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Algebraic Heun Operators with Tetrahedral Monodromy

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Abstract

Our work adds to the picture of second order differential operators with a full set of algebraic solutions, which we will call algebraic. We see algebraic Heun operators as pull-backs of algebraic hypergeometric operators via Belyi functions. We focus on the case when the hypergeometric one has a tetrahedral monodromy group. We find arithmetic conditions for the pull-back functions to exist. For each distribution of the singular points in the ramified fibers, we identify the minimal values of the exponent differences and we explicitly construct the dessin d'enfant corresponding to the pull-back function in the minimal cases. Then by allowing some parameters to vary, we find infinite families of such graphs, hence of Heun operators with tetrahedral monodromy.

1 Introduction

Consider a **second order linear differential operator** L with coefficients rational functions over $\mathbb{C}$. We focus on the situation when the corresponding ordinary linear differential equation:

$$\frac{\mathrm{d}^2 y(z)}{\mathrm{d} z^2} + p(z)\frac{\mathrm{d} y(z)}{\mathrm{d} z} + q(z)y(z) = 0 \tag{1}$$

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has a full set of solutions in an algebraic extension of $\mathbb{C}(z)$. In this case, we call the equation and the operator algebraic.*

This topic has a long history, for details see [8]. Let S be the set of poles of the coefficients, which will be called singular points of the equation. The first requirement is for these singularities to be regular. If the equation has a finite number of regular singular points, it is called **Fuchsian**. The cases when (1) has one or two singular points are trivial. The hypergeometric equation covers the situation of 3 singular points which is more challenging but has been completely solved. The most relevant results were obtained by Schwarz [25] and Klein [12] who provided conditions for the hypergeometric equation to be algebraic.

The algebraicity of a Fuchsian operator is equivalent with the finiteness of the monodromy group. The monodromy group of a differential operator of order 2 is the image of the representation $\pi_1(\mathbb{P}^1 \setminus S) \to GL(2, \mathbb{C})$ given by the analytic continuation of the 2 solutions in a basis along the closed paths that represent the elements of $\pi_1(\mathbb{P}^1 \setminus S)$. The projective monodromy group is defined in a similar way, using the continuation of the ratios of solutions in a basis.

This work comes to complete the picture of algebraic differential operators with 4 singular points, by looking at Heun operators given by (8). We give a detailed description of such operators that are pull-backs of hypergeometric ones with tetrahedral monodromy, by following the strategy described in [19] and analysing the possible distributions of the singular points in the ramified fibers of the pull-back functions, obtaining nine essentially distinct cases. We give arithmetic conditions for existence and construct some families of such operators by allowing some parameters to vary. As a consequence of this study, we obtain the following result:

Proposition 1. *There are pull-back functions of any degree greater than* 2 *from the hypergeometric operator with tetrahedral monodromy to an algebraic Heun operator.*

The structure of the paper is at follows. The second section presents the necessary terminology and results to introduce our work. The third section contains our results. In the first subsection, we describe the minimal cases, that is, we find out the minimal possible degree for the function that realizes an algebraic Heun operator as a pull-back of a hypergeometric one. In the second subsection, we construct some infinite families, starting from the minimal ones and allowing at least one of the parameters involved to vary. We construct the associated graph by "adding" cells. We identify three types of cells that cover

*Throughout the paper, we shall use mostly interchangeably equation and operator when naming the concepts associated to them.

all the cases identified in the previous sections. The existence of these infinite families means that there are infinite families of algebraic Heun operators.

These ideas can be similarly employed for studying Heun operators with any fixed, finite monodromy group.

2 Second order differential operators with algebraic solutions

As mentioned before, in our study, we follow the approach described in [19]. Therefore, in this section, we will just briefly go over the necessary terminology related to differential equations and the Grothendieck correspondence.

Given a differential operator:

$$L = \frac{\mathrm{d}^2}{\mathrm{d}\, z^2} + p(z)\frac{\mathrm{d}}{\mathrm{d}\, z} + q(z), \tag{2}$$

its singularities have to be regular, as stated previously. This translates to $p(z)$ having poles of order at most 1 and $q(z)$ having poles of order at most 2.

The monodromy condition mentioned in the Introduction implies the rationality of the local exponents. They are the roots of a quadratic equation that can be made explicit from the coefficients of L. Moreover, in this case, if the difference of the local exponents is an integer, the singularity is apparent, i.e. can be removed after a suitable change of variable.

To make things more explicit, the algebraicity of equation (1) is characterized by the following result:

Proposition 2. *[29] Given a Fuchsian differential equation, the following statements are equivalent*

- *The equation is algebraic.*
- *The monodromy group of the equation is finite.*
- *The projective monodromy group of the equation is finite and its Wronskian is algebraic.*

In what follows we will focus on the finiteness of the projective monodromy group, which we will denote by $\mathrm{PMG}(L)$. In order to simplify our study, we shall consider some relations between operators.

Definition 1. *[6][29] Let L be a differential operator with derivation with respect to z.*

A differential operator L_0 is said to be ***projectively equivalent*** *to L if there exists θ a radical function (a product of powers of rational functions) such that $L_0 = \theta \circ L \circ \theta^{-1}$.*

If z is replaced by a non-constant rational function $f(x) \in \mathbb{C}(x)$, then L becomes L_f an operator with derivation with respect to x, called the ***proper rational pull-back*** *of L by $z = f(x)$.*

If L' is a differential operator with respect to x that is projectively equivalent with L_f, L' is called a ***rational pull-back*** *of L by $z = f(x)$.*

Preserving the notations in the previous Definition

Proposition 3 ([1], [27])**.** *The following are true:*

1. $\mathrm{PMG}(L_0) \cong \mathrm{PMG}(L)$.
2. $\mathrm{PMG}(L') \leq \mathrm{PMG}(L)$.

It is worth mentioning that any second order differential operator is projectively equivalent to one in normalized form, i.e. one without first order derivatives ([6] and [29]).

Proposition 4. *[29] Two projectively equivalent differential operators have the same differences of local exponents over their corresponding singularities.*

Notation 1. *For an operator L we denote*

$$\Delta_L = \sum_{\alpha \in \mathbb{P}^1(\mathbb{C})} (|\rho_1(\alpha) - \rho_2(\alpha)| - 1), \tag{3}$$

where $\rho_1(\alpha), \rho_2(\alpha)$ are the local exponents of L in α.

Proposition 5. *[1][16] Let L be a Fuchsian differential operator with finite monodromy and L' be a rational pull-back of L by $z = f(x) \in \mathbb{C}(x)$.*

Let

- *$\rho_1(\alpha)$ and $\rho_2(\alpha)$ be the local exponents of L in $\alpha = f(\tilde{\alpha}) \in \mathbb{P}^1(\mathbb{C})$*
- *$e_{\tilde{\alpha}}$ be the ramification index of f in $\tilde{\alpha}$*
- *$\rho_1(\tilde{\alpha})$ and $\rho_2(\tilde{\alpha})$ be the local exponents of L' in $\tilde{\alpha}$*

It follows that:

$$\rho_1(\tilde{\alpha}) - \rho_2(\tilde{\alpha}) = e_{\tilde{\alpha}}(\rho_1(\alpha) - \rho_2(\alpha)). \tag{4}$$

The degree of a pull-back functions is given by:

$$\Delta_{L'} + 2 = \deg f \cdot (\Delta_L + 2). \tag{5}$$

As mentioned in the introduction, the first non-trivial case to look at is that of hypergeometric operators. Their normal form is given by:

$$H_{\lambda,\mu,\nu} = \frac{\mathrm{d}^2}{\mathrm{d}\, z^2} + \frac{1-\lambda^2}{4z^2} + \frac{1-\mu^2}{4(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4z(z-1)} \tag{6}$$

where $\lambda, \mu, \nu \in \mathbb{C}$ are the differences of the local exponents of the points $0, 1, \infty$ and they completely determine the operator up to projective equivalence. The algebraic hypergeometric operators are given by the "basic Schwarz list" which follows from [25], [12]:

$$\begin{array}{cc} (\lambda,\mu,\nu) & \mathrm{PMG}(H_{\lambda,\mu,\nu}) \\ \hline (1/n, 1, 1/n) & C_N,\ N \in \mathbb{N}^* \\ (1/2, 1/n, 1/2) & D_N,\ N \in \mathbb{N}^* \\ (1/2, 1/3, 1/3) & A_4 \\ (1/2, 1/3, 1/4) & S_4 \\ (1/2, 1/3, 1/5) & A_5 \end{array} . \tag{7}$$

For a general algebraic generator, Klein proved the following result [12], [1], [16]:

Theorem 1 (Klein)**.** *Let L be a second order differential operator in normal form on $\mathbb{P}^1(\mathbb{C})$ with $\mathrm{PMG}(L) = G, |G| < \infty$. Then there exists a unique hypergeometric operator H belonging to (7), with $\mathrm{PMG}(H) = G$, such that L is a pull-back of H via a rational function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Moreover, if $|G| \neq D_2$, the function f is also unique, modulo Möbius transformations.*

Furthermore, if L has no apparent singularity, relation (4) implies that the function f is unramified above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [16], making it a **Belyi function**, i.e. a morphism to $\mathbb{P}^1(\mathbb{C})$ with at most 3 critical values [9]. Grothendieck proved that there is a correspondence between such functions and **dessins d'enfants** [10], which we will see as bicoloured planar multi-graphs. The two colours will be assumed to be white and black [9]. The aforementioned correspondence can be interpreted as a dictionary that "translates" the properties of the function into elements of the graph:

Belyi function with critical values $\{0, 1, \infty\}$	**Dessin d'enfant**
$f^{-1}(\{0\})$	White vertices of the graph
$f^{-1}(\{1\})$	Black vertices of the graph
$f^{-1}(\{\infty\})$	Faces of the graph
Branching order of point $P \in f^{-1}(\{0\})$	Degree of corresponding white vertex w_P
Branching order of point $Q \in f^{-1}(\{1\})$	Degree of corresponding black vertex b_Q
Branching order of point $R \in f^{-1}(\{\infty\})$	Order of the corresponding face φ_R
Degree of f	Number of edges of the graph
Sheets of $f^{-1}([0, 1])$	Edges of the graph

For more details on this correspondence and its implications, see [26], [17].

3 Heun Operators

In the following sections, we will apply the methods from [15], [16], [19] for general second order operators with four singular points. Any such operator is projectively equivalent with a Heun operator. As mentioned in the Introduction, we see the rational function that realizes the Heun operator as a pull-back of a hypergeometric operator as a Belyi function and we try to construct the associated dessin d'enfant using information about the multiplicities of the points. If the construction of the dessin is not possible, it follows that the pull-back does not exist which proves that the operator is not algebraic.

Definition 2. *[7] The **Heun equation** is the canonical second-order Fuchsian differential equation on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, with 4 regular singularities:*

$$\frac{\mathrm{d}^2 U(x)}{\mathrm{d}\, x^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-t}\right)\frac{\mathrm{d}\, U(x)}{\mathrm{d}\, x} + \frac{\alpha\beta x - q}{x(x-1)(x-t)} U(x) = 0, \quad (8)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, t \in \mathbb{C}, t \neq 0, t \neq 1$ such that $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$.

The Riemann scheme of (8) is given by:

$$\left(\begin{array}{cccc|c} 0 & 1 & t & \infty & x \\ \hline 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta & \end{array}\right). \quad (9)$$

While hypergeometric operators are completely determined up to projective equivalence by their Riemann scheme, this does not apply to Heun operators, since they also depend on the accessory parameter q. From this point onward, we will denote with $\mathcal{H}(a, b, c, d)$ a Heun operator with local exponent differences: $a, b, c, d \geq 0$. Sometimes, we will refer to these four values as the parameters of the Heun operator.

Robert Maier computed on a machine 192 solutions of the Heun equation [20], an analogue to Kummer's list for the hypergeometric equation. Also, he studied polynomial Heun-to-Hypergeometric transformations [21].

Raimundas Vidūnas, together with Galina Filipuk and Mark van Hoeij, has a series of articles studying Heun to hypergeometric transformations: [11], [27], [28].

In [27], Filipuk and Vidūnas classify pull-back transformations from hypergeometric to Heun equations. They fix some of the local exponent differences and set others as parameters and they exclude cases with Liouvillian solutions, which include algebraic solutions. Studying the possible branching patterns, they obtain 89 possibilities. For 27 of these, there is no corresponding Belyi functions, therefore there are 61 parametric hypergeometric-to-Heun transformations of maximal degree 12 given by the Belyi functions together with the corresponding dessin denfants.

In [28], Vidūnas studies Liouvillian solutions of Heun's equations that are pull-backs of the parametric hypergeometric equations with cyclic or dihedral monodromy groups. For the cyclic ones, he proves that the pull-back functions are unique up to Möbius transformations for any pair (M, N) of positive integers. The pull-back covering is a Belyi polynomial of degree D:

$$\varphi(x) = 1 - (1-x)^N \left(1 + \frac{Nx}{M}\right)^M .$$

There is a similar result regarding the dihedral case, but in this case the pull-back function is not polynomial, but it has the form

$$\varphi(x) = x^3 \Theta_2(x)^2 / \Theta_1(x)^2,$$

where Θ_1 and Θ_2 verify the following equation

$$(1+\sqrt{x})^N \left(1 - \frac{n\sqrt{x}}{M}\right)^M = \Theta_1(x) + x^{3/2}\Theta_2(x).$$

In [11], Vidūnas and van Hoeij study transformations of hyperbolic hypergeometric operators $H_{\lambda,\mu,\nu}$, as in (6), with the sum of the differences of local exponents less or equal to 1: $\lambda+\mu+\nu \leq 1$. They obtain 387 possible branching patterns which give 29 possible triplets with degree of the pull-back function at most 60.

Our study is complementary to the work of Vidūnas and his collaborators. We are interested in finding conditions for the Heun operators to be algebraic. In what follows we will find the corresponding graphs for pull-back functions from the tetrahedral hypergeometric operator to algebraic Heun operators. A similar analysis can be conducted for operators with octahedral and icosahedral monodromy. We are focused on algebraic Heun operators with no apparent singularities. Therefore, based on Fuchs' theorem [2], we remark the following:

Remark 1. *The local exponent differences of an algebraic Heun operator have to be rational non-integers:*

$$a, b, c, d \in \mathbb{Q} \setminus \mathbb{Z}$$

Any Heun operator is projectively equivalent to one in normalized form. Using Klein's Theorem, we search for Heun operators as pull-backs of the hypergeometric operator given by (6):

$$\mathcal{H}_{a,b,c,d} = f^*(H_{\lambda,\mu,\nu}). \tag{10}$$

Since Heun operators are not completely determined by their local exponent differences, equation (10) denotes the existence of one Heun operator with parameters a, b, c, d as the pull-back of the tetrahedral hypergeometric operator.

The degree of the function f given by (10) follows from (5):

$$(S - 4 + 2) = \deg f(\lambda + \mu + \nu - 3 + 2) \iff S - 2 = \deg f(\lambda + \mu + \nu - 1).$$

In our case $(\lambda, \mu, \nu) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, therefore

$$\deg f = 6(S - 2). \tag{11}$$

We will focus on the existence of the associated dessins with the given data. From Proposition 5, we obtain the possible values for the ramification indices of the pull-back function given in Table 3 (in each cell, the other possibility is 0):

$z \backslash x$	0	1	t	∞	
0	$2a$ or 0	$2b$ or 0	$2c$ or 0	$2d$ or 0	points of order 2
1	$3a$ or 0	$3b$ or 0	$3c$ or 0	$3d$ or 0	points of order 3
∞	$3a$ or 0	$3b$ or 0	$3c$ or 0	$3d$ or 0	points of order 3

On each of the columns corresponding to 0, 1, t, ∞, we have only one non-zero value placed on the row corresponding to the image of the point. On each row, the non-zero values indicate the ramification data over that point. The sum of values on each row is equal to the degree of f, because this is the total number of points in each fiber, counting multiplicities. This will help count the number of non-ramified points over each of the critical values.

In this section and the next one we will prove the following result:

Proposition 6. *There are pull-back functions of any degree greater than* 2 *from the tetrahedral hypergeometric operator to an algebraic Heun operator.*

First of all, we make the following simplification to the problem: we ignore permutations between the columns corresponding to 0, 1, t, ∞ and between the second and third line, since those particular possibilities can be obtained from the ones below by composing the pull-back function with a Möbius transformation. Therefore we obtain 9 essentially different situations which we study. In the first subsection, Minimal Cases, we take the following steps:

- We determine arithmetic conditions for the differences of local exponents.
- We find out the minimal possible degree for the function f (from equation (11) this is equivalent to determining the minimal value of S).
- We check if there exists a dessin d'enfant with the given data.
- If there is no possible graph, it follows that the Heun operator with the predetermined differences of local exponents is not algebraic.
- If we can construct the graph, it follows that there exists an algebraic Heun operator with the corresponding data. From Proposition 3, it follows that its projective monodromy group is a subgroup of the tetrahedral group.

In the second subsection, Families of graphs, we start from the minimal graph and allow at least one of the parameters to vary. We construct families of graphs by "adding" cells. We identify three types of cells that cover all the cases identified in the previous sections. In this way, we obtain infinite families of dessins d'enfants parametrized by natural numbers. Correspondingly, this means that there are infinite families of algebraic Heun operators with pull-back functions of any degree.

3.1 Minimal cases

We proceed with the analysis of the 9 cases.

1.

	0	1	t	∞	
∞	$2a$	$2b$	$2c$	$2d$	$2(S-3)$ points of order 2
0					$2(S-2)$ points of order 3
1					$2(S-2)$ points of order 3

Without loss of generality, we can assume that $a \geq b \geq c \geq d$.
All the values in the table have to be positive integers, so it follows from Remark 1 that $a, b, c, d \in \mathbb{N} + \frac{1}{2}$. In addition, $S - 3 \geq 0$.

We see this as the data for a 3-regular bipartite graph. Such a graph cannot have bridges, thus the order of a face cannot exceed the number of vertices of one color: $2a \leq 2(S-2)$ which gives $b+c+d \geq 2$. Since $b, c, d \in \mathbb{N} + \frac{1}{2}$, the minimal solution is given by

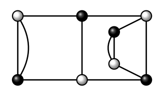


Figure 1: $a = b = \frac{3}{2}, c = d = \frac{1}{2}, \deg f = 12$

As a side note, let us observe that the Lamé operators (Heun operators with parameters: $a = b = c = \frac{1}{2}, d = n + \frac{1}{2}$, where $n \in \mathbb{N}$) would fall under this case. Since we obtained that $b + c + d \geq 2$, it follows that there are no Lamé operators which are pull-backs of the tetrahedral hypergeometric one, recovering the result in [1] and [16].

2.

	0	1	t	∞	
0	$2a$	$2b$	$2c$		$2(S-3)+d$ points of order 2
1				$3d$	$2(S-2)-d$ points of order 3
∞					$2(S-2)$ points of order 3

All the values in the table have to be positive integers, therefore $2(S-2), 2(S-2)-d \in \mathbb{N} \Rightarrow d \in \mathbb{N}$ which contradicts Remark 1. It follows that there are no algebraic Heun operators in this case.

3.

	0	1	t	∞	
0	$2a$	$2b$			$3(S-2)-(a+b)$ points of order 2
∞			$3c$	$3d$	$2(S-2)-(c+d)$ points of order 3
1					$2(S-2)$ points of order 3

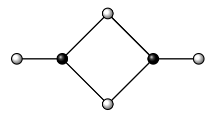
All the values in the table have to be positive integers. This, together with Remark 1, implies that $a, b \in \mathbb{N} + \frac{1}{2}, c, d \in \frac{\mathbb{N}}{3} \setminus \mathbb{N}$ and $2S \in \mathbb{N}$.
It follows that $a+b, c+d \in \mathbb{N}^* \Rightarrow S \in \mathbb{N} \setminus \{0, 1\}$. Since $2(S-2) > 0$, the minimal value for S is 3. Without loss of generality, we can assume that $a \geq b$ and $c \geq d$. There are two possibilities for the values of $a+b$ and $c+d$:

(a) $a + b = 1 \Rightarrow a = b = \frac{1}{2}$,
$c + d = 2 \Rightarrow$

i. $c = \frac{4}{3}, d = \frac{2}{3}$
or

ii. $c = \frac{5}{3}, d = \frac{1}{3}$.

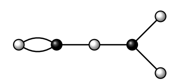
(b) $a + b = 2 \Rightarrow a = \frac{3}{2}, b = \frac{1}{2}$,
$c + d = 1 \Rightarrow c = \frac{2}{3}, d = \frac{1}{3}$.



(a) i.

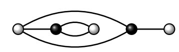
Figure 2: $a = b = \frac{1}{2}, c = \frac{4}{3}, d = \frac{2}{3}, \deg f = 6$

ii. $a = \frac{1}{2}, b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{1}{3}, \deg f = 6$

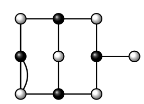


(b)

$a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{2}{3}, d = \frac{1}{3}, \deg f = 6$



$a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{5}{3}, d = \frac{1}{3}, \deg f = 12$



4.

	0	1	t	∞	
0	$2a$	$2b$			$3(S-2) - (a+b)$ points of order 2
∞			$3c$		$2(S-2) - c$ points of order 3
1				$3d$	$2(S-2) - d$ points of order 3

We can assume without loss of generality that $a \geq b$ and $c \geq d$. All the values in the table have to be positive integers. This, together with Remark 1, gives the following implications: $a, b \in \mathbb{N} + \frac{1}{2} \Rightarrow a + b \in \mathbb{N}^* \Rightarrow 3S \in \mathbb{N}^*$.

Also $3(S-2) \geq (a+b)$. Since $a + b \geq 1$ the minimal value for S is $2 + \frac{1}{3}$. In this minimal case $a = b = \frac{1}{2}$. In order to determine c and d, we observe that $c, d \in \frac{1}{3}\mathbb{N} \setminus \mathbb{N}$ and $2S - c, 2S - d \in \mathbb{N} \Rightarrow c - d \in \mathbb{N}$, therefore $c = d = \frac{2}{3}$.

$a = b = \frac{1}{2}, c = \frac{2}{3}, d = \frac{2}{3}, \deg f = 2$

5.

	0	1	t	∞	
∞	$2a$				$3(S-2) - a$ points of order 2
0		$3b$	$3c$	$3d$	$S - 4 + a$ points of order 3
1					$2(S-2)$ points of order 3

We can assume without loss of generality that $b \geq c \geq d$. All the values in the table have to be positive integers, in particular $2S \in \mathbb{N}, b, c, d \in \frac{\mathbb{N}}{3}$. It follows from Remark 1 that $a \in \mathbb{N} + \frac{1}{2}$, therefore $b + c + d \in \mathbb{N}^*$.
We obtain the minimal value for S: $S - 4 + a \geq 0, 3(S-2) - a \geq 0 \Rightarrow 4S - 10 \geq 0 \Rightarrow S \geq \frac{5}{2}$.
For this minimal case, since $S - 4 + a \geq 0$, we get the values for the parameters and the graph below:

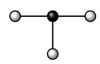


Figure 3: $a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{3}, d = \frac{1}{3}, \deg f = 3$

6.

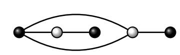
	0	1	t	∞	
∞	$2a$				$3(S-2) - a$ points of order 2
0		$3b$	$3c$		$2(S-2) - (b+c)$ points of order 3
1				$3d$	$2(S-2) - d$ points of order 3

Considering that all the values in Table 6 are positive integers and Remark 1, it follows that $a \in \mathbb{N} + \frac{1}{2}, b, c, d \in \frac{\mathbb{N}}{3} \setminus \mathbb{N}$, therefore $3S \in \mathbb{N} + \frac{1}{2}$. We get two possibilities for S: $S \in \mathbb{N} + \frac{1}{6}$ or $S \in \mathbb{N} + \frac{5}{6}$. In addition, $2(S-2) \geq b + c \geq \frac{2}{3} \Rightarrow S \geq 2 + \frac{1}{3}$.

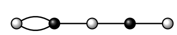
If $S \in \mathbb{N} + \frac{1}{6}$, together with the previous conditions, it follows that the minimal value for S is $\frac{17}{6}$.
If $S = \frac{17}{6}$, then $6S \equiv 5 \pmod 6$. Since $3(b+c) \equiv 3d \pmod 3 \Rightarrow d \in \mathbb{N} + \frac{2}{3}, b, c \in \mathbb{N} + \frac{1}{3}$.

We identify three possibilities:

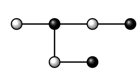
(a) $a = \frac{3}{2}, b = \frac{1}{3}, c = \frac{1}{3}, d = \frac{2}{3}, \deg f = 5.$



(b) $a = \frac{1}{2}, b = \frac{4}{3}, c = \frac{1}{3}, d = \frac{2}{3}, \deg f = 5$

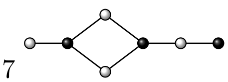


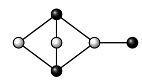
(c) $a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{3}, d = \frac{5}{3}, \deg f = 5$



If $S \in \mathbb{N} + \frac{5}{6}$, it follows that the minimal value for S is $\frac{19}{6}$. There are two additional cases:

(d)

$a = \frac{1}{2}, b = \frac{2}{3}, c = \frac{5}{3}, d = \frac{1}{3}, \deg f = 7$ 

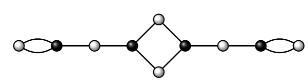
(e) $a = \frac{3}{2}, b = c = \frac{2}{3}, d = \frac{1}{3}, \deg f = 7$. 

7.

	0	1	t	∞	
0					$3(S-2)$ points of order 2
∞	$3a$	$3b$	$3c$	$3d$	$S-4$ points of order 3
1					$2(S-2)$ points of order 3

It is obvious that $S \geq 4$. Since all the values in the table, it follows from Remark 1 that $a, b, c, d \in \frac{\mathbb{N}}{3} \setminus \mathbb{N}$. We can assume without loss of generality that $a \geq b \geq c \geq d$. There are several possibilities.

(a) $a = \frac{8}{3}, b = \frac{2}{3}, c = d = \frac{1}{3}$.
(b) $a = \frac{7}{3}, b = c = \frac{2}{3}, d = \frac{1}{3}$.
(c) $a = b = \frac{5}{3}, c = d = \frac{1}{3}$.
(d) $a = \frac{5}{3}, b = \frac{4}{3}, c = \frac{2}{3}, d = \frac{1}{3}$.
(e) $a = b = \frac{4}{3}, c = d = \frac{2}{3}$.

(a) $a = \frac{8}{3}, b = \frac{2}{3}, c = d = \frac{1}{3}, \deg f = 12$ 

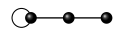
(b)

$$a = \frac{7}{3}, b = c = \frac{2}{3}, d = \frac{1}{3}, S = 4, \deg f = 12.$$

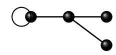
	0	1	t	∞	
0					6 points of order 2
∞	7	2	2	1	0 points of order 3
1					4 points of order 3

Since each white vertex has order 2, we can regard this as the data of a "clean" dessin with 4 vertices of order 3 with faces of orders $7, 2, 2, 1$. We try to construct the graph. Since there is a face of order 1, it follows that the graphs contains a loop:

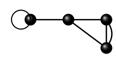
The graph is connected, therefore the vertex with the loop has to be connected with one of the other vertices, which in its turn has to be connected with at least another vertex:



The remaining fourth vertex cannot have a loop therefore it has to be connected to both the vertices without loops.

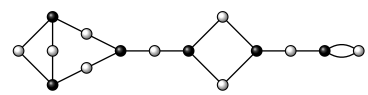


Adding the remaining edges, we get a face of order 3 which did not appear in our data, therefore this case does not yield an algebraic Heun operator.

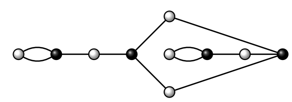


Since $S \in \mathbb{N}$, the next possibility is for $S = 5$. Allowing a to increase, we get the following parameters and graph:

$a = \frac{10}{3}, b = \frac{2}{3}, c = \frac{2}{3}, d = \frac{1}{3}, \deg f = 18$



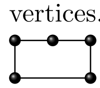
(c) $a = \frac{5}{3}, b = \frac{5}{3}, c = \frac{1}{3}, d = \frac{1}{3}, \deg f = 12$

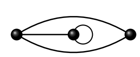


$a = \frac{8}{3}, b = \frac{5}{3}, c = \frac{1}{3}, d = \frac{1}{3}, S = 5, \deg f = 18$

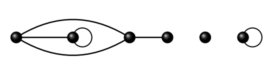
	0	1	t	∞	
0					9 points of order 2
∞	8	5	1	1	1 point of order 3
1					6 points of order 3

In this case there is no corresponding graph. Indeed, let us consider the associated clean dessin: a 3-regular pseudo-hypergraph with 6 vertices. There are two possibilities for a face of order 5:

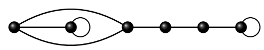




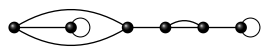
In the first case, we have already drawn 5 vertices, each with two adjacent edges, therefore we cannot add two loops/faces of order 1. In the second case, we try to place the other vertices. The vertex that currently has order 2 has to be connected with one of the unconnected vertices. We also have to have another loop:

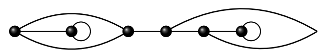


Since there are no more loops, the still unconnected vertex has to be adjacent to the vertices that do not already have degree 3:



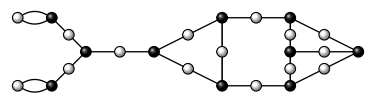
Adding the remaining edge, we obtain a face of order 2 or 5.





This contradicts the data, therefore this case does not yield any algebraic operators. However, we can construct a new starting point:

$a = \frac{14}{3}, b = \frac{5}{3}, c = \frac{1}{3}, d = \frac{1}{3}$,
$\deg f = 30$



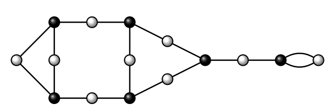
(d)

$$a = \frac{5}{3}, b = \frac{4}{3}, c = \frac{2}{3}, d = \frac{1}{3}, S = 4, \deg = 12.$$

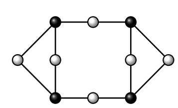
	0	1	t	∞	
0					6 order 2
∞	5	4	2	1	0 order 3
1					4 order 3

Reasoning the same way as we did at point (b), we conclude that there is no graph with this ramification data. Since $S \in \mathbb{N}$, the next possibility is for $S = 5$. Allowing a to increase, we get the following parameters:

$a = \frac{8}{3}, b = \frac{4}{3}, c = \frac{2}{3}, d = \frac{1}{3}, \deg f = 18$



(e) $a = b = \frac{4}{3}, c = d = \frac{2}{3}, \deg f = 12$



8.

	0	1	t	∞	
0					$3(S-2)$ order 2
1	$3a$	$3b$	$3c$		$S-4+d$ order 3
∞				$3d$	$2(S-2)-d$ order 3

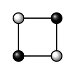
It is obvious that $3S \in \mathbb{N}$ and $2S - d,\ S + d \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. Summing the last two, it follows that $3S \geq 8$, therefore $S \geq \frac{8}{3}$. For this minimal value, it follows that $d = \frac{4}{3}$. Without loss of generality, we can assume that $a \geq b \geq c$. Therefore, we get the following data:

$a = \frac{2}{3}, b = c = \frac{1}{3}, d = \frac{4}{3}, \deg f = 4$

9.

	0	1	t	∞	
0					$3(S-2)$ order 2
1	$3a$	$3b$			$2(S-2)-(a+b)$ order 3
∞			$3c$	$3d$	$2(S-2)-(c+d)$ order 3

It is obvious that $S > 2$ which gives $a + b > 1$ or $c + d > 1$. We assume, without loss of generality, that $a + b > 1$. Since $a, b, c, d \in \frac{\mathbb{N}}{3}$, therefore $a+b \geq \frac{4}{3}$ or $c+d \geq \frac{4}{3}$. Since $2(S-2) \geq a+b$, $2(S-2) \geq c+d$, it follows $S - 2 \geq \frac{2}{3}$. Therefore, the minimal value for S is $\frac{8}{3}$. It follows that



$$a = b = c = d = \frac{2}{3}, \deg f = 4$$

3.2 Families of graphs

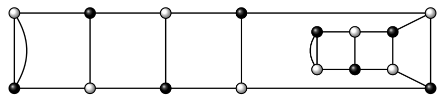
In this subsection, we start from the minimal cases previously outlined and allow some parameters to vary. In order to build the corresponding graphs, we "add" cells to the initial graphs. We identify three types of cells and these situations cover all 9 previous cases.

- Going back to the first case: We want to vary some of the parameters. Allowing $a = \frac{3}{2} + m, b = \frac{3}{2} + n$, $(m, n \in \mathbb{N})$, we get the following data:

$$a = \frac{3}{2} + m, b = \frac{3}{2} + n, c = d = \frac{1}{2}, S = 4 + n + m, \deg f = 12 + 6n + 6m$$

	0	1	t	∞	
∞	$3 + 2m$	$3 + 2n$	1	1	$2 + 2m + 2n$ points of order 2
0					$4 + 2n + 2m$ points of order 3
1					$4 + 2n + 2m$ points of order 3

The graph for $m = 0, n = 0$ is the one in Figure 1. We represent a graph corresponding to the case $m = n = 1$ below.



The graphs corresponding to the different values for m and n are obtained by inserting m respectively n cells of type:

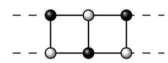


Figure 4: First type of added cell

- In the third case, again, we would like to allow at least one of the parameters to vary, namely c. We proceed with the first subcase:

	0	1	t	∞	
0	1	1			$3n+2$ points of order 2
∞			$4+3n$	2	n points of order 3
1					$2n+2$ points of order 3

We were not able to proceed as in the first case, instead we found ourselves distinguishing between the case when n even and the case when n odd.

1. Taking $n = 2m, m \in \mathbb{N}$, we get: $S = 2m+3, \deg f = 12m+6$. For $m = 0$, the corresponding graph is the one in Figure 2. For $m \in \mathbb{N}^*$, m cells given in Figure 5 will be added:

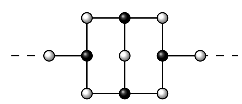
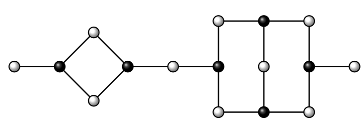
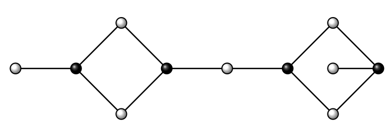


Figure 5: Second type of added cell

We give the graph for $m = 1$:



2. Taking $n = 2m+1, m \in \mathbb{N}$, we get: $S = 2m+3, \deg f = 12m+6$. The graph for $m = 0$ is given below:



The graphs for different values of m are obtained again by adding m cells given in Figure 5.

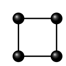
This strategy works for all the remaining cases with the exception of case 5 and 6 e. However, a few problems appear in cases 7e and 9.

Let us look at case 7e:

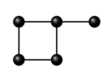
If we want to allow the parameter a to vary we get the following data:

$$a = \frac{4}{3} + n, b = \frac{4}{3}, c = d = \frac{2}{3} \Rightarrow S = 4 + n, \deg f = 12 + 6n.$$

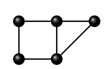
If we allow n to vary and take $n = 1$, we regard this as the data for a clean dessin. It follows that there are 6 vertices of degree 3, and 5 faces of degrees: 7, 4, 3, 2, 2.
Without loss of generality we can assume that the exterior face is the one of order 7. Since the graph has no faces of order 1, it follows that the face of order 4 looks like this:



Now, since the graph is 3-regular and connected, at least one of the vertices has to be connected to a new one:

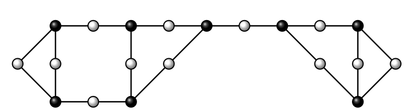


Now, since the graph has a face of order 3, the newly introduces vertex has to be connected to another vertex of the face of order 4:



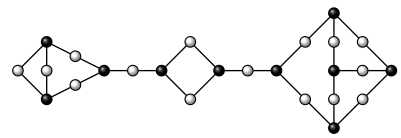
Now, the remaining unconnected vertex has to be connected to the three vertices that have only order 2, and we cannot obtain two faces of order 2, therefore there is no graph in this case.
For $n = 2$ we cannot add cells to the minimal example, therefore we build a new minimal example:

$a = \frac{10}{3}, b = \frac{4}{3}, c = d = \frac{2}{3}, \deg f = 24$



Now we can proceed as in the previous cases, and let $a = \frac{10}{3} + n$, $n \in \mathbb{N}$. For $n = 2m, m \in \mathbb{N}$ we add m cells given in Figure 5 to the figure above. For $n = 2m+1$, $m \in \mathbb{N}$, the minimal example $m = 0$ is given by:

$a = \frac{13}{3}, b = \frac{4}{3}, c = d = \frac{2}{3}, \deg f = 30$



- The fifth case and subcase e of case 6 are studied similarly to the previous cases but the added cell is different.

  We let a vary by an integer $n \in \mathbb{N}$, $a = \frac{3}{2} + n$:

	0	1	t	∞	
∞	$3+2n$				$2n$ order 2
0		1	1	1	$2n$ order 3
1					$2n+1$ order 3

Again, we distinguish between n even and n odd.
If $n = 2m, m \in \mathbb{N}$, we start from the minimal case in Figure 3, adding the following cell:

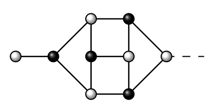
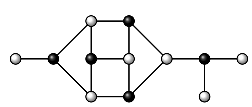
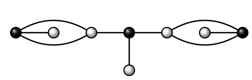


Figure 6: Third type of added cell

We present the graph for $m = 1$:



For $n = 2m+1$, we present the minimal example, i.e. for $n = 1, m = 0$. For different values of m, we add m cells given in Figure 6:



This analysis covers all the cases identified previously. In the minimal cases we found functions of degree 2, 3, 4, 5, 6 and 7. Since the added cells have 12 edges and start from two minimal cases (even and odd), it follows that there exist functions of any degree and therefore Proposition 9 is proven.

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