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The spectral discretization of the second-order wave equation

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Abstract

In this paper we deal with the discretization of the second order wave equation by the implicit Euler scheme for the time and the spectral method for the space. We prove that the time semi discrete and the full discrete problems are well posed. We show an optimal error estimates related to both variables time and space.

1 Introduction

The phenomena of wave propagation characterize many applications. We distinguish essentially three types of waves: acoustic waves, i.e. waves which propagate in a fluid (water or air for example); elastic waves, i.e. waves propagating in a solid and finally electromagnetic waves such as light. In this paper, We will handle the acoustic wave equation for its simplicity (scalar model). In another side, it embodies the main concepts related to all other types (elastic, electromagnetic...). The acoustic waves are expressed by the same type of equations: hyperbolic equations of order 2 of the form:

$$\partial_t^2 \varphi - A\varphi = 0$$

where A is a differential operator in space of order 2. Let Ω an open bounded connected domain of $\mathbb{R}^d,\ d = 2$ or 3. Γ is its Lipschitz continuous boundary

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and T a positive real number. We denote by $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$ according to the dimension. Let the following boundary value problem of the wave equation

$$\begin{cases} \partial_t^2\varphi - \Delta\varphi = 0 & \text{in} \quad \Omega\times]0,\mathrm{T}[, \\ \varphi = 0 & \text{on} \quad \Gamma\times]0,\mathrm{T}[, \\ \varphi(.,0) = \varphi_0 & \text{in} \quad \Omega, \\ \partial_t\varphi(.,0) = \psi_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where the wave φ is the unknown defined on $\Omega\times]0,T[$ and (φ_0, ψ_0) are the data functions defined on Ω.
Many works have been interested in the a priori and a posteriori analysis of hyperbolic partial differential equations such as the wave equations see [3, 7, 8, 9, 10, 14, 15]. In this paper, we propose for the second order wave equation a discretization by an implicit Euler scheme for the time and spectral method for the space. In our previous works, we performed a similar analysis of the spectral element discretization but for the heat equation ([1, 2]). The spectral method is widely used in the numerical resolution of partial differential equations due to its high precision. It was first introduced by Patera [13] to solve an incompressible flow problem by combining the spectral method and the finite element method.

The paper is organized as follows:

• Section 2 is devoted to recalling the characteristics of the wave equation. We prove some energy estimates.

• In section 3, we describe the time semi-discrete problem. We discretize the second time derivative by using a second difference quotient of the solution on a non-uniform temporal grid. The second-order wave equation is transformed as a first-order system. We prove that this time discretization is equivalent to an implicit Euler time discretization of the associated first-order system. We show that the time semi discrete problem is stable and we prove optimal a priori time error estimate.

• Section 4 yields the fully discrete problem where the time discretization is combined with a space spectral discretization. We prove that the fully discrete problem is well posed and we show an unconditional stability condition. We establish an optimal a priori error estimate.

2 Some characteristics of the wave equation

Let $H^s(\Omega)$, $s > 0$, the Sobolev spaces associated with the norm $\| \,.\, \|_{s,\Omega}$ and the semi-norm $| \,.\, |_{s,\Omega}$. The space $H^1_0(\Omega)$ stands for the closure in $H^1(\Omega)$ of the space of infinitely differentiable functions with compact support in Ω and

$H^{-1}(\Omega)$ is its dual space. The scalar product and its associate norm on the space $L^2(\Omega)$ are denoted by $(.,.)$ and $\| \, . \, \|$. $H^{\frac{1}{2}}(\partial\Omega)$ is the space of trace of functions in $H^1(\Omega)$. Let $\gamma \subset \partial\Omega$, $H_{00}^{\frac{1}{2}}(\gamma)$ is the space of functions in $H^{\frac{1}{2}}(\gamma)$ such that their extension by zero to $\partial\Omega/\gamma$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$.

We introduce some notions to clarify the spaces of functions that depend on time. The function $u(\mathbf{x}, t)$, defined on the domain $\Omega\times]0, T[$, can be written as:

$$\begin{array}{ll} u :]0, T[& \longrightarrow X \\ \quad t & \longmapsto u(t) = u(., t) \end{array}$$

where X is a separable Banach space. We define $\mathcal{C}^j(0, T; X)$ the set of time $\mathcal{C}^j$ classes functions with a value on X. $\mathcal{C}^j(0, T; X)$ is a Banach space for the norm :

$$\|u\|_{\mathcal{C}^j(0,T;X)} = \sup_{0\le t\le T} \sum_{l=0}^{j} \| \, \partial_t^l u \, \|_X$$

where $\partial_t^l u$ is the partial derivative of order l in time of the function u. We define also the spaces :

$$L^p(0, T; X) = \{v \text{ mesurable on }]0, T[\text{ such that } \int_0^T \|v(t)\|_X^p dt < \infty\}$$

and

$$H^s(0, T; X) = \{v \in L^2(0, T; X); \partial^k v \in L^2(0, T; X);\ k \le s\}.$$

$L^p(0, T; X)$ is a Banach space for the norm :

$$\|v\|_{L^p(0,T;X)} = \begin{cases} (\displaystyle\int_0^T \|v(t)\|_X^p dt)^{\frac{1}{p}}, & \text{for} \quad 1 \le p < +\infty \\ \\ \displaystyle\sup_{0\le t\le T} \|v(t)\|_X, & \text{for} \quad p = +\infty, \end{cases}$$

and $H^s(0, T; X)$ is an Hilbert space for the following scalar product:

$$(u, v) = ((u, v)_{L^2(0,T;X)} + \sum_{k=0}^{s} (\partial^k u, \partial^k v)_{L^2(0,T;X)})^{\frac{1}{2}}.$$

Finally we define the space $W^{m,1}(0, T, X)$ of function in $L^1(0, T, X)$ such that all their derivatives up to the order m belong to $L^1(0, T, X)$.

We notice that the proofs in this section are formally presented. Their purpose here is simply to motivate their discrete analogous estimate to be considered later in the article.

Herein, we suppose that the data function f is different from zero in order to have a more generalized form of our system.

$$\left\{\begin{array}{lll} \partial_t\Phi - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\Phi = F & \text{in} & \Omega\times]0,\mathrm{T}[, \\ \varphi = 0 & \text{on} & \Gamma\times]0,\mathrm{T}[, \\ \Phi(.,0) = \Phi_0 & \text{in} & \Omega, \end{array}\right. \tag{2}$$

where $\Phi = \begin{pmatrix} \varphi \\ \psi = \partial_t\varphi \end{pmatrix}$, $F = \begin{pmatrix} f \\ g \end{pmatrix}$, and $\Phi_0 = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$.

Lemma 2.1. *We suppose $(f,g) \in L^1(0,T;H_0^1(\Omega)) \times L^1(0,T;L^2(\Omega))$ and $(\varphi_0,\psi_0) \in H_0^1(\Omega) \times L^2(\Omega)$. We have the following estimate for t, $0 \le t \le T$,*

$$\Big(\parallel \psi \parallel^2 + \parallel \nabla\varphi \parallel^2\Big)^{\frac{1}{2}} \le \Big(\parallel \psi_0 \parallel^2 + \parallel \nabla\varphi_0 \parallel^2\Big)^{\frac{1}{2}} + \int_0^t (\parallel f \parallel + \parallel g \parallel)(s)ds. \tag{3}$$

Proof 1. *We make the inner product with the function $\begin{pmatrix} -\Delta\varphi \\ \psi \end{pmatrix}$ in the first equation of system (2), and we integrate by parts the second term which leads to :*

$$\frac{1}{2}\frac{d}{dt}(\parallel \partial_t\varphi \parallel^2) + \frac{1}{2}\frac{d}{dt}(\parallel \nabla\varphi \parallel^2) \le \parallel f \parallel + \parallel g \parallel .$$

Then integrating this inequality between 0 and t, we conclude the estimate (3).

Remark 2.1. *Considering the Laplace equation with a Dirichlet boundary condition*

$$\left\{\begin{array}{lll} -\Delta\varphi = h & \text{in} & \Omega, \\ \varphi = 0 & \text{on} & \Gamma. \end{array}\right. \tag{4}$$

Let $\varphi = (\Delta)^{-1}h$ the solution of problem (4). The operator $(\Delta)^{-1}$ is an isometry of the space $H^{-1}(\Omega)$ into $H_0^1(\Omega)$, self-adjoint and positive defined. Thus, for any data function $h \in H^{-1}(\Omega)$, we have $\|((\Delta)^{-1})^{\frac{1}{2}}h\| = \|h\|_{H^{-1}(\Omega)}$ (see ([12], Chap. 1, Th. 12.3) for the proof). Then if (f,g) belongs to $L^1(0,T,L^2(\Omega)) \times L^1(0,T,H^{-1}(\Omega))$ and making the inner product of the first *equation of system (2) by* $\begin{pmatrix} \varphi \\ (\Delta)^{-1}\psi \end{pmatrix}$, *we conclude the following estimation :*

$$\begin{aligned}\Big(\| \psi \|^2_{H^{-1}(\Omega)} + \| \varphi \|^2 \Big)^{\frac{1}{2}} \leq \quad & \Big(\| \psi_0 \|^2_{H^{-1}(\Omega)} + \| \varphi_0 \|^2 \Big)^{\frac{1}{2}} \\ & + \int_0^t (\| f \| + \| g \|_{H^{-1}(\Omega)})(s) ds. \end{aligned} \tag{5}$$

In conclusion of this section, we deduce the well posedness of the system (1) by using the Cauchy-Lipschitz theorem and the estimate (3).

Proposition 2.1. *For any data* $(\varphi_0, \psi_0) \in H^1_0(\Omega) \times L^2(\Omega)$, *the system (1) has a unique solution* $\varphi \in \mathcal{C}^1(0, T; L^2(\Omega)) \cap \mathcal{C}^0(0, T; H^1_0(\Omega))$. *Moreover this solution satisfies*

$$\| \partial_t \varphi \|^2 + \| \nabla \varphi \|^2 = \| \nabla \varphi_0 \|^2 + \| \varphi_0 \|^2 . \tag{6}$$

3 The time semi discrete problem

For the time discretization, we consider a partition of the interval $[0, T]$ in sub-intervals $[t_k, t_{k+1}]$, $1 \leq k \leq I$, such that $0 = t_0 < t_1 < \ldots < t_K = T$. We denote by $h_k = t_{k+1} - t_k$ by $h = (h_1, \ldots, h_K)$ and by $|h| = \max\limits_{1 \leq k \leq K} |h_k|$.

To formulate the time semi-discrete problem, we apply the Euler implicit method to the system (1). Then it consists to find the sequence functions $(\varphi^k)_{0 \leq k \leq K}$ in the space $H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega)^{K-1}$ such that

$$\left\{ \begin{array}{lll} \dfrac{\varphi^{k+1} - \varphi^k}{h_k} - \dfrac{\varphi^k - \varphi^{k-1}}{h_{k-1}} - h_k \Delta \varphi^{k+1} = 0 & \text{in} \quad \Omega, & 1 \leq \text{k} \leq \text{K}, \\ \varphi^{k+1} = 0 & \text{on} \quad \Gamma, & 1 \leq \text{k} \leq \text{K}, \\ \varphi^0 = \varphi_0 & \text{in} \quad \Omega, & \\ \varphi^1 = \varphi_0 + h_0 \psi_0 & \text{in} \quad \Omega. & \end{array} \right. \tag{7}$$

We suppose that the data $(\varphi_0, \psi_0) \in H^1_0(\Omega) \times L^2(\Omega)$. Then if φ^0 and ψ^0 are known, we easily show that φ^{k+1}; $k \geq 1$ is a solution of the following variational problem:

Find φ^{k+1} in $H^1_0(\Omega)$ such that for any $\psi \in H^1_0(\Omega)$ we have:

$$\begin{aligned} \int_\Omega \varphi^{k+1}(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} + \quad & h_k^2 \int_\Omega \nabla \varphi^{k+1}(\mathbf{x}) \nabla \psi(\mathbf{x}) d\mathbf{x} \\ & = \int_\Omega \Big(\varphi^k + \frac{h_k}{h_{k-1}} (\varphi^k - \varphi^{k-1}) \Big)(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x}. \end{aligned} \tag{8}$$

Proposition 3.1. *For the data* $(\varphi_0, \psi_0) \in H_0^1(\Omega) \times L^2(\Omega)$, *if* φ^0 *and* ψ^0 *are known, problem (8) has a unique solution* φ^{k+1}; $k \geq 1$ *in* $H_0^1(\Omega)$. *Moreover the solution* $(\varphi^k)_{0 \leq k \leq K}$ *of problem (7) verifies for* $0 \leq k \leq K$ *the following stability condition:*

$$\|\frac{\varphi^{k+1} - \varphi^k}{h_k}\|^2 + \|\nabla \varphi^{k+1}\|^2 \leq \|\psi_0\|^2 + 2\|\nabla \varphi_0\|^2 + 2h_0^2 \|\nabla \psi_0\|^2. \tag{9}$$

Proof 2. *It is easy to prove using the Lax-Milgram theorem that problem (8) has a unique solution. Then by iteration on* k, *we deduce that problem (7) has a unique solution.*

To prove the stability condition (9) making the inner product of the first equation of system (7) by $\frac{\varphi^{k+1} - \varphi^k}{h_k}$ *leads to*

$$\|\frac{\varphi^{k+1} - \varphi^k}{h_k}\|^2 + \|\nabla \varphi^{k+1}\|^2 = (\frac{\varphi^{k+1} - \varphi^k}{h_k}, \frac{\varphi^k - \varphi^{k-1}}{h_{k-1}}) + (\nabla \varphi^{k+1}, \nabla \varphi^k). \tag{10}$$

Thanks to the Cauchy-Schwarz inequality we obtain,

$$\|\frac{\varphi^{k+1} - \varphi^k}{h_k}\|^2 + \|\nabla \varphi^{k+1}\|^2 \leq \|\frac{\varphi^k - \varphi^{k-1}}{h_{k-1}}\|^2 + \|\nabla \varphi^k\|^2. \tag{11}$$

Then by iteration on k *we have*

$$\|\frac{\varphi^{k+1} - \varphi^k}{h_k}\|^2 + \|\nabla \varphi^{k+1}\|^2 \leq \|\frac{\varphi^1 - \varphi^0}{h_0}\|^2 + \|\nabla \varphi^1\|^2.$$

Finally, we conclude the desired estimate by using the third and the fourth equations of system (7).

Remark 3.1. *1) We notice that the solution* φ^{k+1}; $k \geq 1$ *of problem (8) belongs to the space* $H^{s+1}(\Omega)$ *for* $s \geq \frac{1}{2}$. *When the domain* Ω *is convex or of dimension* 1, $s \geq 1$ *is explicitly known. In general, for any* $\frac{1}{2} \leq s \leq 1$, *we derive from the stability condition (9) the following inequality:*

$$\| \varphi^{k+1} \|^2 \leq C h_k^{-2s} \Big(\|\psi_0\|^2 + 2\|\nabla \varphi_0\|^2 + 2h_0^2 \|\nabla \psi_0\|^2 \Big), \tag{12}$$

where C *is a constant independent from the step* h.
This inequality is not optimal since $\| \varphi^{k+1} \|^2$ *is not bounded independently of the step* h.

2) The time discretization of problem (2) using implicit Euler method gives us:

Find the two dimensions sequence $\Phi^k = \begin{pmatrix}\varphi^k \\ \psi^k\end{pmatrix}$*, such that,*

$$\left\{\begin{array}{lll} \dfrac{\Phi^{k+1}-\Phi^k}{h_k} - \begin{pmatrix}0 & 1\\ \Delta & 0\end{pmatrix}\Phi^{k+1} = F^{k+1} & \text{in} \quad \Omega, & 0 \le \mathrm{k} \le \mathrm{K}, \\ \varphi^{k+1} = 0 & \text{on} \quad \Gamma, & 0 \le \mathrm{k} \le \mathrm{K}, \\ \Phi^0 = \Phi_0 & \text{in} \quad \Omega, & \end{array}\right. \tag{13}$$

where $F^{k+1} = \begin{pmatrix}f^{k+1} \\ g^{k+1}\end{pmatrix}$*. For* $n \neq 0$*, system (13) coincides with system (7) if* $F^{k+1} = \mathbf{0};\ k \geq 1$*. When* $n = 0$*, we propose the two following cases so that the two systems coincide completely:*
1) We replace the fourth equation of system (7) by the following implicit equation:

$$\left\{\begin{array}{ll} \varphi^1 - h_0^2\Delta\varphi^1 = \varphi_0 + h_0\psi_0 & \text{in} \quad \Omega, \\ \varphi^1 = 0 & \text{on} \quad \Gamma \end{array}\right. \tag{14}$$

2) We replace the third equation of system (13):

$$\left\{\begin{array}{lll} \dfrac{\Phi^{k+1}-\Phi^k}{h_k} - \begin{pmatrix}0 & 1\\ \Delta & 0\end{pmatrix}\Phi^{k+1} = F^{k+1} & \text{in} \quad \Omega, & 1 \le \mathrm{k} \le \mathrm{K}, \\ \varphi^{k+1} = 0 & \text{on} \quad \Gamma, & 1 \le \mathrm{k} \le \mathrm{K}, \\ \psi^1 = \begin{pmatrix}\varphi_0 + h_0\psi_0 \\ \psi_0\end{pmatrix} & \text{in} \quad \Omega, & \end{array}\right. \tag{15}$$

By multiplying the first equation of system (15) with $\begin{pmatrix}-\Delta\varphi^{k+1} \\ \varphi^{k+1}\end{pmatrix}$*, We deduce the following stability condition:*

$$\begin{aligned} \| \psi^{k+1} \|^2 + \| \nabla\varphi^{k+1} \|^2 \leq \quad & 2\Big(\| \psi^1 \|^2 + \| \nabla\varphi^1 \|^2 \Big) \\ & +2\Big(\sum_{j=1}^{k} h_j(\| g^{j+1} \| + \| \nabla f^{j+1} \|)\Big)^2. \end{aligned} \tag{16}$$

Nevertheless, if we make the inner product of the first equation of system (15) with $\begin{pmatrix}\varphi^{k+1} \\ (\Delta)^{-1}\varphi^{k+1}\end{pmatrix}$ *where* $(\Delta)^{-1}$ *is the operator introduced in remark 2.1, we conclude the following stability condition in a weaker norms:*

$$\begin{aligned} \| \psi^{k+1} \|_{H^{-1}(\Omega)}^2 + \| \varphi^{k+1} \|^2 \leq \quad & 2\Big(\| \psi^1 \|_{H^{-1}(\Omega)}^2 + \| \varphi^1 \|^2 \Big) \\ & +2\Big(\sum_{j=1}^{k} h_j(\| g^{j+1} \|_{H^{-1}(\Omega)} + \| f^{j+1} \|)\Big)^2. \end{aligned} \tag{17}$$

Herein, we'll focus on estimating the error. For the solution φ of system (1) and ${(\varphi)^k}_{0\le k\le K}$ solution of system (7), we define the vector error $\Upsilon^k = \begin{pmatrix} e(\varphi)^k \\ e(\psi)^k \end{pmatrix}$ such that $e(\varphi)^k = \varphi(t_k) - \varphi^k$ and $e(\psi)^k = \psi(t_k) - \psi^k$. We easily show that the sequence error vector ${(\Upsilon)^k}_{0\le k\le K}$ is a solution of system (15) where the two components of F^{k+1} are the two following consistency errors:

$$\varepsilon(\varphi)^k = \frac{\varphi(t_{k+1}) - \varphi(t_k)}{h_k} - \partial_t\varphi(t_{k+1}), \quad \varepsilon(\psi)^k = \frac{\psi(t_{k+1}) - \psi(t_k)}{h_k} - \partial_t\psi(t_{k+1}). \tag{18}$$

Theorem 3.1. *Suppose that the solution φ of system (1) belongs to $W^{3,1}(0,T;L^2(\Omega)) \cap W^{2,1}(0,T;H_0^1(\Omega))$. Then the following a priori error estimate between the solution φ and the solution ${(\varphi)^k}_{0\le k\le K}$ of system (7) holds for $0 \le k \le K$:*

$$\|\varepsilon(\varphi)^k\|^2 + \|\nabla(\varphi(t_k) - \varphi^k)\|^2 \le Ch^2\Big(\int_0^{t_k} (\| \partial_t^3\varphi \| + \| \partial_t^2\nabla\varphi \|)(s)ds\Big)^2 \tag{19}$$

where C is a positive constant independent from the step h.

Proof 3. *Since the sequence error vector ${(\Upsilon)^k}_{0\le k\le K}$ is a solution of system (15) where the second member of the equality is the vector F^{k+1} then applying the stability condition (16) leads to:*

$$\begin{aligned} \| e(\varphi)^k \|^2 + \| \nabla e(\varphi)^k \|^2 \le 2\Big(\| e(\psi)^0 \|^2 + \| \nabla e(\varphi)^0 \|^2 \Big) \\ + 2\Big(\sum_{j=1}^{k} h_j(\| \varepsilon(\psi)^j \| + \| \nabla\varepsilon(\varphi)^j \|)\Big)^2. \end{aligned} \tag{20}$$

Thanks to the Taylor's theorem with remainder integral to bound the terms $\| \varepsilon(\psi)^j \|$, $\| \nabla\varepsilon(\varphi)^j \|$, $\| e(\psi)^0 \|$ and $\| \nabla e(\varphi)^0 \|$ which permits us to conclude the desired estimate.

Finally, using the same steps of the proof of Theorem 3.1 and replacing the stability condition (16) by the stability condition (17), we find the error estimate in a weaker norms.

Corollary 3.1. *Suppose that the solution φ of system (1) belongs to $W^{3,1}(0,T;L^2(\Omega)) \cap W^{2,1}(0,T;H_0^1(\Omega))$. Then the following a priori error estimate between the solution φ and the solution ${(\varphi)^k}_{0\le k\le K}$ of system (7) holds for $0 \le k \le K$:*

$$\|\varepsilon(\varphi)^k\|^2_{H^{-1}(\Omega)} + \|\nabla(\varphi(t_k) - \varphi^k)\|^2 \le Ch^2\Big(\int_0^{t_k} (\| \partial_t^3\varphi \|_{H^{-1}(\Omega)} + \| \partial_t^2\nabla\varphi \|)(s)ds\Big)^2 \tag{21}$$

where C is a positive constant independent from the step h.

The estimates (20) and (21) are optimal of order 1 in time since the discretization is based on the implicit Euler scheme which is of order 1.

4 The full spectral discrete problem

In the following henceforth, we assume that Ω is a rectangle for the dimension two or a parallelepiped rectangle for the dimension three.

Let $\mathbb{P}_N(\Omega)$ the polynomials space with degree $\leq N$ for each variable where $N \geq 2$, and $\mathbb{P}_N^0(\Omega) = \mathbb{P}_N(\Omega) \cap H_0^1(\Omega)$. We define the set of nodes $\zeta_i,\ 0 \leq i \leq N$, roots of the polynomial $(1-x^2)L'_N$ such as L_N is the Legendre polynomial and the set weights $\varrho_i,\ 0 \leq i \leq N$ of the Gauss Lobatto quadrature formula on the interval $]-1,1[$. We then recall the following equality:

$$\forall \eta_N \in \mathbb{P}_{2N-1}(]-1,1[) \quad \int_{-1}^{1} \eta_N(x)dx = \sum_{i=0}^{N} \eta_N(\zeta_i)\varrho_i. \tag{22}$$

We also recall the following property (see [6, 5]):

$$\|\eta_N\|^2_{L^2(]-1,1[)} \leq \sum_{i=0}^{N} \eta_N^2(\zeta_i)\varrho_i \leq 3\|\eta_N\|^2_{L^2]-1,1[}, \quad \forall \eta_N \in \mathbb{P}_N(]-1,1[). \tag{23}$$

The reference domain $]-1,1[^d,\ (d=2,3)$ is transformed to the domain Ω using the affine mapping T then the scalar product is defined on continuous functions u and v by:

$$(u,v)_N = \begin{cases} \frac{\text{meas}(\Omega)}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} (u \circ T)(\zeta_i,\zeta_j)(v \circ T)(\zeta_i,\zeta_j)\varrho_i\varrho_j, & \text{if d} = 2, \\ \frac{\text{meas}(\Omega)}{8} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} (u \circ T)(\zeta_i,\zeta_j,\zeta_k)(v \circ T)(\zeta_i,\zeta_j,\zeta_k)\varrho_i\varrho_j\varrho_k, & \text{if d} = 3. \end{cases} \tag{24}$$

Remark 4.1. *To simplify the analysis, we assume that the spectral grid does not change as a function of time which means that the discretization is fixed over time.*

We suppose that φ_0 and ψ_0 are respectively continuous on Ω and on $\Omega \times]0,T[$. The discrete problem is deduced from the system (7) by using the Galerkin method combined with the numerical integration.
Find $\varphi_N^k \in \mathbb{P}_N^0(\Omega) \times \mathbb{P}_N(\Omega) \times (\mathbb{P}_N^0(\Omega))^{K-1}$, $0 \leq k \leq K$ such that:

$$\varphi_N^0 = \mathfrak{I}_N(\varphi_0) \quad \text{and} \quad \varphi_{\mathrm{N}}^1 = \mathfrak{I}_{\mathrm{N}}(\varphi_0) + \mathrm{h}_0\mathfrak{I}_{\mathrm{N}}(\psi_0) \quad \text{in} \quad \Omega \tag{25}$$

and, for $1 \leq k \leq K$,

$$(\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k}-\frac{\varphi_N^k-\varphi_N^{k-1}}{h_{k-1}},\psi_N)_N+h_k(\nabla\varphi_N^{k+1},\nabla\psi_N)_N=0, \quad \forall\psi_N\in\mathbb{P}_N^0(\Omega), \tag{26}$$

where $\mathfrak{I}_N$ is the interpolating operator from $L^2(\Omega)$ into $\mathbb{P}_N(\Omega)$.
In the same way as in (8), we notice that φ_N^{k+1}, $1 \leq k \leq K$ is the solution of the following discrete variational formulation:

$$(\varphi_N^{k+1},\psi_N)_N+h_k^2(\nabla\varphi_N^{k+1},\nabla\psi_N)_N=(\varphi_N^k+\frac{h_k}{h_{k-1}}(\varphi_N^k-\varphi_N^{k-1}),\psi_N)_N. \tag{27}$$

Proposition 4.1. *For the data $(\varphi_0,\psi_0)\in H_0^1(\Omega)\times L^2(\Omega)$. If φ_N^0 and ψ_N^0 are known, problem (27) has a unique solution φ_N^{k+1}; $k\geq 1$ in $H_0^1(\Omega)$. Moreover, the solution $(\varphi_N^k)_{0\leq k\leq K}$ of problem (25)–(26) verifies for $0\leq k\leq K$ the following stability condition:*

$$\|\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k}\|^2+\|\nabla\varphi_N^{k+1}\|^2\leq (3^d)^K\Big(\|\mathfrak{I}_N(\psi_0)\|^2+2\|\nabla\mathfrak{I}_N(\varphi_0)\|^2+2h_0^2\|\nabla\mathfrak{I}_N(\psi_0)\|^2\Big). \tag{28}$$

Proof 4. *By using the Lax-Milgram theorem and the property (23), it is easy to show that the discrete variational formulation (27) has a unique solution. Then, by iteration on k we deduce that system (25)–(26) has a unique solution. Before starting the proof of the stability condition (28), we define $\|\cdot\|_d$ the discrete norm deduced from the discrete scalar product $(.,.)_N$.*
To prove the stability condition (28), making the discrete inner product of the equation (26) by $\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k}$ leads to:

$$\|\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k}\|_d^2+\|\nabla\varphi_N^{k+1}\|_d^2=(\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k},\frac{\varphi_N^k-\varphi_N^{k-1}}{h_{k-1}})_N+(\nabla\varphi_N^{k+1},\nabla\varphi_N^k)_N.$$

Thanks to the Cauchy-Schwarz inequality and property (23), we have

$$\|\frac{\varphi_N^{k+1}-\varphi_N^k}{h_k}\|^2+\|\nabla\varphi_N^{k+1}\|^2\leq 3^d\Big(\|\frac{\varphi_N^k-\varphi_N^{k-1}}{h_{k-1}}\|^2+\|\nabla\varphi^k\|^2\Big).$$

Then by iteration on k, we obtain

$$\|\frac{\varphi^{k+1}-\varphi^k}{h_k}\|^2+\|\nabla\varphi^{k+1}\|^2\leq (3^d)^K\Big(\|\frac{\varphi^1-\varphi^0}{h_0}\|^2+\|\nabla\varphi^1\|^2\Big).$$

Thus, estimate (28) is deduced from (25).

We end this section by estimating the a priori error between the full discrete solution of problem (27) and the semi discrete solution in time of problem (8).

Proposition 4.2. *If $\varphi_0,\ \psi_0$ are continuous on $\overline{\Omega}$ and $\varphi_N^0,\ \psi_N^0$ are known. The error estimate between the solution $\varphi^{k+1}, k \geq 1$ of problem (8) and $\varphi_N^{k+1}, k \geq 1$ solution of problem (27) holds*

$$\|\varphi^{k+1} - \varphi_N^{k+1}\| \leq C\Big(\inf_{\chi_N^{k+1} \in \mathbb{P}_N^0(\Omega)} \|\varphi^{k+1} - \chi_N^{k+1}\| + \\ + \Big[\|\varphi_0 - \varphi_N^0\| + \|\psi_0 - \psi_N^0\| \sum_{j=1}^{k} (E^{1,j} + E^{2,j} + E^{3,j}) \Big] \Big). \tag{29}$$

where

$$\begin{aligned} E^{1,j} &= \frac{1}{h_j^2} \sup_{\psi_N \in \mathbb{P}_N^0(\Omega)} \frac{\displaystyle\int_\Omega (\varphi^{j+1} - \varphi^j)\psi_N \, d\mathbf{x} - (\chi_N^{j+1} - \chi_N^j, \psi_N)_N}{\| \psi_N \|}, \\ E^{2,j} &= \sup_{\psi_N \in \mathbb{P}_N^0(\Omega)} \frac{\displaystyle\int_\Omega \nabla\varphi^{j+1} \nabla\psi_N \, d\mathbf{x} - (\nabla\chi_N^{j+1}, \nabla\psi_N)_N}{\| \psi_N \|}, \\ E^{3,j} &= \sup_{\psi_N \in \mathbb{P}_N^0(\Omega)} \frac{\displaystyle\int_\Omega (\varphi^j - \varphi^{j-1})\psi_N \, d\mathbf{x} - \big(\mathfrak{I}_N(\varphi^j - \varphi^{j-1}), \psi_N\big)_N}{\| \psi_N \|}, \end{aligned}$$

and C is a positive constant independent of N.

Proof 5. *Consider $\chi_N^{k+1} \in \mathbb{P}_N^0(\Omega)$. Using the triangular inequality we have:*

$$\|\varphi^{k+1} - \varphi_N^{k+1}\| \leq \|\varphi^{k+1} - \chi_N^{k+1}\| + \|\chi_N^{k+1} - \varphi_N^{k+1}\|.$$

To estimate the term $\|\varphi_N^{k+1} - \chi_N^{k+1}\|$, we write the two problems (7) and (27). For $\psi_N \in \mathbb{P}_N^0(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega \varphi^{k+1}(\mathbf{x})\psi_N(\mathbf{x}) d\mathbf{x} + \quad & h_k^2 \int_\Omega \nabla\varphi^{k+1}(\mathbf{x}) \nabla\psi_N(\mathbf{x}) d\mathbf{x} \\ &= \int_\Omega \Big(\varphi^k + \frac{h_k}{h_{k-1}}(\varphi^k - \varphi^{k-1})\Big)(\mathbf{x})\psi_N(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

and

$$(\varphi_N^{k+1}, \psi_N)_N + h_k^2 (\nabla\varphi_N^{k+1}, \nabla\psi_N)_N = (\varphi_N^k + \frac{h_k}{h_{k-1}}(\varphi_N^k - \varphi_N^{k-1}), \psi_N)_N.$$

Let $\tau_k = \frac{h_k}{h_{k-1}}$. By doing the difference term by term, leads,

$$(\varphi_N^{k+1}-\chi_N^{k+1},\psi_N)_N+h_k^2(\nabla(\varphi_N^{k+1}-\chi_N^{k+1}),\nabla\psi_N)_N = (\varphi_N^k-\chi_N^k,\psi_N)_N+\tau_k\mathcal{K}^k(\psi_N)$$

such that,

$$\begin{aligned}\mathcal{K}^k(\psi_N) = \quad & \frac{1}{h_k^2}\Big(\int_\Omega(\varphi^{k+1}-\varphi^k)\psi_N\,d\mathbf{x}-(\chi_N^{k+1}-\chi_N^k,\psi_N)_N\Big)\\ & +\int_\Omega\nabla\varphi^{k+1}\nabla\psi_N\,d\mathbf{x}-(\nabla\chi_N^{k+1},\nabla\psi_N)_N\\ & +\int_\Omega(\varphi^k-\varphi^{k-1})\psi_N\,d\mathbf{x}-\big(\mathfrak{I}_N(\varphi^j-\varphi^{j-1}),\psi_N\big)_N.\end{aligned}$$

We simply prove that the operator $\mathcal{K}^k$ is linear and continuous on the space $\mathbb{P}_N^0(\Omega)$ which is an Hilbert space for the discrete scalar product $(.,.)_N$. Thus, thanks to Riesz theorem, we deduce that there exists a unique element ϑ_N^k in $\mathbb{P}_N^0(\Omega)$ such that,

$$\mathcal{K}^k(\psi_N) = (\vartheta_N,\psi_N)_N.$$

Then by applying the result proved in ([11], Prop. 4.1) and [4], we obtain,

$$\|\varphi_N^{k+1}-\chi_N^{k+1}\| \le C\left(\|\varphi_0-\varphi_N^0\|+\|\psi_0-\psi_N^0\|+\sum_{j=1}^k\|\vartheta_N^j\|^2\right)^{1/2},$$

where C is a positive constant independent of N.
We take note there exists C a positive constant independent of N such that,

$$\|\vartheta_N^j\| \le C\sup_{\psi_N\in\mathbb{P}_N^0(\Omega)}\frac{(\vartheta_N^j,\psi_N)_N}{\|\psi_N\|},$$

which permits us to conclude (29).

To find the order of convergence as a function of N, it is necessary to estimate each of the terms of the second member of the inequality (29).

• Estimation of $E^{1,j}$

We consider $\varpi^{j+1} = \varphi^{j+1}-\varphi^j$, and $\chi_N^{j+1}-\chi_N^j = \Pi_{N-1}^{1,0}(\varpi^{j+1})$. By the exactness of the Gauss-Lobatto quadrature formula (22), the two terms $\displaystyle\int_\Omega \Pi_{N-1}^{1,0}(\varpi^{j+1})\,\psi_N\,d\mathbf{x}$ and $(\Pi_{N-1}^{1,0}(\varpi^{j+1}),\psi_N)_N$ are equal, then

$$E^{1,j} \le \|\varpi^j-\Pi_{N-1}^{1,0}(\varpi^j)\|. \tag{30}$$

$\Pi_N^{1,0}$ is the orthogonal projection operator from $H_0^1(\Omega)$ into $\mathbb{P}_N^0(\Omega)$ related to the scalar product defined by the semi norm $|\ .\ |_{1,\Omega}$. See ([6], Lem. VI.2.5) and [5] for all the properties of this operator.

• Estimation of $E^{2,j}$

Using the exactness of the Gauss-Lobatto quadrature formula (22) for a polynomial of degree $\leq 2N-1$, we obtain:

$$\int_\Omega \nabla\varphi^{j+1}\nabla\psi_N\, d\mathbf{x} - (\nabla\chi_N^{j+1}, \nabla\psi_N)_N = \int_\Omega \nabla(\varphi^{j+1} - \Pi_{N-1}^{1,0}\varphi^{j+1})\nabla\psi_N\, d\mathbf{x} - \Big(\nabla(\chi_N^{j+1} - \Pi_{N-1}^{1,0}\chi_N^{j+1}), \nabla\psi_N\Big)_N. \tag{31}$$

Thanks to the triangular and Cauchy-Schwarz inequalities, we have:

$$\sup_{\psi_N\in\mathbb{P}_N^0(\Omega)} \frac{\displaystyle\int_\Omega \nabla\varphi^{j+1}\nabla\psi_N\, d\mathbf{x} - (\nabla\chi_N^{j+1}, \nabla\psi_N)_N}{\|\ \psi_N\ \|} \leq \Big(|\varphi^{j+1} - \Pi_{N-1}^{1,0}\varphi^{j+1}|_{1,\Omega} + |\chi_N^{j+1} - \Pi_{N-1}^{1,0}\chi_N^{j+1}|_{1,\Omega}\Big). \tag{32}$$

Then we conclude by using the properties of operator $\Pi_{N-1}^{1,0}$.

• Estimation of $E^{3,j}$

Let $\theta^j = \varphi^j - \varphi^{j-1}$. We use for this estimation Π_{N-1} the orthogonal projection from $L^2(\Omega)$ into $\mathbb{P}_{N-1}(\Omega)$. Then by the exactness of the Gauss-Lobatto quadrature formula, for a polynomial of degree $\leq 2N-1$, we have:

$$\int_\Omega \theta^j(\mathbf{x})\psi_N(\mathbf{x})\, d\mathbf{x} - (\mathfrak{I}_N\theta^j, \psi_N)_N = \int_\Omega (\theta^j - \Pi_{N-1}\theta^j)(\mathbf{x})\psi_N(\mathbf{x})\, d\mathbf{x} - (\mathfrak{I}_N\theta^j - \Pi_{N-1}\theta^j, \psi_N)_N.$$

Using the inequality (23) in each direction, leads to:

$$\int_\Omega \theta^j(\mathbf{x})\psi_N(\mathbf{x})\, d\mathbf{x} - (\mathfrak{I}_N\theta^j, \psi_N)_N \leq \Big[\|\theta^j - \Pi_{N-1}\theta^j\|^2 + 9\|\theta^j - \mathfrak{I}_N\theta^j\|^2\Big]\|\psi_N\|.$$

Using the approximation properties of operator Π_{N-1} (see [5], Theo. 7.1) and $\mathfrak{I}_N$ (see [5], Theo. 14.2). For $\theta^j \in H^s(\Omega)$; $s > 1$, we obtain:

$$\sup_{\psi_N\in\mathbb{P}_N(\Omega)} \frac{\displaystyle\int_\Omega \theta^j(\mathbf{x})\psi_N(\mathbf{x})\, d\mathbf{x} - (\theta^j, \psi_N)_N}{\|\ \psi_N\ \|} \leq CN^{-2s}\|\theta^j\|_{s,\Omega}^2. \tag{33}$$

Finally, to estimate respectively the best approximation errors

$$\inf_{\chi_N^{k+1}\in\mathbb{P}_N^0(\Omega)} \|\varphi^{k+1}-\chi_N^{k+1}\|, \quad \|\varphi_0-\varphi_N^0\| \quad \text{and} \quad \|\psi_0-\psi_{\mathrm{N}}^0\|, \tag{34}$$

we just choose respectively $\chi_N^{k+1}=\Pi_N^{1,0}\varphi^{k+1}$, $\varphi_N^0=\Pi_N^{1,0}\varphi_0$ and $\psi_N^0=\Pi_N\psi_0$ then we conclude by means of the properties of operators $\Pi_N^{1,0}$ and Π_N.
So, from the estimations (30), (32), (33) and (34), we have the following main theorem about a priori error estimate.

Theorem 4.1. *For a data (φ_0,ψ_0) continuous on $\bar{\Omega}$, and the solution $(\varphi^k)_{0\le k\le K}$ of problem (7) belongs to $H^s(\Omega)$; $s>1$. Then the error between φ^{k+1} solution of problem (8) and φ_N^{k+1} solution of problem (27) holds:*

$$\begin{aligned}\|\varphi^{k+1}-\varphi_N^{k+1}\| \le C\Big[N^{-s}\Big(\|\varphi^{k+1}\|_{s,\Omega}+\sum_{j=1}^{k}\Big(h_j^{-2}\|\varphi^{j+1}-\varphi^j\|_{s,\Omega}+\|\varphi^j-\varphi^{j-1}\|_{s,\Omega}\Big)\Big)\\ +N^{1-s}\sum_{j=1}^{k}\|\varphi^{j+1}\|\Big],\end{aligned} \tag{35}$$

where C is a positive constant independent of N.

Conclusion

This work concerns the numerical analysis of the implicit Euler scheme in time and the spectral discretization in space of the second order wave equation. We prove an optimal error estimate in time and in space. These estimations depend only on the regularity of the solution. The more difficult case where the spectral discretization depends on the time and its numerical implementation will be the subject of a forthcoming work.

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References

[1] M. Abdelwahed, N. Chorfi, *A posteriori analysis of the spectral element discretization of a non linear heat equation.* Adv. Nonlinear Anal. 2021; 10: 477–490.

[2] M. Abdelwahed, N. Chorfi, *On the convergence analysis of a time dependent elliptic equation with discontinuous coefficients*. Adv. Nonlinear Anal. 2020; 9: 1145-1160.

[3] S. Adjerid, *A posteriori finite element error estimation for second–order hyperbolic problems*. Comput. Methods Appl. Mech. Engrg. 191 (2002), 4699–4719.

[4] A. Bergam, C. Bernardi, Z. Mghazli, *A posteriori analysis of the finite element discretization of some parabolic equations*. Math. Comp. 74 (2005), no. 251, 1117-1138.

[5] C. Bernardi, Y. Maday, *Spectral Methods*. in Handbook of Numerical Analysis V, P.G. Ciarlet and J.-L. Lions, eds., North–Holland, Amsterdam, 1997, pp. 209–485.

[6] C. Bernardi, Y. Maday, F. Rapetti, *Discrétisations variationnelles de problèmes aux limites elliptiques. Collection Mathématiques et Application*, 45, Springer-Verlag, Paris, 2004.

[7] W. Bangerth, R. Rannacher, *Finite element approximation of the acoustic wave equation: error control and mesh adaptation*. East-West J. Numer. Math. 7 (1999), 263-282.

[8] W. Bangerth, R. Rannacher, *Adaptive finite element techniques for the acoustic wave equation*. J. Comput. Acoust. 9 (2001), 575-591.

[9] C. Bernardi, E. Süli, *Time and space adaptivity for the second-order wave equation*. Math. Models Methods Appl. Sci. 15(2), 199-225 (2005)

[10] A. Chaoui1, F. Ellaggoune2, A. Guezane-Lakoud, *Full discretization of wave equation*. Boundary Value Problems (2015) 2015:133 DOI 10.1186/s13661-015-0396-3

[11] Y. Daikh, W. Chikouche, *Spectral element discretization of the heat equation with variable diffusion coefficient*. HAL Id: hal-01143558, https://hal.archives-ouvertes.fr/hal-01143558, Apr 2015.

[12] J.L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*. Dunod, 1968.

[13] A. T. Patera, *A spectral element method for fluid dynamics: Laminar flow in a channel expansion*. J. Comput. Phys., 54, 468–488 (1984).

[14] E. Süli, *A posteriori error analysis and global error control for adaptive finite volume approximations of hyperbolic problems.* Numerical Analysis 1995 (Dundee 1995), 169-190, Pitman Res. Notes Math. Ser. 344. Longman, Harlow (1996).

[15] E. Süli, *A posteriori error analysis and adaptivity for finite element approximations of hyperbolic problems.* In: D. Kröner, M. Ohlberger and C. Rohde (Eds.) An Introduction to Recent Developments in Theory and Numerics for Conservation Laws. Lecture Notes in Computational Science and Engineering Volume 5, 123 -194, Springer-Verlag (1998).

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