i!}\frac{1}{(n-i)!}\left(\frac{n}{n+\alpha}\right)^{l-i}x^i. \quad (11)$$

Proof. Let $l \geq 0$.

$$D_n^\alpha(e_l, x) = (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}} t^l p_{n,k}^\alpha(t)\,dt.$$

We will treat the integral separately:

$$\begin{aligned}\int_0^{\frac{n}{n+\alpha}} t^l p_{n,k}^\alpha(t)\,dt &= \left(\frac{n+\alpha}{n}\right)^n\binom{n}{k}\left(\frac{n}{n+\alpha}\right)^{n+l+1}\beta(k+l+1, n-k+1)\\ &= \left(\frac{n}{n+\alpha}\right)^{l+1}\binom{n}{k}\frac{(k+l)!\,(n-k)!}{(n+l+1)!}\\ &= \left(\frac{n}{n+\alpha}\right)^{l+1}\frac{n!}{k!\,(n-k)!}\frac{(k+l)!\,(n-k)!}{(n+l+1)!}\\ &= \left(\frac{n}{n+\alpha}\right)^{l+1}\frac{n!}{(n+l+1)!}\frac{(k+l)!}{k!}\end{aligned}$$

Now, we have that

$$D_n^\alpha(e_l, x) = (n+1)\left(\frac{n}{n+\alpha}\right)^l\frac{n!}{(n+l+1)!}\sum_{k=0}^{n} p_{n,k}^\alpha(x)\frac{(k+l)!}{k!}.$$

In order to get the desired result, we need the derivative of order l with respect to x of the expression $x^l\,(x+y)^n$:

$$\frac{\partial^l}{\partial x^l} x^l\,(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \frac{(k+l)!}{k!}. \tag{12}$$

Now, using Leibniz's rule, we have:

$$\frac{\partial^l}{\partial x^l} x^l\,(x+y)^n = \sum_{i=0}^{l} \binom{l}{i} \frac{l!}{i!} \frac{n!}{(n-i)!} x^i\,(x+y)^{n-i}. \tag{13}$$

Substituting y from (12) and (13), with $\frac{n}{n+\alpha}-x$, and equallying the two relations we get

$$\sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{n}{n+\alpha}-x\right)^{n-k} \frac{(k+l)!}{k!} = \sum_{i=0}^{\min\{n,l\}} \binom{l}{i} \frac{l!}{i!} \frac{n!}{(n-i)!} x^i \left(\frac{n}{n+\alpha}\right)^{n-i},$$

which is

$$\left(\frac{n}{n+\alpha}\right)^n \sum_{k=0}^{n} p_{n,k}^{\alpha}\,(x)\, \frac{(k+l)!}{k!} = \sum_{i=0}^{\min\{n,l\}} \binom{l}{i} \frac{l!}{i!} \frac{n!}{(n-i)!} x^i \left(\frac{n}{n+\alpha}\right)^{n-i}.$$

Going back to $D_n^{\alpha}\,(e_l, x)$, we get

$$D_n^{\alpha}\,(e_l,x) = (n+1)\left(\frac{n}{n+\alpha}\right)^l \frac{n!}{(n+l+1)!} \sum_{k=0}^{n} p_{n,k}^{\alpha}\,(x)\, \frac{(k+l)!}{k!}$$

$$= (n+1)\left(\frac{n}{n+\alpha}\right)^l \frac{n!}{(n+l+1)!} \left(\frac{n+\alpha}{n}\right)^n \sum_{i=0}^{\min\{n,l\}} \binom{l}{i} \frac{l!}{i!} \frac{n!}{(n-i)!} x^i \left(\frac{n}{n+\alpha}\right)^{n-i}$$

$$= (n+1)\, \frac{(n!)^2}{(n+l+1)!} \sum_{i=0}^{\min\{n,l\}} \binom{l}{i} \frac{l!}{i!} \frac{1}{(n-i)!} \left(\frac{n}{n+\alpha}\right)^{l-i} x^i,$$

which is the desired result. □

Now, we can prove the announced result:

Proposition 10. *For all $l \geq 0$, we have*

$$D_n^{\alpha}\,(e_l,x) \to e_l\,(x) \text{ uniformly on } [0,1]. \tag{14}$$

Proof. We rewrite $D_n^\alpha(e_l, x)$ from 11 as follows:

$$D_n^\alpha(e_l, x) = (n+1)\frac{(n!)^2}{(n+l+1)!}\frac{1}{(n-l)!}x^l + O\left(\frac{1}{n}\right)x^{l-1} + \cdots + O\left(\frac{1}{n^l}\right)$$

Now, as $n \to \infty$, we have:

$$D_n^\alpha(e_l, x) \to e_l(x), \text{ uniformly for } x \in [0,1],$$

which completes the proof. □

Remark 11. *From Proposition 10 and the linearity of the operators D_n^α, we conclude that for all polynomials $P \in \mathcal{P}_{[0,1]}$, $D_n^\alpha(P, x) \to P(x)$ uniformly for $x \in [0,1]$, where with $\mathcal{P}_{[0,1]}$ we denote the polynomials on $[0,1]$.*

Now, we are one step closer to prove the uniform convergence for all continous functions on $[0,1]$. In order to demonstrate the result, we need to show that the norm of the operators are bounded.

Proposition 12. *We have that:*

$$\|D_n^\alpha\| \le e^{2\alpha}, \tag{15}$$

for all $n \in \{1,2,\dots\}$, and $\alpha \ge 0$.

Proof. Let $f \in C[0,1]$. We have that $|f(x)| \le \|f\|$ for all $x \in [0,1]$, where $\|\cdot\|$ is the supremum norm. We get:

$$\begin{aligned}
|D_n^\alpha(f,x)| &= \left|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)\, f(t)\, dt\right| \\
&\le (n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}}\left|p_{n,k}^\alpha(t)\right| |f(t)|\, dt \\
&\le (n+1)\left(\frac{n+\alpha}{n}\right)\|f\|\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)\, dt \\
&= \|f\|\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|.
\end{aligned}$$

Now, we consider the following two cases:

Case 1. For $x \in \left[0, \frac{n}{n+\alpha}\right]$, we have

$$\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right| = \sum_{k=0}^{n} p_{n,k}^\alpha(x) = 1,$$

therefore $|D_n^\alpha(f,x)| \le \|f\|, \ \forall\, x \in \left[0, \frac{n}{n+\alpha}\right]$.

Case 2. For $x \in \left(\frac{n}{n+\alpha}, 1\right]$ we have:

$$\begin{aligned}
\sum_{k=0}^{n} \left|p_{n,k}^\alpha(x)\right| &= \left(\frac{n+\alpha}{n}\right)^n \sum_{k=0}^{n} \binom{n}{k} x^k \left(x - \frac{n}{n+\alpha}\right)^{n-k} \\
&= \left(\frac{n+\alpha}{n}\right)^n \left(2x - \frac{n}{n+\alpha}\right)^n \\
&= \left(2\frac{n+\alpha}{n}x - 1\right)^n .
\end{aligned}$$

Now, for $\frac{n}{n+\alpha} < x \le 1$, we have that

$$\left(1 + \frac{2\alpha}{n}\right)^n \ge \left(2\frac{n+\alpha}{n}x - 1\right)^n > 1.$$

Taking $a_n = \left(1 + \frac{2\alpha}{n}\right)^n$, we have that a_n is a increasing sequence bounded by $e^{2\alpha}$. For $\alpha \ge 0$, we have that

$$\sum_{k=0}^{n} \left|p_{n,k}^\alpha(x)\right| \le e^{2\alpha},$$

therefore $|D_n^\alpha(f,x)| \le e^{2\alpha}\|f\|, \ \forall\, x \in \left(\frac{n}{n+\alpha}, 1\right]$, and $\alpha \ge 0$.

To conclude, for $\alpha \ge 0$ we have that $\max\{1, e^{2\alpha}\} = e^{2\alpha}$, which imply that

$$|D_n^\alpha(f,x)| \le e^{2\alpha}\|f\|, \ \forall\, x \in [0,1].$$

□

Theorem 13. *For all $f \in C[0,1]$, we have:*

$$D_n^\alpha(f,x) \to f(x) \ \textit{uniformly on } [0,1]. \tag{16}$$

Proof. From Remark 11 we have that operators D_n^α approximate uniformly polynomials on $[0,1]$, and from Proposition 12 we have that the norms of the operators D_n^α are bounded on $[0,1]$. From these observations, and the fact that $\mathcal{P}_{[0,1]}$ is dense in $C[0,1]$, we conclude that $D_n^\alpha(f,x) \to f(x)$ uniformly on $[0,1]$, $\forall f \in C[0,1]$. □

In the following we will present a result in terms of the usual modulus of continuity $\omega(f,\delta)$.

Theorem 14. *For $f \in C[0,1]$ and $x \in [0,1]$ we have:*

$$\left|D_n^{\alpha}\left(f,x\right)-f\left(x\right)\right| \leq \left\{1+\frac{1}{\delta}\sqrt{\frac{2}{n+2}\left[x\left(\frac{n}{n+\alpha}\right)+\frac{1}{n+3}\right]}\right\}\omega\left(f,\delta\right), \quad (17)$$

for $x \in \left[0, \frac{n}{n+\alpha}\right]$, *and*

$$\left|D_n^{\alpha}\left(f,x\right)-f\left(x\right)\right| \leq \left\{e^{2\alpha}+\frac{e^{2\alpha}}{\delta'}\left[\frac{2\alpha}{n}+\frac{n}{\left(n+\alpha\right)\left(n+2\right)}\right]\right\}\omega\left(f,\delta'\right), \quad (18)$$

for $x \in \left(\frac{n}{n+\alpha}, 1\right]$.

Proof. In order to prove the stated result we will consider two cases:

Case 1. Consider $x \in \left[0, \frac{n}{n+\alpha}\right]$. In this case there is no problem with positivity of the operators. Take:

$$\begin{aligned}
\left|D_n^{\alpha}\left(f,x\right)-f\left(x\right)\right| &= \left|\left(n+1\right)\frac{n+\alpha}{n}\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}\left(t\right)f\left(t\right)dt-f\left(x\right)\right| \\
&\leq \left(n+1\right)\frac{n+\alpha}{n}\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}\left(t\right)\left|f\left(t\right)-f\left(x\right)\right|dt \\
&\leq \left[\left(n+1\right)\frac{n+\alpha}{n}\sum_{k=0}^{n}p_{n,k}^{\alpha}\left(x\right)\int_0^{\frac{n}{n+\alpha}}p_{n,k}^{\alpha}\left(t\right)\left(1+\frac{\left|t-x\right|}{\delta}\right)dt\right]\omega\left(f,\delta\right).
\end{aligned}$$

Now, by applying Hölder's inequality twice and arranging the terms, we get:

$$\begin{aligned}
\left|D_n^{\alpha}\left(f,x\right)-f\left(x\right)\right| &\leq \left\{1+\frac{1}{\delta}\left[D_n^{\alpha}\left(\left(t-x\right)^2,x\right)\right]^{\frac{1}{2}}\right\}\omega\left(f,\delta\right) \\
&= \left\{1+\frac{1}{\delta}\left[\mu_{n,2}\left(x\right)\right]^{\frac{1}{2}}\right\}\omega\left(f,\delta\right) \\
&= \left\{1+\frac{1}{\delta}\sqrt{\frac{2}{n+2}\left[\frac{n-3}{n+3}x\left(\frac{n}{n+\alpha}-x\right)+\frac{1}{n+3}\left(\frac{n}{n+\alpha}\right)^2\right]}\right\}\omega\left(f,\delta\right) \\
&\leq \left\{1+\frac{1}{\delta}\sqrt{\frac{2}{n+2}\left[x\left(\frac{n}{n+\alpha}-x\right)+\frac{1}{n+3}\right]}\right\}\omega\left(f,\delta\right),
\end{aligned}$$

which completes the first assessment.

Case 2.In this case we will consider $x \in \left(\frac{n}{n+\alpha}, 1\right]$, interval on which the operator is not positive. We will proceed by direct calculation:

$$\left|D_n^\alpha(f,x) - f(x)\right| \left|(n+1)\frac{n+\alpha}{n}\sum_{k=0}^{n} p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t) f(t)\,dt - f(x)\right|$$

$$\leq (n+1)\frac{n+\alpha}{n}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)\left|f(t)-f(x)\right| dt$$

$$\leq \left[(n+1)\frac{n+\alpha}{n}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)\left(1+\frac{|t-x|}{\delta'}\right) dt\right]\omega\left(f,\delta'\right)$$

$$= \Big[(n+1)\frac{n+\alpha}{n}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)\,dt +$$

$$+\frac{(n+1)}{\delta'}\frac{n+\alpha}{n}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)(x-t)\,dt\Big]\omega\left(f,\delta'\right)$$

$$= \left[\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right| + \frac{x}{\delta'}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right| - \frac{1}{\delta'}\frac{n}{n+\alpha}\sum_{k=0}^{n}\left|p_{n,k}^\alpha(x)\right|\frac{k+1}{n+2}\right]\omega\left(f,\delta'\right)$$

$$= \left\{\left(\frac{2x(n+\alpha)}{n}-1\right)^n + \frac{x}{\delta'}\left(\frac{2x(n+\alpha)}{n}-1\right)^n\right.$$

$$\left. -\frac{1}{\delta'}\frac{n}{(n+2)(n+\alpha)}\left(\frac{2x(n+\alpha)}{n}-1\right)^n - \frac{nx}{\delta'(n+2)}\left(\frac{2x(n+\alpha)}{n}-1\right)^{n-1}\right\}\omega(f,\delta).$$

Taking into account that $x \leq 1$, we have

$$\left(\frac{2x(n+\alpha)}{n}-1\right)^n \leq \left(\frac{2(n+\alpha)}{n}-1\right)^n = \left(1+\frac{2\alpha}{n}\right)^n \leq e^{2\alpha},$$

and with the same idea we have

$$\left(\frac{2x(n+\alpha)}{n}-1\right)^{n-1} \leq \left(\frac{2(n+\alpha)}{n}-1\right)^{n-1} =$$

$$\left(1+\frac{2\alpha}{n}\right)^{\frac{n}{2\alpha}\frac{2\alpha}{n}(n-1)} \leq e^{2\alpha\frac{n-1}{n}} \leq e^{2\alpha}.$$

With these remarks, we have:

$$|D_n^\alpha(f,x)-f(x)| \leq \left\{e^{2\alpha}+\frac{e^{2\alpha}}{\delta'}\times\right.$$

$$\left.\left[x\left(\frac{2x(n+\alpha)}{n}-1\right)-\frac{n}{(n+\alpha)(n+2)}\left(\frac{2x(n+\alpha)}{n}-1\right)-\frac{nx}{(n+2)}\right]\right\}\omega(f,\delta')$$

$$=\left\{e^{2\alpha}+\frac{e^{2\alpha}}{\delta'}\left[2x^2\frac{(n+\alpha)}{n}-2x+\frac{n}{(n+\alpha)(n+2)}\right]\right\}\omega(f,\delta').$$

We denote by $P(x)$ the quadratic function:

$$P(x)=2x^2\frac{(n+\alpha)}{n}-2x+\frac{n}{(n+\alpha)(n+2)},$$

for $x\in\left(\frac{n}{n+\alpha},1\right]$. $P(x)$ attains its minimum at $x_0=\frac{n}{2(n+\alpha)}\leq\frac{n}{n+\alpha}$, which implies that $P(x)$ is an increasing function on $\left(\frac{n}{n+\alpha},1\right]$ with the maximum value $P(1)=\frac{2\alpha}{n}+\frac{n}{(n+\alpha)(n+2)}$.

Hence,

$$|D_n^\alpha(f,x)-f(x)| \leq \left\{e^{2\alpha}+\frac{e^{2\alpha}}{\delta'}\left[\frac{2\alpha}{n}+\frac{n}{(n+\alpha)(n+2)}\right]\right\}\omega(f,\delta'),$$

for $x\in\left(\frac{n}{n+\alpha},1\right]$, which completes the proof. □

Remark 15. *If we take*

$$\delta=\delta'=\max\left\{\sqrt{\frac{2}{n+2}\left[\frac{n}{n+\alpha}+\frac{1}{n+3}\right]},\frac{2\alpha}{n}+\frac{n}{(n+\alpha)(n+2)}\right\}, \tag{19}$$

then $\| D_n^\alpha(f)-f \|\leq 2e^{2\alpha}\omega(f,\delta)$.

4 Voronovskaja type result

In this section we will prove a Voronovskaja type theorem.

Theorem 16. *Let $f\in C[0,1)$ be a bounded, two times derivable function at the point $x\in(0,1)$. Then, the following holds:*

$$\lim_{n\to\infty} n\left[D_n^\alpha(f,x)-f(x)\right]=(1-2x)f'(x)+x(1-x)f''(x). \tag{20}$$

Proof. We take $x \in (0,1)$ fixed and take the Taylor expansion of f of order two at point x :

$$f(t) = f(x) + (t-x) f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 g(t-x), \tag{21}$$

where g is a bounded function having the property $g(t-x) \to 0$ when $t \to x$.

We denote by $\delta(x,t) = (t-x)^2 g(t-x)$, $t \in [0,1]$.

Now, we apply the operators D_n^α to $f(t)$ in (21) and we get:

$$D_n^\alpha(f,x) = f(x)\mu_{n,0}(x) + f'(x)\mu_{n,1}(x) + \frac{f''(x)}{2}\mu_{n,2}(x) + D_n^\alpha(\delta(x,\cdot),x) \tag{22}$$

$$= f(x) + f'(x)\left(-\frac{2}{n+2}x + \frac{n}{(n+\alpha)(n+2)}\right) + \frac{f''(x)}{2}\frac{2}{(n+2)(n+3)} \times$$
$$\times \left[\left(\frac{n}{n+\alpha} - x\right)\left(\frac{n}{n+\alpha} + (n-2)x\right) + x^2\right] + D_n^\alpha(\delta(x,\cdot),x).$$

Then

$$\lim_{n\to\infty} n\left[D_n^\alpha(f,x) - f(x)\right] = (1-2x) f'(x) + x(1-x) f''(x) + \lim_{n\to\infty} n D_n^\alpha(\delta(x,\cdot),x). \tag{23}$$

Now, we will prove that $\lim_{n\to\infty} n D_n^\alpha(\delta(x,\cdot),x) = 0$. To achieve this, we will use Hölder's inequality

$$|n D_n^\alpha(\delta(x,\cdot),x)| =$$
$$= n\left|(n+1)\left(\frac{n+\alpha}{n}\right)\sum_{k=0}^{n} p_{n,k}^\alpha(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)(t-x)^2 g(t-x)\,dt\right|$$
$$\le n(n+1)\left(\frac{n+\alpha}{n}\right)\sqrt{\sum_{k=0}^{n} p_{n,k}^\alpha(x)\sum_{k=0}^{n}\left(\int_0^{\frac{n}{n+\alpha}} p_{n,k}^\alpha(t)(t-x)^2 g(t-x)\,dt\right)^2}.$$

We treat the integral part separately using again Hölder's inequality:

$$\left(\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-x)^2 g(t-x)\,dt\right)^2 =$$
$$=\left(\int_0^{\frac{n}{n+\alpha}} \sqrt{p_{n,k}^{\alpha}(t)}\,(t-x)^2\,\sqrt{p_{n,k}^{\alpha}(t)}g(t-x)\,dt\right)^2$$
$$\leq\left(\sqrt{\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-x)^4\,dt\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\,g^2(t-x)\,dt}\right)^2$$
$$=\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-x)^4\,dt\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\,g^2(t-x)\,dt$$

Getting back with this result, we have:

$$|nD_n^{\alpha}(\delta(x,\cdot),x)| \leq n(n+1)\left(\frac{n+\alpha}{n}\right)\times$$
$$\times\sqrt{\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)(t-x)^4\,dt\sum_{k=0}^{n} p_{n,k}^{\alpha}(x)\int_0^{\frac{n}{n+\alpha}} p_{n,k}^{\alpha}(t)\,g^2(t-x)\,dt}$$
$$=n\sqrt{\mu_{n,4}(x)}\sqrt{D_n^{\alpha}\left(g^2(\cdot-x),x\right)}.$$

Taking a look at the 4-th moment of the operator, we observe that $\mu_{n,4}(x) = O\left(\frac{1}{n^2}\right)$. Then, we have that $n\sqrt{\mu_{n,4}(x)} = O(1)$.

From the convergence theorem we know that

$$\lim_{n\to\infty} D_n^{\alpha}\left(g^2(\cdot-x),x\right) = g^2(x-x) = 0,$$

which implies that

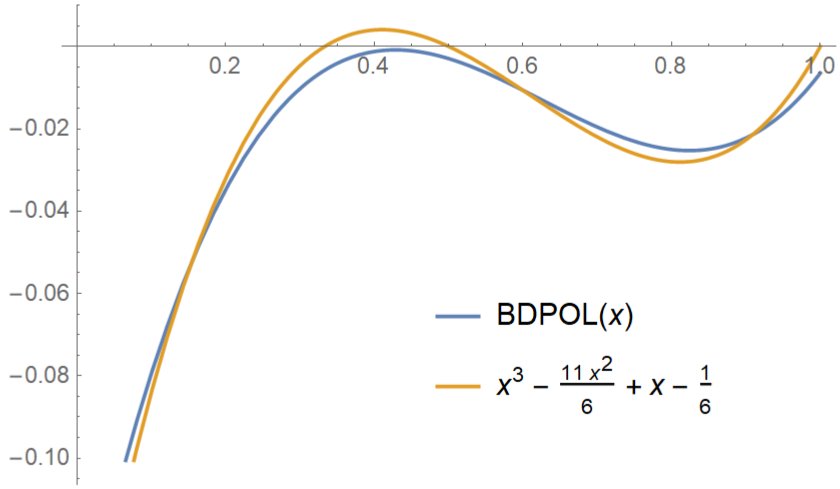
$$\lim_{n\to\infty} nD_n^{\alpha}\left(g^2(\cdot-x),x\right) = 0,$$

so our proof is complete. □

5 Some graphs

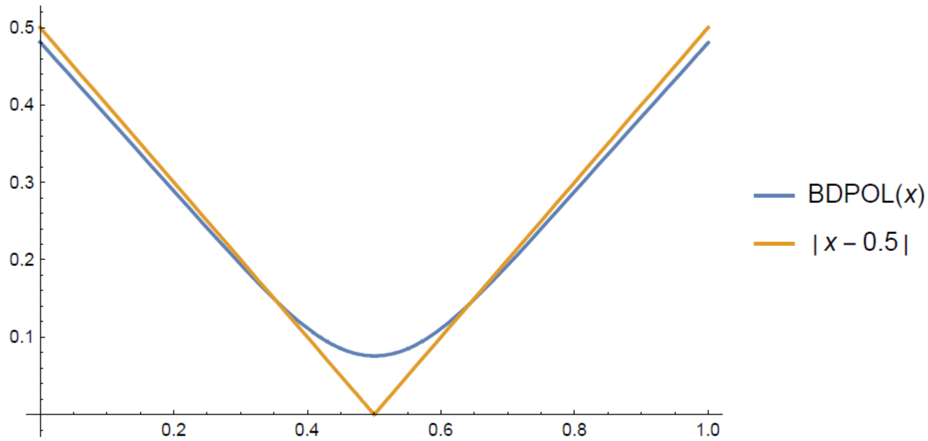
For the first example we considered the function $f(x) = x^3 - (11/6)x^2 + x - 1/6$ for $x \in [0, 1]$. In this case we have obtained Figure 1.

Figure 1: $\alpha = 1$ and $n = 50$



Secondly we took the function $f(x) = |x - 0.5|$ for $x \in [0, 1]$ and we got Figure 2.

Figure 2: $\alpha = 1$ and $n = 50$



6 Conclusions

As it can be seen in the figures above, as well as in the presented results, the operators D_n^α have good properties of approximation even though they are not positive operators on the entire $[0, 1]$.

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