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Properties of n-ary hypergroups relevant for modelling trajectories in HD maps

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Abstract

In the paper we show that trajectories used in HD maps of autonomous vehicles can be well modelled by means of n-ary hyperoperations and hypergroups. We investigate some properties of such hypergroups.

1 Introduction

Our paper deals with a challenging issue relevant for the self-navigation of autonomous vehicles. It is a continuation of studies in which automata theory, rough sets and various generalizations of algebraic concepts as well as concepts of linear algebra are used to model some aspects of self-navigation such as modelling trajectories or processing information. It is to be noted that autonomous driving requires a set of advanced technologies both in the vehicles themselves and in the infrastructure. Since these technologies must continuously communicate, they need to be linked or even integrated. Obviously, it is a high definition (HD) map, real-time and as precise as possible – and discussed in this paper, that is the key component of autonomous driving.

Since our paper relies on the tools of the broad field of algebra, we give a selection of similarly oriented papers. For our own results concerning the topic of self-navigation of autonomous vehicles cf e.g. [12, 13].

Paper [15] gives a real-time analysis of building *Normal Distribution Transformation* (NDT). There, in one of the steps of the transformation, the point

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cloud M is discretized into a grid β with regularly sized cubic cells $\beta_i, i = 1, 2, ..., m$, according to a predefined cell size. In [26], a novel system architecture including a massive *multi-input multi-output* (MIMO) or a *reconfigurable intelligent surface* (RIS) and multiple autonomous vehicles is considered in vehicle location systems. By means of geometrical algebraic notions and properties such as associativity, commutativity and distributivity, the authors transform the global coordinate system into a local one. In [10], the algebraic specification of activities performed during network security risk management is given. This is a useful tool to consider the automation of the process of risk management without considering the implementation issues. The authors of [9] formalise IT security risk management as an algebraic datatype specification to check the consistency of security risk analyses viewed as algebras. Probability of occurrence and severity of consequence are modelled as metrics over attack actions to select optimal countermeasures using multi-objective optimisation. While focusing on security management, their algebraic datatype is an inspiration for a further formalisation of consequences and assets. However, their action abstraction and the use of communicating sequential processes offers a design method for safety controllers as opposed to the more abstract attack model proposed in [10], which focuses on risk assessment. In [9], "*RISK STRUCTURES*" are considered as an algebraic framework of risk modelling, which is meant to support the design of safe controllers for risk aware machines. Using the concept of *risk factor* as a primitive modelling tool, such a framework provides equipment for construction, investigation and safeguarding of such controllers. The authors of [9] prove desired algebraic properties of such equipment and show their applicability by using them to specify key aspects of safety control units for autonomous driving and risk-cautious collaborative robots. In order to handle the issue of directional planning in motion planning of autonomous driving, [1] makes use of the global path information by proposing a conditional *deep Q-network* (DQN) with fuzzy logic. The aim [1] is to make directional planning for an end-to-end autonomous driving systems. In the paper, the architecture of the proposed conditional DQN with fuzzy logic is described. In [23], the design and implementation of a fuzzy logic system for the steering control of autonomous vehicles inside the roundabout is proposed. Cascade architecture for lateral control and parametric trajectory generation are used. Paper [3] presents a model for lane changing in highway driving. It consists of two main parts: threat assessment by assessing the interaction between traffic participants captured in terms of fuzzy logic and a decision making approach based on the concept of Markov Decision Process (MDP). The combined system forms a predictive *Fuzzy Markov Decision Process* (FMDP). A *model predictive control* (MPC) scheme for trajectory generation/control complements this decision process. Study [27] aims at developing a heterogeneous model of traffic flow which could be used to study the possible influence of *connected autonomous vehicles* (CAV) on the flow. The authors of the proposed model make use of cellular automata, which had been used to construct a two-state safe-speed model. Notice that the autonomous vehicle (AV) intelligence is not sufficient to solve the complex traffic situations. However, the concept of *exclusive traffic lanes* bypasses the so far insufficient technology. In [16], the advanced model of cellular automata is used to study the influence of exclusive traffic lanes setting on the traffic flow and stability on highways. Finally, [25] introduces a cellular automaton design for single lane highway sections in order to model automated and human vehicle agents in heterogeneous as well as homogeneous traffic.

In our paper we reflect the fact that no matter how precise and accurate HD maps are, autonomous vehicles are not capable of exact copying the suggested trajectory because of effects such as inaccuracies of sensorics, vehicle speed, etc. Instead, they move in its relative vicinity. Therefore we suggest a solution using algebraic hypercompositional structures in which the hyperoperation (or hypercomposition) includes all trajectories acceptable for safe drive. The necessity of such a hyperoperation for modelling the trajectory of the autonomous vehicle is explained in Section 2. In Section 3 we summarize the basic algebraic terminology required to study our model. Our main results are included in Section 4, in which we study the n-ary hyperoperations in the context of HD maps. We also define some new properties of n-ary hypergroupoids which are motivated by the context of trajectories. In Section 5 we shortly summarize our results and outline possiblities of future study such as the use of fractions and n-ary transposition axiom to model trajectories back in time.

2 Motivation

Our paper is motivated by the vibrant and so far still ongoing process of development and tailoring HD maps, i.e. high-definition maps, which are being developped as an advanced component and sensor used in the course of movement of autonomous vehicles. These very precise HD maps consist of five layers: *base layer*, *geometric layer*, *semantic map layer*, *map priors layer* and *real-time layer*; see [2] for a scheme giving all these layers, or [14, 11, 8] for more details.

In spite of the fact that HD maps are intended as high precision map data, the process of their creation is not error-free. The errors, or imprecisions, include *global accuracy error* such as GPS error, error of localization of the inspection vehicle with recording set for creating the HD maps, *local accuracy error* caused e.g. by the speed used for recording, accuracy in objects localization, hardware precision, etc., and *local sampling error*, i.e. precission of sampling. For details, see [6, 24, 17, 15]. One of the intended purposes of HD maps is to facilitate localization of autonomous vehicles at places with weak GNSS availability. This is an important task assigned to the the first three layers of HD maps. Recall that the remaining two layers, i.e. *map priors* layer and *real-time layer*, provide information on real-time traffic such as traffic density, traffic jams, car parks occupancy, etc.

As a result, autonomous vehicles process enormous amount of data. This is the reason why elements such as IOT (internet of things) or various cloud storages are considered to be employed in the process. However, it becomes obvious that, when driving, the autonomous vehicle will not have read its planned course in its entirety but will be reading it by parts instead. This implies demands in telecommunication standards of mobile networks. However, not even this means perfection in localization of autonomous vehicles because imprecissions must be still be counted with.

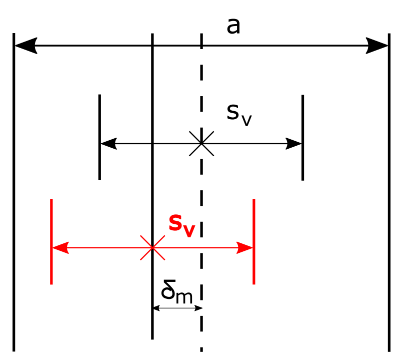

Figure 1: Errors is HD maps: a... lane width, s_v... car width, δ_m... HD map error

When driving, the autonomous vehicle, represented in the course of its movement and localization by a mass point, moves in a certain band (or strip) induced by the errors suggested in Fig. 1. Therefore, in this paper, we regard algebraic structures which can be used to conveniently model the ideal trajectory of the vehicle taking into account, for safety purposes, its immediate vicinity. Our structures work with an adjustable HD map error δ_m. Therefore the error already includes car width, which allows us to model a corridor for a safe passage of the autonomous vehicle. Further on, we will write r (disc radius) instead of δ_m.

3 Basic notions

In this section we collect definitions and trivia regarding concepts that will be used further on. For furher details or properties of the below mentioned notions cf e.g. [4] for a deeper insight and context see overview papers [18, 19]. Notice that$n$-ary hypergroups were introduced in [7] and studied e.g. from the point of view n-ary relations, see [5]. Recently, e.g. [22] studied them in the context of composition hyperrings, i.e. hypercompositional structures with two (hyper)operations.

Definition 1. *Denote by H^n the Cartesian product $H \times \ldots \times H$, where H appears n times. A mapping $f : H^n \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the power set of H excluding the empty set $\emptyset$, is called an n-ary hyperoperation (or n-ary hypercomposition); with n being the arity of the hyperoperation. For all $x, y \in H$ we by $f(x_1, \ldots, x_n)$ mean a subset of $\mathcal{P}^*(H)$ and by $f(A_1, \ldots, A_n)$ the set $\bigcup f(x_1, \ldots, x_n) \subseteq \mathcal{P}^*(H)$, where, for all $i = 1, \ldots, n$, there is $x_i \in A_i$. A non-empty set H with an n-ary hyperoperation $f : H^n \to \mathcal{P}^*(H)$ is called n-ary hypergroupoid and is denoted (H, f).*

Notation 1. *In order to simplify notation, one may use lower and upper indices. Thus one may write $f(x_1^n)$ instead of $f(x_1, \ldots, x_n)$ or e.g. $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$, where $i < j$, instead of $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$. However, at most places we prefer not to use this type of notation.*

Commutativity of n-ary hypergroupoids is not defined while associativity is defined as equality of sets for all possibilities of bracketing elements.

Definition 2. *By an n-ary semihypergroup we mean an n-ary hypergroupoid (H, f) such that there is*

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1}$$

for all $i, j \in \{1, 2, \ldots, n\}$ and all $x_1, x_2 \ldots, x_{2n-1} \in H$.

n-ary hypergroups can be defined in two different ways – by means of the reproductive law or by means of existence of a solution of equations. We present both alternatives.

Definition 3. *[7] By an n-ary hypergroup we mean such an n-ary semihypergroup (H, f) in which the equation*

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \tag{2}$$

has the solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leq i \leq n$.

Remark 1. *Notice that, in accordace with Notation 1, by* (2) *we mean that* b *is an element of* $f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots a_n)$*.*

Remark 2. *Alternatively, we may write that* n*-ary hypergroup is such a* n*-ary semihypergroup* (H, f) *that the* n*-ary reproductive law holds, i.e. that there is*

$$f(H^{i-1}, x, H^{n-i}) = H \tag{3}$$

for all $x \in H$ *and all* $i = \{1, 2, \ldots, n\}$*.*

In the context of binary hyperoperations we mention the *extensivity* of the hyperoperation. Notice that the hyperoperation $f : H \times H \to \mathcal{P}(H)$ is called *extensive* if, for all $a, b \in H$, there is $\{a, b\} \subseteq f(a, b)$. It is easy to verify that every extensive semihypergroup is a hypergroup.

In our paper we work with points in plane. Notice that we denote these as $P = [x, y]$.

4 Modelling trajectories in HD maps using n-ary hyperstructures

Consider discs with center $C = [m, n]$ and perimeter r in real plane, i.e. sets of points

$$disc_{C=[m,n],r} = \{[x, y] \in \mathbb{R}^2 \mid (x-m)^2+(y-n)^2 \leq r^2; m, n \in \mathbb{R}, r \in \mathbb{R}^+\}. \tag{4}$$

Denote $\mathcal{S}_{disc}$ the set of all such discs. On $\mathcal{S}_{disc}$ we define a binary hyperoperation $\circ : \mathcal{S}_{disc} \times \mathcal{S}_{disc} \longrightarrow \mathcal{P}^*(\mathcal{S}_{disc})$ by:

$$disc_{A,r} \circ disc_{B,r} = \{disc_{C,r} \in \mathcal{S}_{disc} \mid C \in |AB|\} \tag{5}$$

In other words, we fix the perimeter and move the disc along the line segment $|AB|$ from A to B.

The following, rather obvious lemma, will make some of our further considerations more easily explicable.

Lemma 1. *The set of points induced by* $disc_{A,r} \circ disc_{B,r}$ *is convex regardless of* A, B *or* r*.*

Proof. Obvious. □

Before we proceed, we explain the difference between "$\circ$" defined by (5) and the below hyperoperation "$*$" (6) which one might consider as well (and which was, in fact, considered at a very preliminary stage of our research at the 14^{th} *Conference on Algebraic Hyperstructures and Applications* in 2020; unpublished):

$$[x_1, y_1] * [x_2, y_2] = \tag{6}$$
$$\left\{ [x, y]; \sqrt{(x-u)^2 + (y-v)^2} \leq r, r \in \mathbb{R}, u \in tx_1 + (1-t)x_2, v \in ty_1 + (1-t)y_2, t \in \langle 0, 1 \rangle \right\}$$

If we examine the resulting sets of points in the real plane, the hyperoperations seem to be the same (see the area of $A * B$, $A \circ B$ in Fig. 2 and 3). However, they are not because they work with different kinds of objects: points vs discs. Indeed, in "$*$" the hyperproduct is defined for *points* and results in the *set of points* while "$\circ$" is defined for *discs* and results in the *set of discs*.

In Fig. 2 we explain why "$*$" is only weakly associative and *not* associative. Assume points $A = [x_1, y_1], B = [x_2, y_2], C = [x_3, y_3]$ and construct $(A*B)*C$. In this case $A*B$ is the set of all *points* of discs with centers on the line segment $|AB|$. As a result, $(A * B) * C$ will include points of discs with centers *beyond* A and B which will expand the set. However, this expansion will be different for $A * (B * C)$ because in $B * C$ we include discs with points beyond B and C instead.

However, as depicted in Fig. 3, the hyperoperation "$\circ$" *is* associative because we work with discs with centers in the line respective line segments only (and do not move beyond the points).

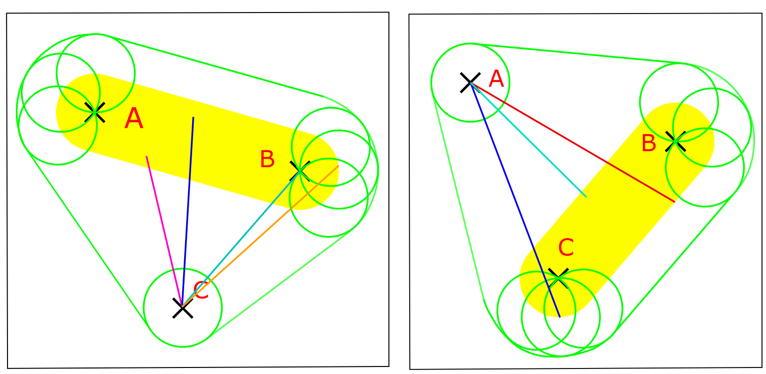

Figure 2: Weak associativity of "$*$" (for clarity reasons circles depicted instead of discs)

Further on we will study the properties of $(\mathcal{S}_{disc}, \circ)$. First of all we show that it is a hypergroup.

Theorem 1. $(\mathcal{S}_{disc}, \circ)$ *is a hypergroup.*

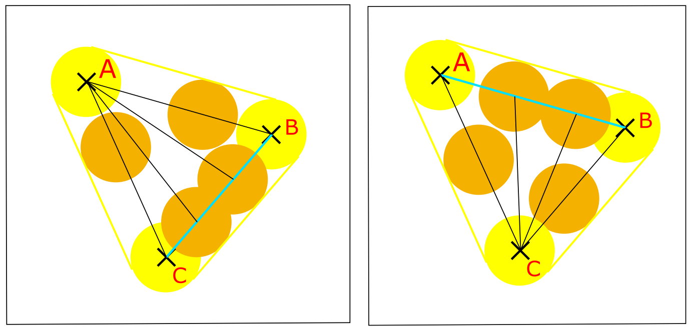

Figure 3: Associativity of "$\circ$"

Proof. Proof of associativity is obvious given the considerations above and Fig. 3. Indeed, for all $disc_{A,r}, disc_{B,r}, disc_{C,r}$, where $A = [x_1, y_1]$, $B = [x_2, y_2]$, $C = [x_3, y_3]$, we have

$$(disc_{A,r} \circ disc_{B,r}) \circ disc_{C,r} =$$

$$= \bigcup_{D=[x_4,y_4]\in\{[x,y] | x\in x_3t+(1-t)u, y\in y_3t+(1-t)v, u\in x_1l+(1-l)x_2, v\in y_1l+(1-l)y_2, t,l\in[0,1]\}} disc_{D,r},$$

i.e. the union of all discs on sides and inside the triangle ABC. The calculation for $disc_{A,r} \circ (disc_{B,r} \circ disc_{C,r})$ is analogous, the sets are obviously equal.

The reproductive axiom holds automaticly because the hyperoperation is extensive. □

From our "Motivation" section it is obvious that the binary hypereoperation of Theorem 1 is not sufficient for modelling the trajectory of autonomous vehicles. In HD maps such vehicles move along curves and follow certain boundaries within which they move. However, the binary operation of Theorem 1 describes movement along line segments only which prevents us to construct *reservation fields*. If we want to model real-life trajectories, we would have to consider curves instead of straight lines. However, constructing such algebraic (hyper)structures would be complicated. Therefore we will employ n-ary hyperstructures in which curves connecting n points will be approximated by $n-1$ line segments as suggested in Fig 4.

Next, we denote by $\mathcal{S}^n_{disc}$ the Cartesian product $\mathcal{S}_{disc} \times \ldots \times \mathcal{S}_{disc}$, where $\mathcal{S}_{disc}$ appears n times. We define the n-ary hyperoperation $f : \mathcal{S}^n_{disc} \to \mathcal{P}^*(\mathcal{S}_{disc})$ by:

$$f\left(disc_{A_1,r}, \ldots, disc_{A_n,r}\right) = \{disc_{B,r} \mid B \in |A_iA_{i+1}|, i \in \{1, 2, \ldots n-1\}\}. \quad (7)$$

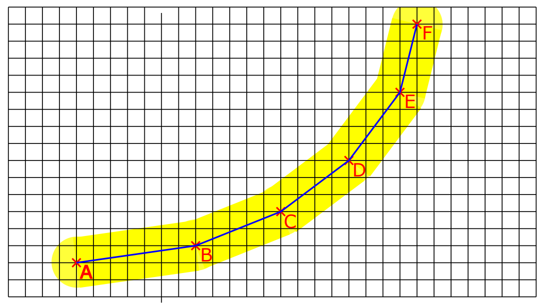

Figure 4: Trajectory: a curve approximated by line segments; the yellow area corresponds to the trace left by the moving disc

Such a hyperoperation is obviously suitable for modelling trajectories of an object in a given grid. Moreover, the parameter r is the width of the footprint of the object, or rather the width the object needs to pass its trajectory.

Before giving a theorem for such an n-ary case, we recall one definition and include two simple lemmas which will facilitate our proof of associativity.

Definition 4. *Let (H, f) be an n-ary hypergroupoid. If there is $\{a_1, \ldots, a_n\} \subseteq f(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in H^n$, we say that H, or f, is* extensive.

Lemma 2. *Sets $f\left(disc_{A_1,r}, \ldots, disc_{A_n,r}\right)$ depend on the order of points $A_1, \ldots A_n$.*

Proof. Obvious, see Fig. 4. □

Lemma 3. *For hyperoperation f defined by (7) there is*

$$f\left(disc_{A_1,r}, \ldots, disc_{A_n,r}\right) = \tag{8}$$
$$= disc_{A_1,r} \circ disc_{A_2,r} \cup disc_{A_2,r} \circ disc_{A_3,r} \cup \ldots \cup disc_{A_{n-1},r} \circ disc_{A_n,r}.$$

Proof. It can be easily seen that the result of the n-ary hyperoperation f is the union of discs with circles on the broken line where the endpoint of each line segment is the starting point of the following line segment. □

Theorem 2. *For all $n > 2$, the pair $(\mathcal{S}_{disc}, f)$, where $f : \mathcal{S}^n_{disc} \to \mathcal{P}^*(\mathcal{S}_{disc})$, is an n-ary hypergroup.*

Proof. The associativity axiom holds – it is obvious thanks to Theorem 1 and Lemma 3. Out of reasons analogous to the binary case, the reproductive axiom follows from the extensivity of f. The fact that $(\mathcal{S}_{disc}, f)$ is, for an arbitrary arity n, extensive, is obvious. □

In the "Motivation" section we mention that, while driving, the car will work with sections of the map only. Therefore, we are going to show that the set $\mathcal{S}_{disc}$ can be restricted as much as to a square of size $2r$, where r is the radius of discs that we work with. In the following theorem notice that ${}_{sub}\mathcal{S}_{disc}$ is an arbitrary subset of $\mathcal{S}_{disc}$, i.e. the elements of ${}_{sub}\mathcal{S}_{disc}$ are discs. Therefore, we have to speak of "the set of points being inside or on the boundary of the discs".

Theorem 3. *If the set of points being inside or on the boundary of the discs of ${}_{sub}\mathcal{S}_{disc}$ is a convex set, then the pair $({}_{sub}\mathcal{S}_{disc}, f)$, where f is defined as* (8), *is, for an arbitrary arity n, an n-ary subhypergroup of* $(\mathcal{S}_{disc}, f)$.

Proof. We must show that $({}_{sub}\mathcal{S}_{disc}, f)$ is a subhypergroupoid of $(\mathcal{S}_{disc}, f)$ which is associative and reproductive. However, for all convex sets, all these properties hold while for all nonconvex at least some do not – see Fig. 5 in which $f(A, B, C) \not\subseteq_{sub} \mathcal{S}_{disc}$.

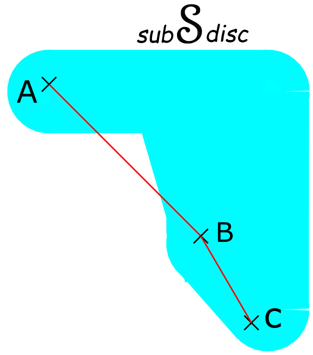

Figure 5: A nonconvex set ${}_{sub}\mathcal{S}_{disc}$ (in blue) which is not a subhypergroupoid of $\mathcal{S}_{disc}$ (notice the line segment $|AB|$).

□

For our purposes it seems convenient to generalize the notions of *reflexive*, *invariant* and *invertible* sets, well-established in the context of binary hypergroups (or, generally speaking, binary hypercompositional structures), to the context of n-ary hypergroups. In the following definition notice that by $f\,(\mathrm{perm}\{x_1, \ldots, x_n\})$ we mean that f is applied on an arbitrary of all possible permutations of elements $\{x_1, \ldots, x_n\}$ while $f(\mathrm{perm}\{\underbrace{x_1, \ldots, x_n, A}_{n}\})$ means that f is applied on an arbitrary of all possible choices of $n-1$ elements from the set $\{x_1, \ldots, x_n\}$ and elements of a fixed set A, i.e. in fact $f(\text{perm}\{\underbrace{x_1, \ldots, x_n, A}_{n}\}) = \bigcup_{a_i \in A} f(\text{perm}\{\underbrace{x_1, \ldots, x_n, a_i}_{n}\})$.

Definition 5. *Let (H, f) be an n-ary hypergrupoid and A a non-empty subset of H. We say that A is:*

- *reflexive in H if, for all $(x_1, \ldots, x_n) \in H^n$, the fact that $f(x_1, \ldots, x_n) \cap A \neq \emptyset$ implies that $f(\text{perm}\{x_1, \ldots, x_n\}) \cap A \neq \emptyset$.*
- *invariant (or normal) in H if, for all $(x_1, \ldots, x_n) \in H^n$, all the sets $f(\text{perm}\{\underbrace{x_1, \ldots, x_n, A}_{n}\})$ are the same.*
- *invertibile in H if, for all $(x_1, \ldots, x_n) \in H^n$, the fact that*

$$x_1 \in f(A, x_2, \ldots, x_n)$$

implies that, simultaneously,

$$x_2 \in f(A, x_1, x_3, \ldots, x_n), \quad \ldots, \quad x_{n-1} \in f(A, x_1, x_2, \ldots, x_{n-2}, x_n).$$

In the "Motivation" section we regard a hyperoperation as a tool to model the trajectory of an autonomous vehicle by means of a certain section of an HD map. Assume that this section is rather small – call it a *microsection*. Thus, the trajectory from point A to C via B can be described by means of $f(disc_{A,r}, disc_{B,r}, disc_{C,r})$. Such a microsection can be represented by the carrier set of an n-ary hypergroup $({}_{sub}\mathcal{S}_{disc}, f)$. However, not all sets $f(\text{perm}\{x_1, \ldots, x_n\})$ are meaningful in the context of trajectories.

Therefore, we introduce the notion of *weak reflexivity*, in which we do not require the property to hold for all sets $f(\text{perm}\{x_1, \ldots, x_n\})$ but only for two specific ones: the forward and backward trajectory.

Definition 6. *Let (H, f) be an n-ary hypergrupoid and A a non-empty subset of H. We say that A is weak reflexive in H if, for all $(x_1, \ldots, x_n) \in H^n$, the fact that $f(x_1, x_2 \ldots, x_{n-1}, x_n) \cap A \neq \emptyset$ implies that $f(x_n, x_{n-1} \ldots, x_2, x_1) \cap A \neq \emptyset$.*

Lemma 4. *For an arbitrary arity $n \geq 2$, in an n-ary hypergroup $(\mathcal{S}_{disc}, f)$, where $f : \mathcal{S}_{disc}^n \to \mathcal{P}^*(\mathcal{S}_{disc})$, there is $f(x_1, x_2, \ldots x_{n-1}, x_n) = f(x_n, x_{n-1}, \ldots, x_2, x_1)$.*

Proof. Obvious using mathematical induction. □

Remark 3. *In Lemma 4 holds for "the way there and back again (in the same footsteps)". Obviously, two arbitrary permutations are not the same. Providing an example is trivial.*

Corolary 1. *An arbitrary subset of $_{sub}\mathcal{S}_{disc}$ is weak reflexive in $(\mathcal{S}_{disc}, f)$, where $f : \mathcal{S}^n_{disc} \to \mathcal{P}^*(\mathcal{S}_{disc})$.*

Proof. Obvious due to Lemma 4. □

Notice that since we are discussing a binary case in the following theorem, we could have used "$\circ$" instead of f.

Theorem 4. *For arity $n = 2$ of f, an arbitrary set $_{sub}\mathcal{S}_{disc}$ is invariant in $(\mathcal{S}_{disc}, f)$.*

Proof. For an arbitrary $disc_{A,r} \in \mathcal{S}_{disc}$ and an arbitrary subset $_{sub}\mathcal{S}_{disc} \subseteq \mathcal{S}_{disc}$ there is

$$f(disc_{A,r},\ {}_{sub}\mathcal{S}_{disc}) = \bigcup_{disc_{B,r} \in_{sub} \mathcal{S}_{disc}} f(disc_{A,r}, disc_{B,r}) =$$

$$= \bigcup_{disc_{B,r} \in_{sub} \mathcal{S}_{disc}} \{disc_{C,r} \in \mathcal{S}_{disc} | C \in |AB|, \text{ for all } disc_{B,r} \in_{sub} \mathcal{S}_{disc}\} =$$

$$\bigcup_{disc_{B,r} \in_{sub} \mathcal{S}_{disc}} f(disc_{B,r}, disc_{A,r},) = f({}_{sub}\mathcal{S}_{disc}, disc_{A,r},)$$

Thus $_{sub}\mathcal{S}_{disc}$ is invariant in $(\mathcal{S}_{disc}, f)$. □

Definition 7. *Let (H, f) be an n-ary hypergroupoid and A a non-empty subset of H. We say that A is weakly invariant in H if, for all $(x_1, \ldots, x_n) \in H^n$, there is*

$$f(A, x_1, x_2 \ldots, x_{n-1}, x_n) \cap f(x_1, A, x_2 \ldots, x_{n-1}, x_n) \cap \ldots \tag{9}$$
$$\ldots \cap f(x_1, x_2 \ldots, x_{n-1} A, x_n) \cap f(x_1, x_2 \ldots, x_{n-1}, x_n, A) \neq \emptyset.$$

Theorem 5. *Let $(\mathcal{S}_{disc}, f)$ be an n-ary hypergroupoid and A a non-empty subset of $\mathcal{S}_{disc}$. If f is extensive, then A is weakly invariant in $\mathcal{S}_{disc}$.*

Proof. Suppose that in $(\mathcal{S}_{disc}, f)$ there is $(x_1, \ldots, x_n) \in f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \mathcal{S}^n_{disc}$. Further, suppose an arbitrary set $A \subseteq \mathcal{S}_{disc}$. Recall that $f(\text{perm}\{\underbrace{x_1, \ldots, x_n, A}_{n}\}) = \bigcup_{a_i \in A} f(\text{perm}\{\underbrace{x_1, \ldots, x_n, a_i}_{n}\})$. Then there is

$$\{(x_1, \ldots, x_n, a_i) \mid a_i \in A\} \subseteq f(\text{perm}\{\underbrace{x_1, \ldots, x_n, A}_{n}\}).$$

In other words,

$$f(A, x_1, x_2 \dots, x_{n-1}, x_n) \cap f(x_1, A, x_2 \dots, x_{n-1}, x_n) \cap \dots \\ f(x_1, x_2 \dots, , x_{n-1} A, x_n) \cap f(x_1, x_2 \dots, x_{n-1}, x_n, A) = T, \quad (10)$$

where $T = \{(x_1, \dots, x_n)\} \cup A$. □

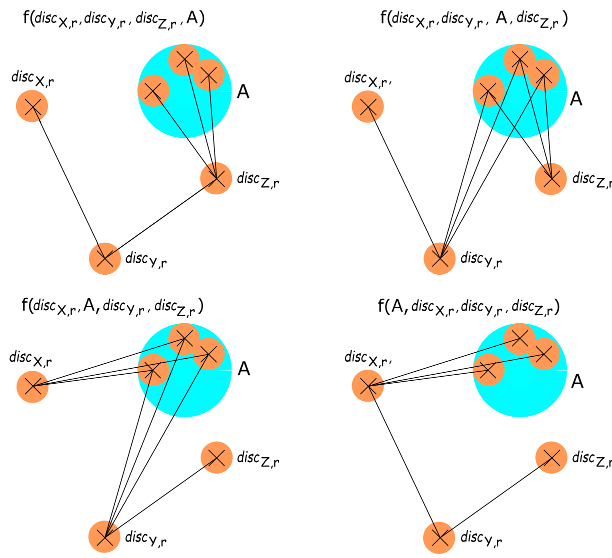

Figure 6: Weakly invariant subhypergroup: in fact, the line segments are places where discs with the same diameters as those with centers x, y, z are positioned; one can see that the intersection of all highlighted sets is T only. Notice that, once again, only line segments along which the discs move are depicted for clarity reasons.

Theorem 6. *Regardless of the arity of f, no subhypergroupoid $({}_{sub}\mathcal{S}_{disc}, f)$ of $(\mathcal{S}_{disc}, f)$ is invertible in $(\mathcal{S}_{disc}, f)$.*

Proof. Suppose that for some $disc_{B,r} \in_{sub} \mathcal{S}_{disc}$ there is

$$disc_{B,r} \in f(disc_{A_1,r}, disc_{A_2,r}, \dots, disc_{A_n,r}, A),$$

where B and $A_i, i = 1, 2, \dots, n$ are points while A is a set. This means, by Lemma 3, that $disc_{B,r} \in disc_{A_1,r} \circ disc_{A_2,r} \cup disc_{A_2,r} \circ disc_{A_3,r} \cup \dots \cup disc_{A_n,r} \circ A$. In other words,

$$B \in |A_1 A_2| \vee B \in |A_2 A_3| \vee \dots \vee B \in |A_n X|,$$

where $disc_{X,r} \in A$. Yet if B is a point of a line segment $|A_iA_{i+1}|$, $i = 1, 2, \ldots, n-1$, then $A_i \notin |BA_{i+1}|$, which means that at least one of the conditions in the definition of invertibility is brokem. See also Fig. 7. □

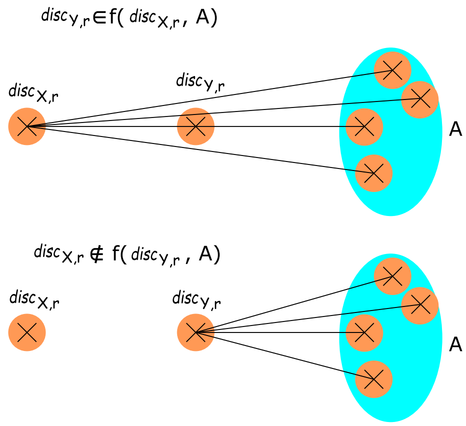

Figure 7: Broken invertibility; notice that in this depicted – binary – case, f could be substituted by "∘"

5 Suggestions for future research

In the context of HD maps, the hyperoperation $disc_{A,r} \circ disc_{B,r}$ means modelling the trajectory of a vehicle from A to B. Thus, when describing the trajectory the vehicle has already passed in order to get to A, one can use the fraction $disc_{A,r}/disc_{B,r} = \{disc_{C,r} \mid disc_{A,r} \in disc_{C,r} \circ disc_{B,r}\}$. This is depicted in Fig. 8, where the vehicle is moving from left to the right. The yellow area is $disc_{A,r} \circ disc_{B,r}$ and the blue area, i.e. the path the vehicle already passed in order to get to A, is $disc_{A,r}/disc_{B,r}$. The blue area continues beyond the figure (this is suggested by the red arrow). Given such fractions it seems very relevant to study the transposition axiom and n-ary equivalents of transposition hypergroups.

Notice that the some attempts to study n-ary transposition can be seen in [20, 21]. Below we propose a new definition that seems to be useful in future considerations, which might be the direction of some future research.

Definition 8. *An n-ary hypergrup (H, f) is called* transposition n-ary hypergroup *if the n-ary transposition axiom holds, i.e. if for all $a, b, c, d, x_1, \ldots, x_n \in$*

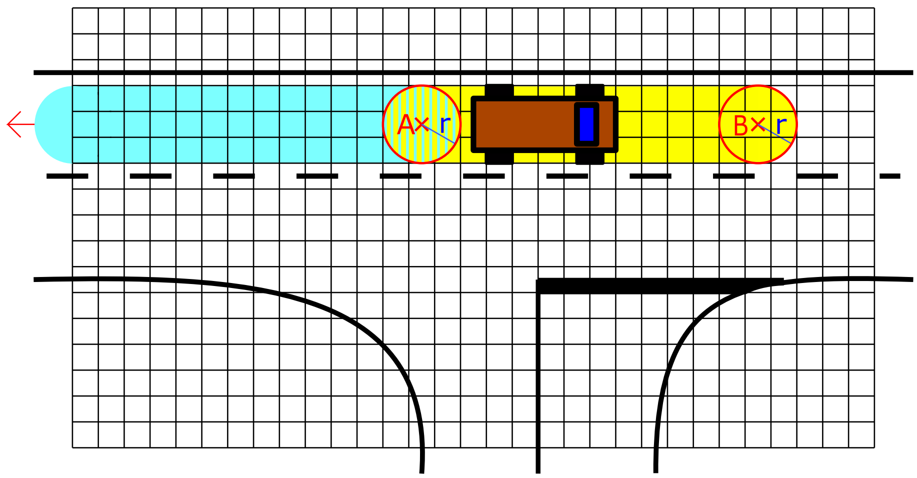

Figure 8: A vehicle moving from left to the right along a crossroads

H the fact that

$$a/(x_1, \dots, x_n) \approx (y_1, \dots y_n) \setminus b \tag{11}$$

impies that each set $f(perm\{\underbrace{y_1, \dots, y_n, a}_{n}\})$ *has a nonempty intersection with each set* $f(perm\{\underbrace{x_1, \dots, x_n, b}_{n}\})$.

Sets $a/(x_1, \dots, x_n) = \{c \in H, x_i \in f(x_1^{i-1}, c, x_{i+1}^n)\}$ *and* $(y_1, \dots y_n) \setminus b = \{d \in H, y_i \in f(y_1^{i-1}, d, y_{i+1}^n)\}$ *are called* left *and* right extensions *or* fractions*, respectively.*

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