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About the B-concavity of functions with many variables

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Abstract

The paper deals with the study of the property of B-concavity and BB concavity in the bi-dimesional case and with the relation between these properties and the Bernstein operators defined on a simplex.

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Key Words: B concavity, BB-concavity, Bernstein operators

1 Introduction

The notion of $\mathbb{B}-$concavity for functions with many variables was introduced in paper [3].

In paper [6], authors proved that a function of one variable which is increasing and $\mathbb{B}-$ concave is transformed by Bernstein operators in a function with the same properties. It is well known that Bernstein operators preserve other types of properties of functions like the property of higher order convexity, see [5], or the property of higher order quasiconvexity, see [4].

In this paper, we introduce the notion of $\mathbb{BB}-$concavity which is a slight modification of the $\mathbb{B}-$concavity for functions of two variables. Also, we prove that Bernstein operators on bidimensional simplex transform a function which is increasing and $\mathbb{BB}-$concave in a $\mathbb{B}-$concave function.

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2 $\mathbb{B}$-concavity and $\mathbb{BB}$-concavity

For $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$, let

$$a \vee b := (\max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_n, b_n\}).$$

Definition 2.1. *[3] A function $f : M \subset \mathbb{R}^n_+ \to (0, \infty)$ is $\mathbb{B}-$concave function if it has the following properties:*

i) $\lambda a \vee b \in M, \forall a, b \in M, \lambda \in [0, 1]$;

ii) $f(\lambda a \vee b) \geq \lambda f(a) \vee f(b), \forall a, b \in M, \lambda \in [0, 1]$.

As usually, we define a partial order $'' \leq ''$ on $\mathbb{R}^n$ by: $a \leq b$ iff $a_i \leq b_i, \forall i = \overline{1, n}$, when $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n)$.

A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is named increasing if $f(a) \leq f(b), \forall a, b \in D, a \leq b$.

Theorem 2.1. *Let be $M \subset R^2_+$ a domain which satisfies condition i) from Definition 2.1. If $f : M \longrightarrow (0, \infty)$ is an increasing function, then the following affirmations are equivalent:*

i) f is a $\mathbb{B}$-concave function

ii) For all $v = (v_1, v_2) \in M$ function

$$\varphi_v : [0, 1] \longrightarrow \mathbb{R}, \varphi_v(t) = \frac{t}{f(tv)}, t \in [0, 1]$$

is increasing.

Proof. For the first part of the theorem, we suppose that f is $\mathbb{B}-$ concave and increasing function.

Let be $v \in M, x, y \in [0, 1]$ and $\lambda \in [0, 1]$.

Because f is $\mathbb{B}-$ concave function, then:

$$f\Big((\lambda x \vee y)v\Big) = f\Big((\lambda x)v \vee (yv)\Big) \geq \lambda f(xv) \vee f(yv).$$

We denote $g(t) = f(tv), g : [0, 1] \longrightarrow \mathbb{R}_+, t \in [0, 1]$, so

$$g(\lambda x \vee y) \geq \lambda g(x) \vee g(y). \tag{1}$$

Let $t_1 \leq t_2$. We have to prove that $\frac{t_1}{g(t_1)} \leq \frac{t_2}{g(t_2)}$ which ca be rewritten:

$$\frac{t_1}{t_2} \cdot g(t_2) \leq g\Big(\frac{t_1}{t_2} \cdot t_2\Big). \tag{2}$$

In relation (1) we replace $y = 0$. So we obtain $g(\lambda x) \geq \lambda g(x) \vee g(0)$.
But $\lambda g(x) \vee g(0) \geq \lambda g(x)$, so $g(\lambda x) \geq \lambda g(x)$.
In previous relation we replace $x = t_2$ and $\lambda = \dfrac{t_1}{t_2}$.
Then relation (2) is true, so we can conclude that g is increasing.

For the second part of the theorem, we suppose that f and $\dfrac{t}{f(tv)}$ are increasing functions.

Let be $a, b \in M, \lambda \in [0,1]$. We have $\dfrac{\lambda}{f(\lambda a)} \leq \dfrac{1}{f(a)}$.
Further, $\lambda f(a) \leq f(\lambda a) \leq f(\lambda a \vee b)$ and $f(b) \leq f(\lambda a \vee b)$, so

$$f(\lambda a \vee b) \geq \lambda f(a) \vee f(b)$$

and we can conclude that f is $\mathbb{B}-$concave function. □

Corollary 2.1. *Let be $f : M \to \mathbb{R}_+^*$ an increasing and differentiable function in both variables, where $M \subset R_+^2$ is a domain which satisfies condition i) from Definition 2.1. The inequality*

$$f(tv) - t\Big(\frac{\partial f(tv)}{\partial x} \cdot v_1 + \frac{\partial f(tv)}{\partial y} \cdot v_2\Big) > 0 \tag{3}$$

holds $\forall v \in M, \forall t \in [0,1]$, where $v = (v_1, v_2)$ if and only if f is a $\mathbb{B}$-concave function.

Proof. First, we prove (3) in condition f is increasing and $\mathbb{B}-$concave.
If we derive the function φ_v, where

$$\varphi_v : [0,1] \to \mathbb{R}, \varphi_v(t) = \frac{t}{f(tv)}$$

we obtain:

$$\frac{d}{dt}\varphi_v(t) = \frac{f(tv) - t \cdot \Big(\dfrac{\partial f(tv)}{\partial x} \cdot v_1 + \dfrac{\partial f(tv)}{\partial y} \cdot v_2\Big)}{f^2(tv)}. \tag{4}$$

Because f is $\mathbb{B}-$concave, the function φ_v is increasing from the previous theorem, so the fraction (4) is greater or equal than 0, so relation (3) is true.

The converse part of the proof can be obtained immediately from Theorem 2.1, because relation (3) implies that function $\dfrac{t}{f(tv)}, t \in [0,1]$ is increasing. Then from the converse part of Theorem 2.1, it follows that f is $\mathbb{B}$-concave function. □

Remark 2.1. *The inequality (3) is equivalent to:*

$$f(x,y) - x \cdot \frac{\partial f}{\partial x}(x,y) - y \cdot \frac{\partial f}{\partial y}(x,y) \geq 0, \forall (x,y) \in M.$$

Indeed, we can take $v = (x,y)$ and $t = 1$.

Definition 2.2. *Let be*

$$\Delta = \{(x,y) | x \geq 0, y \geq 0, x + y \leq 1\}. \tag{5}$$

A function $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}-$concave function if it is true the following:

$$f(x,y) - \alpha f(z,y) - \beta f(x,w) \leq 0, \tag{6}$$

$\forall (x,y) \in \Delta, (z,y) \in \Delta, (x,w) \in \Delta$, *such that* $z \leq x, w \leq y, z \cdot w < x \cdot y$, *where*

$$\alpha = \frac{x \cdot y - x \cdot w}{x \cdot y - z \cdot w} \text{ and } \beta = \frac{x \cdot y - y \cdot z}{x \cdot y - z \cdot w}. \tag{7}$$

Remark 2.2. *It is simple to show that Δ satisfies condition $i)$ from Definition 2.1.*

Remark 2.3. *The numbers α and β are the unique coefficients for which:*

$$(x,y) = \alpha \cdot (z,y) + \beta \cdot (x,w).$$

Remark 2.4. *A function $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}-$concave function if it is true the following:*

$$f\Big(\alpha \cdot (z,y) + \beta \cdot (x,w)\Big) \leq \alpha f(z,y) + \beta f(x,w), \tag{8}$$

$\forall (x,y) \in \Delta, (z,y) \in \Delta, (x,w) \in \Delta$ *and* $z \leq x, w \leq y, z \cdot w < x \cdot y$, *where* $\alpha \in [0,1]$ *and* $\beta \in [0,1]$ *are indicated in (7) and Δ is indicated in (5).*

Theorem 2.2. *If $f : \Delta \to \mathbb{R}$ is a differentiable $\mathbb{BB}$-concave function then it is true the following:*

$$f(x,y) - x \cdot \frac{\partial f}{\partial x}(x,y) - y \cdot \frac{\partial f}{\partial y}(x,y) \geq 0, \forall (x,y) \in \Delta. \tag{9}$$

Proof. The inequality from (8), which is the condition of a function to be $\mathbb{BB}$-concave is equivalent to:

$$\alpha\Big(f(x,y) - f(z,y)\Big) + \beta\Big(f(x,y) - f(x,w)\Big) + (1 - \alpha - \beta) f(x,y) \leq 0. \tag{10}$$

Using (7), we have:

$$1-\alpha-\beta=-\frac{(y-w)\cdot(x-z)}{x\cdot y-z\cdot w}. \tag{11}$$

Then, we replace (7) and(11) in relation (10) and we have:

$$-\frac{(y-w)\cdot(x-z)}{x\cdot y-z\cdot w}\cdot f(x,y)-\frac{x\cdot y-x\cdot w}{x\cdot y-z\cdot w}\cdot\frac{f(z,y)-f(x,y)}{z-x}\cdot(z-x)- \\ -\frac{x\cdot y-y\cdot z}{x\cdot y-z\cdot w}\cdot\frac{f(x,w)-f(x,y)}{w-y}\cdot(w-y)\le 0.$$

From the Mean value theorem there are the points ξ and η, $z<\xi<x$, $w<\eta<y$ such that he previous relation can be rewritten in the following form:

$$\frac{(y-w)\cdot(x-z)}{x\cdot y-z\cdot w}\cdot f(x,y)-(x-z)\cdot\frac{x\cdot y-x\cdot w}{x\cdot y-z\cdot w}\cdot\frac{\partial f}{\partial x}(\xi,y) \\ -(y-w)\frac{x\cdot y-y\cdot z}{x\cdot y-z\cdot w}\cdot\frac{\partial f}{\partial y}(x,\eta)\ge 0.$$

After simplification and passing to limit $z\to x$ and $w\to y$ we obtain:

$$f(x,y)-x\cdot\frac{\partial f}{\partial x}(x,y)-y\cdot\frac{\partial f}{\partial y}(x,y)\ge 0$$

which is exactly relation (9). □

3 $\mathbb{B}$-concavity of Bernstein operators on two dimensional simplex

Let B_n be the Bernstein operators of two variables defined on $C(\Delta)$, given by:

$$B_n(f)(x,y)=\sum_{k\ge 0,l\ge 0,k+l\le n} p_{n,k,l}(x,y)\cdot f\Big(\frac{k}{n},\frac{l}{n}\Big), f\in C(\Delta),(x,y)\in\Delta, \tag{12}$$

where:

$$p_{n,k,l}(x,y)=\frac{n!}{k!l!(n-l-k)!}\cdot x^k\cdot y^l\cdot(1-x-y)^{n-k-l}. \tag{13}$$

The equation (12) can be rewritten:

$$B_n(f)(x,y)=\sum_{k=0}^{n} p_{n,k}(x)\sum_{l=0}^{n-k} p_{n-k,l}\Big(\frac{y}{1-x}\Big)\cdot f\Big(\frac{k}{n},\frac{l}{n}\Big), \tag{14}$$

where:

$$p_{r,s}(z) = \frac{r!}{s!(r-s)!} \cdot z^s \cdot (1-z)^{r-s}, z \in [0,1]. \tag{15}$$

Theorem 3.1. *If $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}$-concave and increasing, then*

$$B_n(f)(x,y) - x \cdot \frac{\partial}{\partial x} B_n(f)(x,y) - y \frac{\partial}{\partial y} B_n(f)(x,y) \geq 0.$$

Proof. We have:

$$\begin{aligned}
\frac{\partial}{\partial z} p_{r,s}(z) &= \binom{r}{s} \Big[s \cdot z^{s-1} \cdot (1-z)^{r-s} - (r-s) \cdot z^s \cdot (1-z)^{r-s-1} \Big] \\
&= r \cdot \Big[\binom{r-1}{s-1} \cdot z^{s-1} \cdot (1-z)^{r-s} - \binom{r-1}{s} \cdot z^s \cdot (1-z)^{r-s-1} \Big] \\
&= r \cdot \Big[p_{r-1,s-1}(z) - p_{r-1,s}(z) \Big].
\end{aligned}$$

We compute the derivate of $B_n(f)$ with regard to y:

$$\begin{aligned}
&\frac{\partial}{\partial y} B_n(f)(x,y) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \sum_{l=0}^{n-k} \frac{1}{1-x} (p_{n-k,l})' \Big(\frac{y}{1-x} \Big) \cdot f\Big(\frac{k}{n}, \frac{l}{n} \Big) \\
= \;& \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{n-k}{1-x} \cdot \sum_{l=0}^{n-k} \Big[p_{n-k-1,l-1}\Big(\frac{y}{1-x} \Big) - p_{n-k-1,l}\Big(\frac{y}{1-x} \Big) \Big] \times \\
&\times f\Big(\frac{k}{n}, \frac{l}{n} \Big) \\
= \;& n \cdot \sum_{k=0}^{n} p_{n-1,k}(x) \cdot \sum_{l=0}^{n-k-1} p_{n-k-1,l}\Big(\frac{y}{1-x} \Big) \cdot \Big[f\Big(\frac{k}{n}, \frac{l+1}{n} \Big) - f\Big(\frac{k}{n}, \frac{l}{n} \Big) \Big] \\
= \;& n \cdot \sum_{k=0}^{n} \binom{n-1}{k} \cdot x^k \cdot (1-x)^{n-1-k} \cdot \sum_{l=0}^{n-k-1} \binom{n-k-1}{l} \cdot \Big(\frac{y}{1-x} \Big)^l \times \\
&\times \Big(1 - \frac{y}{1-x} \Big)^{n-k-1-l} \cdot \Big[f\Big(\frac{k}{n}, \frac{l+1}{n} \Big) - f\Big(\frac{k}{n}, \frac{l}{n} \Big) \Big] \\
= \;& n \cdot \sum_{k \geq 0, l \geq 0, k+l \leq n-1} \binom{n-1}{k, l, n-1-k-l} \cdot x^k \cdot y^l \cdot (1-x-y)^{n-k-1-l} \times \\
&\times \Big[f\Big(\frac{k}{n}, \frac{l+1}{n} \Big) - f\Big(\frac{k}{n}, \frac{l}{n} \Big) \Big],
\end{aligned}$$

In a similar way, we obtain the derivative of $B_n(f)$ with regard to x:

$$\begin{aligned}\frac{\partial}{\partial x}B_n(f)(x,y) &= n\cdot\sum_{k\geq 0,l\geq 0,k+l\leq n-1}\binom{n-1}{k,l,n-1-k-l}\cdot\\ &\cdot\quad x^k\cdot y^l\cdot(1-x-y)^{n-k-1-l}\cdot\Big[f\Big(\frac{k+1}{n},\frac{l}{n}\Big)-f\Big(\frac{k}{n},\frac{l}{n}\Big)\Big],\end{aligned}$$

where:

$$\binom{n-1}{k,l,n-1-k-l}=\frac{(n-1)!}{k!l!(n-1-k-l)!}.$$

We substract from $B_n(f)(x,y)$, the derivative $\frac{\partial}{\partial x}B_n(f)(x,y)$ multipled by x and derivative $\frac{\partial}{\partial y}B_n(f)(x,y)$ multipled by y and we obtain:

$$\sum_{k\geq 0,l\geq 0,k+l\leq n} p_{n,k,l}(x,y)\Big\{f\Big(\frac{k}{n},\frac{l}{n}\Big)-k\cdot\Big[f\Big(\frac{k}{n},\frac{l}{n}\Big)-f\Big(\frac{k-1}{n},\frac{l}{n}\Big)\Big]-$$
$$-l\cdot\Big[f\Big(\frac{k}{n},\frac{l}{n}\Big)-f\Big(\frac{k}{n},\frac{l-1}{n}\Big)\Big]\Big\}.$$

It is sufficient to have

$$f\Big(\frac{k}{n},\frac{l}{n}\Big)-k\cdot\Big[f\Big(\frac{k}{n},\frac{l}{n}\Big)-f\Big(\frac{k-1}{n},\frac{l}{n}\Big)\Big]-$$
$$-l\cdot\Big[f\Big(\frac{k}{n},\frac{l}{n}\Big)-f\Big(\frac{k}{n},\frac{l-1}{n}\Big)\Big]\geq 0$$

which is equivalent to

$$k\cdot f\Big(\frac{k-1}{n},\frac{l}{n}\Big)+l\cdot f\Big(\frac{k}{n},\frac{l-1}{n}\Big)\geq(k+l-1)\cdot f\Big(\frac{k}{n},\frac{l}{n}\Big). \tag{16}$$

We denote:

$$(x,y):=\Big(\frac{k}{n},\frac{l}{n}\Big),(z,y):=\Big(\frac{k-1}{n},\frac{l}{n}\Big),(x,w):=\Big(\frac{k}{n},\frac{l-1}{n}\Big).$$

Hence $z<x$ and $w<y$.

But, f is $\mathbb{BB}$-concave, so considering relation (6), relation (16) is true. □

Corollary 3.1. *If $f\in C(\Delta)$ is $\mathbb{BB}-$concave and increasing in both variables, then $B_n(f)$ is $\mathbb{B}-$concave.*

Proof. Because f is increasing, $B_n(f)$ is also increasing.

So, from Remark (2.1) and Theorem 3.1, $B_n(f)$ is $\mathbb{B}-$concave. □

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