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Characterization of second type plane foliations using Newton polygons

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Abstract

In this article we characterize the foliations that have the same Newton polygon that their union of formal separatrices, they are the foliations called of the second type. In the case of cuspidal foliations studied by Loray [**Lo**], we precise this characterization using the Poincaré-Hopf index. This index also characterizes the cuspidal foliations having the same process of singularity reduction that the union of its separatrices. Finally we give necessary and sufficient conditions when these cuspidal foliations are generalized curves, and a characterization when they have only one separatrix.

1 Introduction

Camacho, Lins-Neto and Sad [**Cam-Li-Sad**] introduced and studied the singularities of foliations of the generalized curve type, these are the foliations without saddle-nodes in their reduction of singularities. These foliations receive this name because they have a behavior similar to the union of their separatrices, where a separatrix is an irreducible analytical curve invariant for the foliation. For these foliations the Poincaré-Hopf index coincides with the

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Milnor number of the union of their separatrices [**Cam-Li-Sad**]) and the reduction of singularities of these foliations coincides with the desingularization of the union of their separatrices.
The singularities of generalized curved type verify that their Gómez Mont - Seade - Verjovski index [**Go-Sea-Ve**] is zero and their Camacho - Sad index [**Cam-Sad2**] and the Baum-Bott index are equal [**Br**].

The foliations of the second type can be thought of as a generalization of the singularities of the generalized curve type, in which we will allow the existence of formal separatrices. In order to add the formal separatrix we must admit that we have saddle-nodes in their resolution of singularities that generate formal separatrices, they could not be a corner of two divisors, nor could saddle-nodes outside the corners with weak separatrix contained in the divisor. Note that with these restrictions the singularities of a second type foliation with a single separatrix will have to be a generalized curve foliation. The singularities of second type were introduced by Mattei and Salem [**Ma-Sal**]. They characterized this type of singularities by means of the coincidence of the multiplicity of the foliation with the multiplicity of differential of the union of their formal separatrices. For these singularities, the reduction of singularities coincides with the desingularization of its separatrix. It should be noted that the proof given in [**Cam-Li-Sad**] to prove this property for generalized curve type foliations also proves this property for second type singularities. There are other characterizations of these singularities (see [**Can-Co-Mol**] and [**FP-Mol**]).

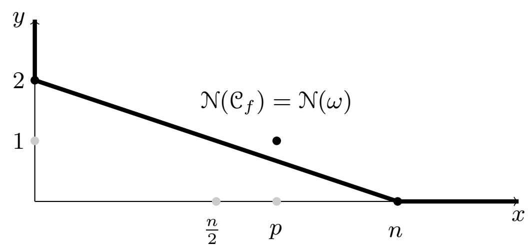
Merle [**Mer**] gives a decomposition of the polar curve of an irreducible curve $\mathcal{C}$ that determines the topology of $\mathcal{C}$. This theorem was generalized for foliations by Rouillé [**R**], where he gives a decomposition of the polar of a foliation, of the generalized curve type, which determines the topology of its separatrix. The main ingredient for his decomposition is Dulac's Theorem [**Du**], this theorem tells us that the Newton polygon of the foliation coincides with the Newton polygon of the separatrix. Merle's theorem has been generalized for reduced curves by [**GB**] and for branches and their approximate roots in [**GB-Gw**], the first of which was generalized for foliations by [**Co**] and the second by [**Sar**]. There are examples of foliations where their Newton polygon coincides with that of their separatrices and is not a foliation of the generalized curve type, however these foliations are of the second type.

The form of the nilpotent singularities was given by F. Takens [**Ta**, Page 55]

$$\omega = d(x^n + y^2) + \alpha x^p U(x) dy, \tag{1}$$

where $\alpha \in \mathbb{C}^*$, $U(x) \in \mathbb{C}\{x\}$, $U(0) \neq 0$, $p \geq 2$ and $n \geq 3$. According to the reduction of singularities, three cases are considered $2p > n$, $2p = n$ and $2p < n$.

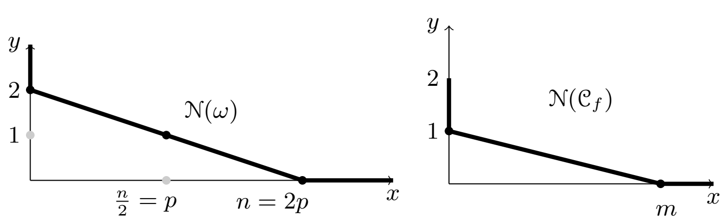
D. Cerveau and R. Moussu [**Ce-Mou**] study the forms given in (1) when $2p > n$. In this case, the foliation has an invariant curve of equation $\mathcal{C}_f : u(x,y)(y^2 + x^n + \cdots) = 0$, where $u(x,y) \in \mathbb{C}[[x,y]]$ with $u(0,0) \neq 0$. In addition, the foliation turns out to be of the generalized curve type and the polygon of Newton of the foliation and of its separatrix coincide (see definition in Section 2).



Newton polygon of $\omega = d(x^n + y^2) + \alpha x^p U(x) dy$ and $\mathcal{C}_f = u(x,y)(y^2 + x^n + \cdots)$.

R. Meziani [**Mez**] studies the form given in (1) when $n = 2p$. Meziani distinguishes the following cases:

- If $\alpha = \pm 4$, the foliation given in (1) it is non-dicritical and has a singularity Dulac saddle-node. In this case, the separatrix is $\mathcal{C}_f : y = a_m x^m + \cdots$, $m \geq 2$ and the foliation is not of the second type. As the multiplicity of the foliation and the separatrix are invariant by changing coordinates, there can not be a change of coordinates in which the polygons of the foliation and that of its separatrix coincide.



Newton polygons of $\omega = d(x^{2p} + y^2) + \alpha x^p U(x) dy$ and $\mathcal{C}_f : y = a_m x^m + \cdots$, $m \geq 2$, $a_m \neq 0$.

- If $\alpha \neq \pm 4$ and (1) is of a generalized curve type, the separatrix is $\mathcal{C}_f$: $u(x,y)(y^2 + x^{2p} + \cdots) = 0$, where $u(x,y) \in \mathbb{C}[[x,y]]$ with $u(0,0) \neq 0$. As in the case $2p > n$ we have also that the polygon of Newton of the foliation and that of its separatrix coincide.

- If $\alpha \neq \pm 4$ and (1) is not of a generalized curve type. It could arrive the dicritical case (which is not interesting in this context) or the non-dicritical case. If the foliation is non-dicritical, it has a singularity of the Dulac-resonant type and with a single separatrix that passes through a saddle-node tangent to the divisor, in particular the foliation is not of the second type and its separatrix is $\mathcal{C}_f : y = a_m x^m + \cdots,\ m \geq 2$. As in the singularity Dulac saddle-node, the polygon of the foliation and its separatrix are different under any change of coordinates.

In the case $2p < n$, the foliation is of the second type that is not generalized curve. In this case the union of formal separatrices is $\mathcal{C}_f : u(x,y)(y^2 + x^{2p} + \cdots) = 0$, where $u(x,y) \in \mathbb{C}[[x,y]]$ with $u(0,0) \neq 0$. The foliation (1) and $\mathcal{C}_f$ have different polygons.

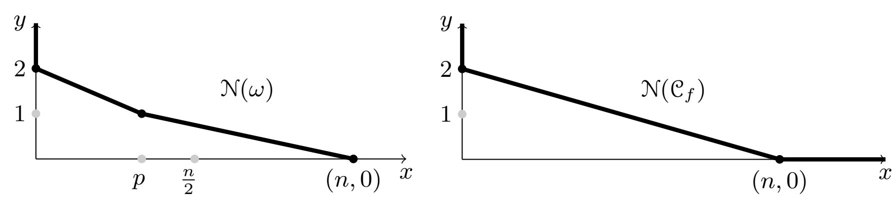


Figure 1:
Newton polygons of $\omega = d(x^n + y^2) + \alpha x^p U(x) dy$ and $\mathcal{C}_f = y^2 + x^n$.

Nevertheless, after [**St**] , we can express the foliation given in (1) by a change of coordinates by

$$yx^{n-p-1}C(x)dx - (y - x^p)dy,\ C(x) \neq 0, \tag{2}$$

where $C(x) \in \mathbb{C}\{x\}$ and the foliation has a formal separatrix $\mathcal{C}_f : y(y - x^p + \cdots) = 0$. In addition, the polygon of the foliation and that of its separatrix coincide.

In this paper we give a new characterization of the singularities of the second type in terms of the Newton polygon of their union of separatrices.

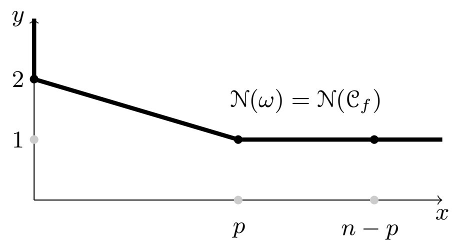


Figure 2:
Newton polygon of $\omega = yx^{n-p-1}C(x)dx - (y - x^p)dy$ and $\mathcal{C}_f = y(y - x^p + \cdots)$.

Theorem 1.1. *A non-dicritical foliation is of the second type if and only if there is a change of coordinates under which its Newton polygon coincides with that of their union of separatrices.*

We will prove Theorem 1.1 in Section 4.

According to Loray [**Lo**], a foliation with a cuspidal singularity is given by

$$\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = \mathrm{d}(y^p - x^q) + \Delta(x, y)(px\mathrm{d}y - qy\mathrm{d}x), \tag{3}$$

where p, q are positive natural numbers and $\Delta(x, y) \in \mathbb{C}\{x, y\}$.

We will use Theorem 1.1 to characterize, in terms of the Poincaré-Hopf index for foliations, when the foliations with cuspidal singularities studied by Loray [**Lo**] are of the second type.

Denote by $\mathrm{PH}(\mathcal{F})$ the Poincaré-Hopf index of the foliation $\mathcal{F}$. For the foliation $\mathrm{d}(y^p - x^q)$ we have $\mathrm{PH}_{(p,q)} := \mathrm{PH}(\mathrm{d}(y^p - x^q)) = (p - 1)(q - 1)$.

Theorem 1.2. *Let $\mathcal{F}_{\omega_{p,q,\Delta}}$ be a cuspidal foliation as in (3) and suppose that $\mathcal{F}_{\omega_{p,q,\Delta}}$ is non-dicritical. The next statements are equivalents:*

- (a) *The cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type.*
- (b) *The intersection number $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$.*
- (c) *The cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ has the same reduction of singularities that $\mathrm{d}(y^p - x^q)$.*

In general, if a foliation has the same reduction of singularities as its union of separatrices then the foliation is not of the second type (see Example 2.3).

However, after Theorem 1.2, for the cuspidal foliations this property characterizes foliations of the second type.

We also give, in the next theorem, necessary and sufficient conditions when the cuspidal foliations are of the generalized curve type.

Theorem 1.3. *Let $\mathcal{F}_{\omega_{p,q,\Delta}}$ be a cuspidal foliation as in (3) and suppose that $\mathcal{F}_{\omega_{p,q,\Delta}}$ is non-dicritical. We have:*

(a) If the intersection number $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$, then $\mathcal{F}_{p,q,\Delta}(x,y)$ is of the generalized curve type.

(b) If p, q are coprime then the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of generalized curve type if and only if $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$.

We will prove Theorem 1.2 and Theorem 1.3 in Section 5.

2 Basic Definitions and Notations

In order to fix the notation, we will remember basic concepts of local foliation theory and plane curves. Denote by $\mathbb{C}[[x,y]]$ the ring of formal powers series in two variables with coefficients in $\mathbb{C}$ and $\mathbb{C}\{x,y\}$ the sub-ring of $\mathbb{C}[[x,y]]$ formed by formal powers series that converge in a neighborhood of $0 \in \mathbb{C}^2$. Consider the maximal ideals $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$ of $\mathbb{C}\{x,y\}$ and $\mathbb{C}[[x,y]]$ respectively. The *order* of a power series $\widehat{h}(x,y) = \sum_{ij} a_{ij}x^iy^j \in \mathbb{C}[[x,y]]$ is $\mathrm{ord}(\widehat{h}) := \min\{i+j \ : \ a_{ij} \neq 0\}$.

A *singular formal foliation* $\widehat{\mathcal{F}_\omega}$ of codimension one over $\mathbb{C}^2$ is locally given by a 1-form $\widehat{\omega} = \widehat{A}(x,y)\mathrm{d}x + \widehat{B}(x,y)\mathrm{d}y$, where $\widehat{A}, \widehat{B} \in \widehat{\mathfrak{m}}$ are coprime. The power series $\widehat{A}$ and $\widehat{B}$ are called the *coefficients* of $\widehat{\omega}$. The *multiplicity* of the foliation $\widehat{\mathcal{F}_\omega}$ is defined as $\mathrm{mult}(\widehat{\omega}) := \min\{\mathrm{ord}(\widehat{A}), \mathrm{ord}(\widehat{B})\}$.

Let $T \subseteq \mathbb{N}^2$. Denote by $D(T)$ the convex hull of $(T + \mathbb{R}^2_{\geq 0})$, where $+$ is the Minkowski sum, and by $\mathcal{N}(T)$ the polygonal boundary of $D(T)$, which will call *Newton polygon* determined by T.

Let $\widehat{h}(x,y) = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{C}[[x,y]]$. The *support* of $\widehat{h}$ is

$$\mathrm{supp}(\widehat{h}) := \{(i,j) \in \mathbb{N}^2 \ : \ a_{ij} \neq 0\},$$

and the Newton polygon of $\widehat{h}$ is by definition the Newton polygon $\mathcal{N}(\mathrm{supp}(\widehat{h}))$.

Let $\widehat{\omega}=\widehat{A}(x,y)\mathrm{d}x+\widehat{B}(x,y)\mathrm{d}y$ be a one-form, where $\widehat{A},\widehat{B}\in\widehat{\mathfrak{m}}$. The *support* of $\widehat{\omega}$ is

$$\operatorname{supp}(\widehat{\omega})=\operatorname{supp}(x\widehat{A})\cup\operatorname{supp}(y\widehat{B}).$$

If we write $\widehat{\omega}=\sum_{i,j}\widehat{\omega}_{ij}$, where $\widehat{\omega}_{ij}=\widehat{A}_{ij}x^{i-1}y^j\mathrm{d}x+\widehat{B}_{ij}x^iy^{j-1}\mathrm{d}y$, then

$$\operatorname{supp}(\widehat{\omega})=\{(i,j):(\widehat{A}_{ij},\widehat{B}_{ij})\neq(0,0)\}.$$

Let $\widehat{\mathcal{F}_\omega}:\widehat{\omega}=0$ be a foliation given by the one-form $\widehat{\omega}$. The Newton polygon of $\widehat{\mathcal{F}_\omega}$, denoted by $\mathcal{N}(\widehat{\mathcal{F}_\omega})$ or $\mathcal{N}(\widehat{\omega})$ is the Newton polygon $\mathcal{N}(\operatorname{supp}(\widehat{\omega}))$.

Let $\widehat{f}(x,y)\in\mathbb{C}[[x,y]]$. We say that the $\widehat{\mathcal{S}}_f:\widehat{f}(x,y)=0$ is *invariant* by $\widehat{\mathcal{F}_\omega}$ if $\widehat{\omega}\wedge\mathrm{d}\widehat{f}=\widehat{f}.\widehat{\eta}$, where $\widehat{\eta}$ is a two-form (i.e. $\widehat{\eta}=\widehat{g}\mathrm{d}x\wedge\mathrm{d}y$, for some $\widehat{g}\in\mathbb{C}[[x,y]]$). If $\widehat{\mathcal{S}}_f$ is irreducible then we will say that $\widehat{\mathcal{S}}_f$ is a *formal separatrix* of $\widehat{\mathcal{F}_\omega}:\widehat{\omega}=0$.

We will consider *non-dicritical* foliations, that is, foliations having a finite set of separatrices (see [**Cam-Li-Sad**, page 158 and page 165]). Let $(\widehat{\mathcal{S}}_{f_j})_{j=1}^r$ be the set of all formal separatrices of the non-dicritical foliation $\widehat{\mathcal{F}_\omega}:\widehat{\omega}=0$. Each separatrix $\widehat{\mathcal{S}}_{f_j}$ corresponds to an irreducible formal power series $\widehat{f_j}(x,y)$. Denote by $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})$ the union $\bigcup\widehat{\mathcal{S}}_{f_j}$ of all separatrices of the foliation $\widehat{\mathcal{F}_\omega}$, which we will call *union of formal separatrices* of $\widehat{\mathcal{F}_\omega}$. In the following we will denote by $\mathcal{F}_\omega$ a holomorphic foliation and by $\mathcal{S}(\mathcal{F}_\omega)$ its union of convergent separatrices.

The *dual vector field* associated to $\widehat{\mathcal{F}_\omega}$ is $X=\widehat{B}(x,y)\frac{\partial}{\partial x}-\widehat{A}(x,y)\frac{\partial}{\partial y}$. We say that the origin $(x,y)=(0,0)$ is a *simple or reduced singularity* of $\widehat{\mathcal{F}_\omega}$ if the matrix associated with the linear part of the field

$$\begin{pmatrix}\frac{\partial\widehat{B}(0,0)}{\partial x} & \frac{\partial\widehat{B}(0,0)}{\partial y}\\ -\frac{\partial\widehat{A}(0,0)}{\partial x} & -\frac{\partial\widehat{A}(0,0)}{\partial y}\end{pmatrix}\tag{4}$$

has two eigenvalues λ,μ, with $\frac{\lambda}{\mu}\notin\mathbb{Q}^+$.
It could happen that

a) $\lambda\mu\neq0$ and $\frac{\lambda}{\mu}\notin\mathbb{Q}^+$ in which case we will say that the *singularity is not degenerate* or

b) $\lambda\mu=0$ and $(\lambda,\mu)\neq(0,0)$ in which case we will say that the singularity is a *saddle-node*.

In the *b*) case, the *strong separatrix* of a foliation with singularity P is an analytic invariant curve whose tangent at the singular point P is the eigenspace associated with the non-zero eigenvalue of the matrix given in (4). The zero eigenvalue is associated with a formal separatrix called *weak separatrix*.

From now on $\pi : M \to (\mathbb{C}^2, 0)$ represents *the process of singularity reduction* of $\widehat{\mathcal{F}_\omega}$ [**Ma-Mou**], obtained by a finite sequence of point blows-up, where $\mathcal{D} := \pi^{-1}(0) = \bigcup_{j=1}^{n} D_j$ is the *exceptional divisor*, which is a finite union of projective lines with normal crossing (that is, they are locally described by one or two regular and transversal curves). In this process, any separatrix of $\widehat{\mathcal{F}_\omega}$ is smooth, disjoint and transverse to $D_j \subset \mathcal{D}$, and it does not pass through a corner (intersection of two components of the divisor $\mathcal{D}$). Let $\widehat{\mathcal{F}}_\omega$ be a non-dicritical formal foliation and consider the minimal reduction of singularities $\pi : M \to (\mathbb{C}^2, 0)$ of $\widehat{\mathcal{F}}_\omega$ (this is, a reduction with the minimal number of blows-up that reduces the foliation). The *strict transform* of the foliation $\widehat{\mathcal{F}}_\omega$ is given by $\widehat{\mathcal{F}}'_\omega = \pi^* \widehat{\mathcal{F}}_\omega$ and the *exceptional divisor* is $\mathcal{D} = \pi^{-1}(0)$.

A foliation $\widehat{\mathcal{F}_\omega}$ is a *generalized curve* if in its reduction of singularities there are no saddle-node points.

If in the process of singularity reduction of $\widehat{\mathcal{F}}_\omega$, the exceptional divisor $\mathcal{D}$ at point P contains the weak invariant curve of the saddle-node, then the singularity is called *saddle-node tangent*. Otherwise we will say that $\widehat{\mathcal{F}}_\omega$ is a *saddle-node transverse* to $\mathcal{D}$ at point P.

Definition 2.1. The foliation $\widehat{\mathcal{F}}_\omega$ is of the *second type* with respect to the divisor $\mathcal{D}$ if no singular points of $\widehat{\mathcal{F}}'_\omega$ is of tangent node type.

Non-dicritical foliations of the second type were studied by Mattei and Salem [**Ma-Sal**], also by Cano, Corral and Mol [**Can-Co-Mol**] and in the dicritical case by Genzmer and Mol [**Ge-Mol**] and Fernández Pérez-Mol [**FP-Mol**]. Mattei and Salem gave the next characterization of foliations of the second type in terms of the multiplicity of their union of formal separatrices:

Theorem 2.2. *[***Ma-Sal***, Théorème 3.1.9] Let $\widehat{\mathcal{F}_\omega}$ be a non-dicritical foliation and let $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega}) : \hat{f}(x, y) = 0$ be a reduced equation of its union of separatrices. Consider the minimal reduction of singularities $\pi : (M, D) \to (\mathbb{C}^2, 0)$ of $\widehat{\mathcal{F}_\omega}$. Then*

1. *π is a reduction of singularities of $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})$. Furthermore, if $\widehat{\mathcal{F}_\omega}$ is of the second type then π is the minimal reduction of singularities of $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})$.*

2. $\mathrm{mult}(\widehat{\omega}) \geq \mathrm{mult}(\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})) - 1$ *and the equality holds if and only if $\widehat{\mathcal{F}_\omega}$ is of the second type.*

The reciprocal of the first statement of Theorem 2.2 is not true, that is, if the reduction of singularities of the foliation and that of its union of separatrices coincide does not guarantee that the foliation is of the second type, as shown in the following example.

Example 2.3. The union of the separatrices of the foliation $\mathcal{F}_\omega = (xy+y^2)\mathrm{d}x - x^2\mathrm{d}y$ is $\mathcal{S}(\mathcal{F}_\omega) = xy$. The foliation $\mathcal{F}_\omega$ is reduced and its union of separatrices is desingularized after a blow-up but the foliation is not of the second type because the strong separatrix that passes through the saddle-node is not contained in the divisor.

3 Foliations and Newton polygones

Non-dicritical generalized curve foliations are those in which no saddle-node points appear in their process of singularity reduction [**Sei**] and they have a finite number of separatrices [**Cam-Li-Sad**]. These foliations were studied by Camacho, Lins Neto and Sad who proved that

Theorem 3.1. *[**Cam-Li-Sad**, Theorem 2] Let $\mathcal{F}_\omega$ be a non-dicritical generalized curve foliation and $\mathcal{S}(\mathcal{F}_\omega)$ its union of separatrices. Then $\mathcal{F}_\omega$ and $\mathcal{S}(\mathcal{F}_\omega)$ have the same reduction of singularities.*

Rouillé obtained the following result on non-dicritical generalized curve foliations. In [**R**] it is indicated that Mattei reported that this result was known by Dulac [**Du**].

Proposition 3.2. *[**R**, Proposition 3.8] Let $\mathcal{F}_\omega$ be a non-dicritical generalized curve foliation and $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ be a reduced equation of its union of separatrices. Then $\mathcal{N}(\omega) = \mathcal{N}(df)$.*

Observation 3.3. Rouillé proved that the Newton polygons of ω and df coincide for some coordinate system but he did not explicit in his proof what is such system.

Example 3.4. In the reduction of nilpotent singularities given in (1), for the case $2p < n$, after a change of coordinates the foliation becomes (2). In these new coordinates the Newton polygon of the foliation and of its separatrix coincide (see Figure 2), but with the coordinates given initially, the foliation (1) and its separatrix $\mathcal{C}_f : y^2 + x^n = 0$ have different polygons (see Figure 1).

The reciprocal of Theorem 3.1 and Proposition 3.2 are not true, as the following example shows:

Example 3.5. Consider $b \notin \mathbb{Q}$ and the foliation defined by $\omega = ((b-1)xy - y^3)\mathrm{d}x + (xy - bx^2 + xy^2)\mathrm{d}y$. A reduced equation of its union of separatrices is $f(x,y) = xy(x-y) = 0$. The foliation $\mathcal{F}_\omega$ is reduced and the curve $f(x,y)=0$ is desingularised after a blow-up. The Newton polygons $\mathcal{N}(\omega)$ and $\mathcal{N}(f)$ are equal but $\mathcal{F}_\omega$ is not a curve generalized type foliation since a saddle-node point appears in its reduction of singularities.

In [**Br**, pag 532] was introduced the *Gómez-Mont-Seade-Verjovsky index* denoted by $GSV(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega))$, where $\mathcal{F}_\omega : \omega = 0$ and $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ is a reduced equation of union of convergent separatrices of $\mathcal{F}_\omega$. For $f \in \mathbb{C}\{x,y\}$, there are $g, h \in \mathbb{C}\{x,y\}$, with h and f coprime, and an analytic one-form η such that $g\omega = h\mathrm{d}f + f\eta$. In [**Br**], Brunella defines

$$GSV(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \frac{1}{2\pi i}\int_{\partial\mathcal{S}(\mathcal{F}_\omega)} \frac{g}{h} d\left(\frac{h}{g}\right),$$

when $\mathcal{S}(\mathcal{F}_\omega) : f = 0$ is irreducible and $\omega = A(x,y)dx + B(x,y)dy$. We get

$$GSV(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \mathrm{ord}_t\left(\frac{B}{f_y}(\gamma(t))\right),$$

where $\gamma(t)$ is a parametrization of $\mathcal{S}(\mathcal{F}_\omega)$. Now, we remember some results on the index $GSV(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega))$.

Theorem 3.6. *[***Cav-Le***, Théorème 3.3][***Br***, Proposition 7] Let $\mathcal{F}_\omega$ be a non-dicritical foliation and let $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ be a reduced equation of its union of separatrices. Then $\mathcal{F}_\omega$ is a generalized curve foliation if and only if $GSV(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = 0$.*

The next result on generalized curve foliations was obtained by Rouillé [**R**] and it will be very useful in this paper. Denote by $\mathbb{C}[[t]]$ the ring of formal power series in the variable t and coefficients in $\mathbb{C}$, and $\mathbb{C}\{t\}$ the subring of $\mathbb{C}[[t]]$ of convergent power series.

Lemma 3.7. *[***R***, Lemme 3.7] Let $\mathcal{F}_{\omega_1}$ and $\mathcal{F}_{\omega_2}$ two non-dicritical generalized curve foliations with the same union of separatrices. If $\gamma(t) = (x(t), y(t)) \in (\mathbb{C}\{t\})^2$ with $\gamma(0) = 0$, then*

$$\mathrm{ord}_t\gamma^*\omega_1 = \mathrm{ord}_t\gamma^*\omega_2.$$

We deduce from Example 3.5 that there are foliations having the same polygon as their union of separatrices but they are not generalized curve foliations.

Our objective in this paper is to characterize the foliations having the same Newton polygon that its union of separatrices. They will be the non-dicritical foliations of the second type.

4 Characterization of a foliation of the second type in terms of the Newton polygon

In this section, a new characterization of the second-type non-dicritical foliations is given in terms of the Newton polygon of the foliation and that of its union of separatrices.

Lemma 4.1. *Let $\widehat{\mathcal{F}_\omega}$ be a non-dicritical foliation and $\widehat{f}(x,y)=0$ be a reduced equation of its union of separatrices. If $\mathcal{N}(\widehat{\omega})=\mathcal{N}(\widehat{f})$ then $\widehat{\mathcal{F}_\omega}$ is a foliation of the second type.*

Proof. Consider the foliation $\widehat{\mathcal{F}_\omega}$ given by $\widehat{\omega}=\sum_{i,j}\widehat{A}_{ij}x^{i-1}y^j\,\mathrm{d}x+\sum_{i,j}\widehat{B}_{ij}x^iy^{j-1}\mathrm{d}y$.
Since $\mathrm{mult}(\widehat{\omega})=\min\{\mathrm{ord}(\widehat{A}),\mathrm{ord}(\widehat{B})\}$ then

$$\begin{array}{rcl}\mathrm{mult}(\widehat{\omega}) & = & \min\{i+j-1:(i,j)\in\mathcal{N}(\widehat{\omega})\}\\ & = & \min\{i+j-1:(i,j)\in\mathcal{N}(\mathrm{d}\widehat{f})\}\\ & = & \mathrm{mult}(\mathrm{d}\widehat{f}).\end{array}\tag{5}$$

From (5) and the second statement of Theorem 2.2 we finish the proof. □

As a consequence of Lemma 4.1 and Theorem 2.2 we conclude that if $\mathcal{N}(\widehat{\omega})=\mathcal{N}(\widehat{f})$ then the foliation $\widehat{\mathcal{F}_\omega}$ and its union of separatrices $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})$ have the same resolution.
In the following proposition we generalize Lemma 3.7 to second type foliations.

Proposition 4.2. *Let $\widehat{\mathcal{F}_\omega}$ be a non-dicritical second type foliation and $\widehat{\mathcal{S}}(\widehat{\mathcal{F}_\omega})$ its union of separatrices . If $\gamma(t)=(x(t),y(t))\in(\mathbf{C}[[t]])^2$, with $\gamma(0)=(0,0)$, then*

$$\mathrm{ord}_t\gamma^*\widehat{\omega}=\mathrm{ord}_t\gamma^*\mathrm{d}\widehat{f}.$$

Proof. If $\gamma(t)=(x(t),y(t))$ is a parameterization of a separatrix of $\widehat{\omega}$ and $\mathrm{d}\widehat{f}$ then $\widehat{\omega}(\gamma(t)).\gamma'(t)=0=\mathrm{d}\widehat{f}(\gamma(t)).\gamma'(t)$ and we conclude the proposition in such a case.

Suppose now that $\gamma(t)$ is not a parameterization of any separatrix of $\widehat{\omega}$ and $\mathrm{d}\widehat{f}$. We proceed by induction on the number of blows-up n needed in the process of the reduction of the foliation $\widehat{\mathcal{F}_\omega}$. First we suppose that the number of blows-up is $n = 0$. Then the foliations defined by the one-forms $\widehat{\omega}$ and $\mathrm{d}\widehat{f}$ are reduced. If $\widehat{\mathcal{F}_\omega}$ is a generalized curve foliation then the proposition follows from Lemma 3.7.

Suppose now that $\widehat{\mathcal{F}_\omega}$ is a reduced foliation with a saddle-node. Under a change of coordinates we can consider the normal form of the saddle-node given by the equation (see [**Cam-Sad2**, Page 66]):

$$\widehat{\omega} = x(1+\lambda y^p)\mathrm{d}y - y^{p+1}\mathrm{d}x, \quad \text{with } p \geq 1 \text{ and } \quad \lambda \in \mathbb{C},$$

and the reduced equation of its union of formal separatrices given by $\widehat{f}(x,y) = xy$. We can write $\gamma(t) = (x(t), y(t)) = (t^a n_1(t), t^b n_2(t))$, where a, b are positive integers and $n_i(t)$ are units of $\mathbb{C}[[t]]$ (that is $n_i(0) \neq 0$ for $i = 1, 2$). We have

$$\begin{array}{rcl} \gamma^*\widehat{\omega} & = & [bt^{a+b-1}n_1(t)n_2(t) + t^{a+b}n_1(t)n_2'(t) \\ & + & t^{a+b+pb-1}n_1(t)(n_2(t))^{p+1}(\lambda b - a) \\ & + & \lambda t^{a+b+pb}n_1(t)(n_2(t))^p n_2'(t) - t^{a+b+pb}n_1'(t)(n_2(t))^{p+1}]\mathrm{d}t, \end{array}$$

so $\mathrm{mult}(\gamma^*\widehat{\omega}) = a + b - 1$. On the other hand $\mathrm{d}\widehat{f} = y\mathrm{d}x + x\mathrm{d}y$, hence

$$\begin{array}{rcl} \gamma^*\mathrm{d}\widehat{f} & = & t^b n_2(t)(at^{a-1}n_1(t) + t^a n_1'(t))\mathrm{d}t \\ & + & t^a n_1(t)(bt^{b-1}n_2(t) + t^b n_2'(t))\mathrm{d}t \\ & = & [t^{a+b-1}n_1(t)n_2(t)(a+b) + t^{a+b}(n_1'(t)n_2(t) + n_1(t)n_2'(t))]\mathrm{d}t, \end{array}$$

so $\mathrm{mult}(\gamma^*\mathrm{d}\widehat{f}) = a + b - 1$. Therefore, if $\widehat{\mathcal{F}_\omega}$ is a reduced foliation with a saddle-node, then $\mathrm{ord}_t(\gamma^*\widehat{\omega}) = \mathrm{ord}_t(\gamma^*\mathrm{d}\widehat{f})$.

Suppose, by induction hypothesis, that the proposition is true for any non-dicritical second type foliation with $n-1$ steps in their process of singularity reduction.
Now, consider that the foliations defined by the one-forms $\widehat{\omega}$ and $\mathrm{d}\widehat{f}$ are not reduced and $n > 0$. Let E be the blow-up at the origin $(x,y) = (0,0)$ given by $E : (x,u) = (x, xu)$, so $E^*\widehat{\omega} = x^m\widetilde{\widehat{\omega}}$, where m is the multiplicity of $\widehat{\omega}$ and $\widetilde{\widehat{\omega}}$ is the strict transform of $\widehat{\omega}$, which union of separatrices has as equation $x\widetilde{\widehat{f}}$. The process of singularity reduction of $\widetilde{\widehat{\omega}}$ and $\mathrm{d}(\widetilde{\widehat{f}})$ has $n-1$ steps. Denote by $\widetilde{\gamma} = (\widetilde{x}(t), \widetilde{y}(t)) \in (\mathbf{C}[[t]])^2$ the strict transformation, by E, of the curve γ, that is $\gamma = E \circ \widetilde{\gamma}$. So, $(\widetilde{x}(t), \widetilde{y}(t)) = \left(x(t), \frac{y(t)}{x(t)}\right) = \left(t^p n_1(t), t^{q-p}\frac{n_2(t)}{n_1(t)}\right)$ and $\widetilde{\gamma}(0) = (0,0)$ (since $q > p$). Hence applying the induction hypothesis to $\widetilde{\widehat{\omega}}$ and $\widetilde{\gamma}$ we get

$$\operatorname{ord}_t(\widetilde{\gamma})^*\widetilde{\widehat{\omega}} = \operatorname{ord}_t(\widetilde{\gamma})^*\mathrm{d}(x\widetilde{\widehat{f}}). \tag{6}$$

Since $\operatorname{ord}_t(\widetilde{\gamma}^*\widetilde{\mathrm{d}\widehat{f}}) = \operatorname{ord}_t(\widetilde{\gamma}^*\mathrm{d}(x\widetilde{\widehat{f}}))$ the equality (6) becomes

$$\operatorname{ord}_t(\widetilde{\gamma})^*\widetilde{\widehat{\omega}} = \operatorname{ord}_t\widetilde{\gamma}^*\widetilde{\mathrm{d}\widehat{f}}. \tag{7}$$

On the other hand we have
$\gamma^*\widehat{\omega} = (E\circ\widetilde{\gamma})^*\widehat{\omega} = (\widetilde{\gamma})^*E^*\widehat{\omega} = x(t)^m(\widetilde{\gamma})^*\widetilde{\widehat{\omega}}$, so

$$\operatorname{ord}_t\gamma^*\widehat{\omega} = \operatorname{mult}(x(t))\operatorname{mult}(\widehat{\omega}) + \operatorname{ord}_t\widetilde{\gamma}^*\widetilde{\widehat{\omega}}. \tag{8}$$

Similarly we get

$$\operatorname{ord}_t\gamma^*\mathrm{d}\widehat{f} = \operatorname{mult}(x(t))\operatorname{mult}(\mathrm{d}\widehat{f}) + \operatorname{ord}_t\widetilde{\gamma}^*\widetilde{\mathrm{d}\widehat{f}}. \tag{9}$$

Since the foliation $\widehat{\mathcal{F}_\omega}$ is of the second type, by Theorem 2.2 we have $\operatorname{mult}(\widehat{\omega}) = \operatorname{mult}(\mathrm{d}\widehat{f})$. Hence we finish the proof from (7), (8) and (9). □

Proposition 4.2 was also given in [**Can-Co-Mol**, Corollary 1], but with other proof.

Proof of Theorem 1.1. First, suppose that $\widehat{\mathcal{F}_\omega}$ is a second type foliation. Using Theorem 2.2, we have that $\widehat{\mathcal{F}_\omega}$ has the same reduction of singularities as its union of formal separatrices and $\operatorname{mult}(\widehat{\omega}) = \operatorname{mult}(\mathrm{d}\widehat{f})$. Reasoning analogously as in the proof given by Rouillé [**R**] in order to prove the Proposition 3.2, by Proposition 4.2 and Observation 3.3, there is a coordinate system such that $\mathcal{N}(\widehat{\omega}) = \mathcal{N}(\mathrm{d}\widehat{f}) = \mathcal{N}(\widehat{f})$.

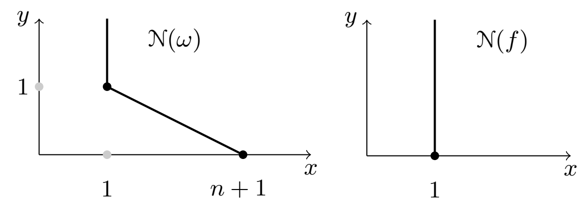
Suppose now that $\mathcal{N}(\widehat{\omega}) = \mathcal{N}(\widehat{f})$. Since $\mathcal{N}(\widehat{f}) = \mathcal{N}(\mathrm{d}\widehat{f})$, we finish the proof after Lemma 4.1. □

Theorem 1.1 gives a new characterization of the non-dicritical second type foliations using its Newton polygon.

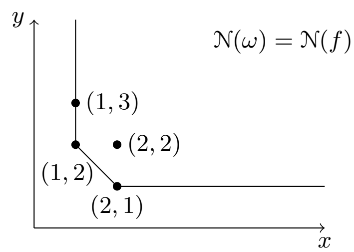
As a consequence we have:

Corollary 4.3. *Let $\widehat{\mathcal{F}_{\omega_1}}$ and $\widehat{\mathcal{F}_{\omega_2}}$ be two non-dicritical second type foliations. If $\widehat{\mathcal{F}_{\omega_1}}$ and $\widehat{\mathcal{F}_{\omega_2}}$ have the same union of formal separatrices, then there is a coordinate system such that $\mathcal{N}(\widehat{\omega_1}) = \mathcal{N}(\widehat{\omega_2})$.*

Example 4.4. The foliation $\mathcal{F}_\omega$ given by $\omega = (ny + x^n)\mathrm{d}x - x\mathrm{d}y$, $n \geq 1$ is not a foliation of the second type. The union of separatrices of $\mathcal{F}_\omega$ is $\mathcal{S}(\mathcal{F}_\omega) : x = 0$. We observe that $\operatorname{supp}(\omega) = \{(1,1),(n+1,0)\}$ and $\operatorname{supp}(f) = \{(1,0)\}$, hence the Newton polygons, in coordinates (x,y), of $\mathcal{F}_\omega$ and $\mathcal{S}(\mathcal{F}_\omega)$ are different.



Example 4.5. Let us go back to Example 3.5. The second type foliation given by $\omega = ((b-1)xy - y^3)dx + (xy - bx^2 + xy^2)dy$ with $-b, 1-b \notin \mathbb{Q}^+$ has as union of separatrices to $\mathcal{S}(\mathcal{F}) = xy(x-y)$. We observe that the polygons $\mathcal{N}(\omega)$ and $\mathcal{N}(f)$ are equal.



5 Cuspidal Foliations

Cuspidal foliations are inspired by *nilpotent foliations.* A foliation $\mathcal{F}_\omega$ in $(\mathbb{C}^2, 0)$ is called a nilpotent singularity if it is generated by a vector field X with a non-zero nilpotent linear part (that is, the matrix associated with the linear part of the field is nilpotent). The nilpotent singularities were generalized to cuspidal singularities by Loray [**Lo**], as we shall see below.

In this section we characterize when foliations with cuspidal singularities are of the second type in terms of weighted order. Furthermore, by means of the weighted order, we give necessary and sufficient conditions for these foliations to be of generalized curve type.
Given $p, q \in \mathbb{N}^*$, we define the *weighted degree* of a monomial $x^i y^j$ as

$$\deg_{(p,q)}(x^i y^j) = \frac{ip + jq}{\gcd(p,q)},$$

and the *weighted order* of a power series $f(x,y) = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{C}\{x,y\}$ as

$$\operatorname{ord}_{(p,q)}(f(x,y)) = \min\left\{\deg_{(p,q)}(x^i y^j) : a_{i,j} \neq 0\right\}.$$

Remember that according to Loray [**Lo**], a foliation with a cuspidal singularity is given as in (3), that is by

$$\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = \mathrm{d}(y^p - x^q) + \Delta(x,y)(px\mathrm{d}y - qy\mathrm{d}x),$$

where p, q are positive natural numbers and $\Delta(x,y) \in \mathbb{C}\{x,y\}$.

On the other hand, remember that $\mathrm{PH}_{(p,q)} := \mathrm{PH}(\mathrm{d}(y^p - x^q)) = pq - p - q + 1$.

Cuspidal foliations are nilpotent foliations when $p = 2$.

For Loray, the foliations $\mathcal{F}_{\omega_{p,q,\Delta}}$ and $\mathrm{d}(y^p - x^q)$ have the same resolution of singularities if and only if $\mathrm{ord}_{(p,q)}(\Delta) > \frac{pq-p-q}{\gcd(p,q)} = \frac{\mathrm{PH}_{(p,q)}-1}{\gcd(p,q)}$. Fernández, Mozo and Neciosup [**F-Moz-N**], find an imprecision in the characterization originally proposed by Loray. These authors mention that the condition is sufficient but not necessary, as it can be seen from the following example.

Example 5.1. The foliation $\omega = \mathrm{d}(y^6 - x^3) + axy(6x\mathrm{d}y - 3y\mathrm{d}x)$ with $a \notin \{-(6r+1)\zeta : r \in \mathbb{Q}_{>0} \textit{ and } \zeta^3 = 1\}$ has the same resolution as the foliation $\mathrm{d}(y^6 - x^3) = 0$, but the function $\Delta(x,y) = axy$ satisfies $\mathrm{ord}_{(6,3)}\Delta = 3$, so the inequality $\mathrm{ord}_{(6,3)}\Delta > \frac{\mathrm{PH}_{(p,q)}-1}{\gcd(p,q)}$ does not hold.

Lemma 5.2. *If the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is a non-dicritical foliation, then $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) = y^p - x^q = 0$ is its union of separatrices.*

Proof. The curve $\mathcal{S}_f : y^p - x^q = 0$ is an invariant curve of the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$. Put $\alpha = \mathrm{ord}(\Delta)$. Then

$$\mathrm{mult}(\omega_{p,q,\Delta}) = \min\{q-1, p-1, \alpha+1\}. \tag{10}$$

Suppose that $p < q$. The multiplicity of the curve $\mathcal{S}_f$ is p. If we assume that the curve $\mathcal{S}_f$ is not the only invariant curve of the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$, then $\mathrm{mult}(\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}})) > p$. Using (10), we have $\mathrm{mult}(\omega_{p,q,\Delta}) = \min\{p-1, \alpha+1\}$. But $\mathrm{mult}(\omega_{p,q,\Delta}) \geq \mathrm{mult}(\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}})) - 1 > p - 1 \geq \min\{p-1, \alpha+1\} = \mathrm{mult}(\omega_{p,q,\Delta})$, which is a contradiction.

Therefore the union of separatrices of the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) = y^p - x^q$. The same reasoning happens when $p \geq q$ and we conclude that $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) = y^p - x^q$. □

Proposition 5.3. *Suppose that the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is non-dicritical. Then, the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type if and only if $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$.*

Proof. First, suppose that the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type. Given that the intersection multiplicity is invariant under change of coordinates, we will suppose, without lost of generality that (x, y) are the coordinates, where $\mathcal{N}(\omega_{p,q,\Delta}) = \mathcal{N}(\mathrm{d}f)$, after hypothesis.
From Lemma 5.2 we get $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) = y^p - x^q$. Put $d := \gcd(p, q)$. The line containing the only compact side of Newton polygone $\mathcal{N}(\mathrm{d}f)$ is $\mathcal{L} : \frac{p}{d}i + \frac{q}{d}j = \frac{pq}{d}$. Since $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type, using Theorem 1.1 we have $\mathcal{N}(\omega_{p,q,\Delta}) = \mathcal{N}(f)$. Therefore, the line $\mathcal{L}$ also contains the only compact side of the Newton polygon of $\mathcal{N}(\omega_{p,q,\Delta})$, that is any $(a, b) \in \mathrm{supp}(\omega_{p,q,\Delta})$ verifies $a\frac{p}{d} + b\frac{q}{d} \geq \frac{pq}{d}$. Suppose that $\Delta(x, y) = \sum_{i,j} a_{ij}x^i y^j \in \mathbb{C}\{x, y\}$, then

$$\omega_{p,q,\Delta} = \left(-qx^{q-1} - q\sum_{i,j} a_{ij}x^i y^{j+1}\right)\mathrm{d}x + \left(py^{p-1} + p\sum_{i,j} a_{ij}x^{i+1}y^j\right)\mathrm{d}y,$$

and $\mathrm{supp}(\omega_{p,q,\Delta}) = \{(q, 0), (i + 1, j + 1)\} \cup \{(0, p)(i + 1, j + 1)\}$, for $(i, j) \in \mathrm{supp}(\Delta)$. If $(i + 1, j + 1) \in \mathrm{supp}(\omega_{p,q,\Delta})$ then $(i + 1)\frac{p}{d} + (j + 1)\frac{q}{d} \geq \frac{pq}{d}$, so we conclude that $(\Delta, y^p - x^q)_0 = ip + jq \geq \mathrm{PH}_{(p,q)} - 1$.

Suppose now that $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$. Suppose without lost of generality that $p < q$ and $\mathrm{ord}\Delta = i_0 + j_0$, for some $a_{i_0,j_0} \neq 0$. Since $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$ we have $i_0p + j_0q \geq \mathrm{PH}_{(p,q)} - 1$. After $p < q$ we get

$$i_0q + j_0q > i_0p + j_0q \geq \mathrm{PH}_{(p,q)} - 1,$$

so $i_0 + j_0 > p - 1 - \frac{p}{q} > p - 2$ and $\alpha = \mathrm{ord}\Delta \geq p - 1$. Since $\mathrm{mult}(\mathrm{d}f) = p - 1$ for $\mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}}) : f(x, y) = y^p - x^q = 0$, using (10) we have $\mathrm{mult}(\omega_{p,q,\Delta}) = p - 1$. Hence $\mathrm{mult}(\omega_{p,q,\Delta}) = \mathrm{mult}(\mathrm{d}f)$ and we conclude that the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type. □

Proposition 5.4. *Suppose that the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is non-dicritical. The foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ has the same reduction of singularities that $\mathrm{d}(y^p - x^q)$, if and only if, $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$.*

Proof. Suppose that $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$. By Proposition 5.3 the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the second type and by Theorem 2.2 we conclude that $\mathcal{F}_{\omega_{p,q,\Delta}}$ and $\mathrm{d}(y^p - x^q)$ have the same reduction of singularities.
Suppose now that $\mathcal{F}_{\omega_{p,q,\Delta}}$ and $\mathrm{d}(y^p - x^q)$ have the same reduction of singularities. The curve $y^p - x^q = 0$ with $p > q$ and $d = \gcd(p, q)$ is desingularized by

$$E : (x, y) = (u^n v^{\frac{p}{d}}, u^m v^{\frac{q}{d}}), \tag{11}$$

such that $mp - nq = d$ and $m, n \in \mathbb{N}^*$. The transformation of

$$\omega_{p,q,\Delta} = (-qx^{q-1} - qy\Delta(x,y))\mathrm{d}x + (py^{p-1} + px\Delta(x,y))\mathrm{d}y,$$

by E defined as (11) is

$$\begin{aligned} E^*\omega_{p,q,\Delta} = & \left[u^{nq-1}v^{\frac{pq}{d}}\left(-nq + mpu^{mp-nq}\right) + dv^{\frac{pq}{d}}u^{nq-1}(u^{m+n-nq}v^{\frac{p+q-pq}{d}}E^*(\Delta(x,y)))\right]\mathrm{d}u \\ + & \left[\frac{pq}{d}u^{nq}v^{\frac{pq}{d}-1}(-1+u^{mp-nq})\right]\mathrm{d}v \\ = & \left(u^{nq-1}v^{\frac{pq}{d}-1}\right)\left[v(-qn + mpu^d + \widetilde{\Delta}(u,v))\mathrm{d}u + \frac{pq}{d}u(u^d - 1)\mathrm{d}v\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} \widetilde{\Delta}(u,v) & = dE^*(\Delta(x,y))u^{m+n-nq}v^{\frac{p+q-pq}{d}} \\ & = \sum_{i,j} da_{ij}u^{ni+mj+m+n-nq}v^{\frac{pi+qj+p+q-pq}{d}}, \end{aligned}$$

is the equation of the strict transform of Δ. So, $\mathrm{ord}_v\widetilde{\Delta} \geq 0$, so $pi + qj + p + q - pq \geq 0$ for all $(i,j) \in \mathrm{supp}(\Delta)$. Therefore $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$. □

Proof of Theorem 1.2. The equivalence of statements (a) and (b) is Proposition 5.3. The equivalence of statements (b) and (c) is Proposition 5.4. □

Corollary 5.5. *Suppose that the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is non-dicritical. If the foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is of the generalized curve type then $(\Delta, y^p - x^q)_0 \geq \mathrm{PH}_{(p,q)} - 1$.*

The fact that the foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of generalized curve type does not imply that $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$, as the next example shows:

Example 5.6. The foliation

$$\omega = \mathrm{d}(y^6 - x^3) + axy(6x\mathrm{d}y - 3y\mathrm{d}x),$$

with $a \in \{-(6r+1)\zeta : r \in \mathbb{Q}_{>0} \textit{ and } \zeta^3 = 1\} \subseteq \mathbb{C}^*$, and $a^3 \neq -1$ is of the generalized curve type, but $(\Delta, y^p - x^q)_0 = 3 = \mathrm{PH}_{(p,q)} - 1$, where $p = 6, q = 3$ and $d = \gcd(p,q) = 3$.

Proposition 5.7. *Suppose that the cuspidal foliation $\mathcal{F}_{\omega_{p,q,\Delta}} : \omega_{p,q,\Delta} = 0$ is non-dicritical and p and q are coprimes. The foliation $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of generalized curve type, if and only if $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$.*

Proof. Let us consider $\omega_{p,q,\Delta} = (-qx^{q-1} - qy\Delta)\mathrm{d}x + (py^{p-1} + px\Delta)y$, $f(x,y) = y^p - x^q$ and $\gamma(t) = (t^p, t^q)$ a parameterization of $f(x,y) = 0$. Thus

$$GSV(\mathcal{F}_{\omega_{p,q,\Delta}}, \mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}})) \quad = \quad \mathrm{ord}_t\left(\tfrac{py^{p-1}+px\Delta}{py^{p-1}}(t^p,t^q)\right) = \mathrm{ord}_t\left(1 + \tfrac{t^p\Delta(t^p,t^q)}{t^{q(p-1)}}\right), \tag{13}$$

where $\Delta(t^p,t^q) = \sum_{ij} a_{ij}t^{pi+qj}$. Note that

$$GSV(\mathcal{F}_{\omega_{p,q,\Delta}}, \mathcal{S}(\mathcal{F}_{\omega_{p,q,\Delta}})) = 0, \quad \text{if and only if } \mathrm{ord}_t\left(1 + \frac{t^p\Delta(t^p,t^q)}{t^{q(p-1)}}\right) = 0,$$

what is equivalent to $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$. From Theorem 3.6 we conclude that $\mathcal{F}_{\omega_{p,q,\Delta}}$ is of the generalized curve type, if and only if $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$. $\square$

Suppose now that p and q are not coprime. We will analyze if the strict inequality $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$ is a sufficient condition for $\mathcal{F}_{\omega_{p,q,\Delta}}$ to be a foliation is of generalized curve type. We begin studying what happens when $d = \gcd(p,q) = 2$.

Let us consider $\mathcal{S} : f = f_1 f_2$ and $g\omega = h\mathrm{d}(f_1f_2) + f_1f_2\eta$. For $\mathcal{S}_i : f_i(x,y) = 0$, we have $GSV(\mathcal{F}, \mathcal{S}_1) = \frac{1}{2\pi i}\int_{\partial\mathcal{S}_1} \frac{\mathrm{d}(\frac{h}{g})}{\frac{h}{g}} + (f_2, f_1)_0$. Analogously, $GSV(\mathcal{F}, \mathcal{S}_2) = \frac{1}{2\pi i}\int_{\partial\mathcal{S}_2} \frac{\mathrm{d}(\frac{h}{g})}{\frac{h}{g}} + (f_1, f_2)_0$. We have

$$\frac{1}{2\pi i}\int_{\partial\mathcal{S}_1\cup\partial\mathcal{S}_2} \frac{\mathrm{d}(\frac{h}{g})}{\frac{h}{g}} = GSV(\mathcal{F},\mathcal{S}_1) + GSV(\mathcal{F},\mathcal{S}_2) - 2(f_1,f_2)_0.$$

Therefore (see [**Br**, page 532]),

$$GSV(\mathcal{F},\mathcal{S}) = GSV(\mathcal{F},\mathcal{S}_1) + GSV(\mathcal{F},\mathcal{S}_2) - 2(f_1,f_2)_0. \tag{14}$$

For $d = \gcd(p,q) = 2$, we have

$$y^p - x^q = \prod_{i=1}^{2}(y^{\frac{p}{2}} - \zeta^i x^{\frac{q}{2}}), \quad \text{with} \quad \zeta^2 = 1.$$

Let $\mathcal{S}_i : f_i(x,y) = (y^{\frac{p}{2}} - \zeta^i x^{\frac{q}{2}})$ and $\gamma_i(t) = (t^{\frac{p}{2}}, A_i t^{\frac{q}{2}})$ with $A_i^{\frac{p}{2}} = \zeta^i$ a parameterization of $\mathcal{S}_i$. Then

$$(f_1, f_2)_0 = \mathrm{ord}_t(f_1(\gamma_2(t))) = \mathrm{ord}_t(t^{\frac{pq}{4}}(1-\zeta)) = \tfrac{pq}{4}. \tag{15}$$

Remember that $\omega_{p,q,\Delta} = (-qx^{q-1} - qy\Delta)\mathrm{d}x + (py^{p-1} + px\Delta)\mathrm{d}y$, thus

$$GSV(\mathcal{F}, \mathcal{S}_1) = \mathrm{ord}_t\left(t^{\frac{pq}{4}}(\zeta - 1) + \tfrac{(\zeta-1)}{A_1^{p-1}} t^{\frac{p}{2}+\frac{q}{2}-\frac{pq}{4}} \Delta(t^{\frac{p}{2}}, A_1 t^{\frac{q}{2}})\right). \tag{16}$$

If we consider $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$, from (16) we have that $GSV(\mathcal{F}, \mathcal{S}_1) = \frac{pq}{4}$. Similarly, it turns out that $GSV(\mathcal{F}, \mathcal{S}_2) = \frac{pq}{4}$.
After (15) and (14) we have $GSV(\mathcal{F}, \mathcal{S}) = 0$, which is equivalent to $\omega_{p,q,\Delta}$, so the foliaction $\mathcal{F}$ generalized curve type when $d = 2$.
In general [**Br**], when $\mathcal{S} : f = f_1 \cdots f_d$, we have to

$$GSV(\mathcal{F}, \mathcal{S}) = \sum_{i=1}^{d} GSV(\mathcal{F}, \mathcal{S}_i) - 2\sum_{\substack{i \neq j \\ i=1}}^{N} (f_i, f_j)_0, \tag{17}$$

where $N = \begin{pmatrix} d \\ 2 \end{pmatrix}$, $GSV(\mathcal{F}, \mathcal{S}_i) = \dfrac{(d-1)pq}{d^2}$, and $(f_i, f_j)_0 = \dfrac{pq}{d^2}$. Therefore, from (17) we get

$$GSV(\mathcal{F}, \mathcal{S}) = 0. \tag{18}$$

Hence the following proposition holds.

Proposition 5.8. *Let $\mathcal{F}_{p,q,\Delta}$ be a non dicritical foliation and suppose that $(\Delta, y^p - x^q)_0 > \mathrm{PH}_{(p,q)} - 1$. Then $\mathcal{F}_{p,q,\Delta}$ is of the generalized curve type.*

Proof of Theorem 1.3. It is an immediate consequence of Propositions 5.7 and 5.8. □

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