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COLLECTIVELY FIXED POINT THEORY IN THE COMPACT AND COERCIVE CASES

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Abstract

We present collectively fixed point results for multivalued maps which automatically generate analytic alternatives and minimax inequalities. As an application we consider equilbrium type problems for generalized games.

1. Introduction.

In this paper we begin in Section 2 by presenting a variety of new collectively fixed point results for multivalued maps in both the compact and coercive case; we refer the reader to [1, 5, 6, 9] for some results in the literature. Our goal is to obtain results which are natural when one is considering equilbrium type problems for generalized games. Along the way we will also consider new analytic alternatives and minimax inequalities.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with

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compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [8]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to Kakutani (and we write $\phi \in Kak(X, Y)$); here $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following class of maps will play a major role in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in DKT(Z, W)$ [6, 9] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : \, w \in S(z)\}$ is open (in Z) for each $w \in W$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathcal{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:
(i). $\mathcal{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;
(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii). $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathfrak{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps F, $G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X,K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X,K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous functions $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

X is said to be q– Schauder admissible if any nonempty compact convex subset Ω of X is Schauder admissible.

Theorem 1.1. *[2, 10] Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X,X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

Remark 1.2. Other variations of Theorem 1.1 can be found in [11].

2. Fixed point results

In this section we present a variety of collectively fixed point results in both the compact and coercive case. These fixed point results will general analyic alternatices and minimax inequalities so automatically they generate equilibrium type results in generalized games. We begin with a result which illustrates our approach. One of the conditions in our first theorem can be a little restrictive (from the generalized game point of view) but this condition will be removed in some later results in this paper.

Theorem 2.1. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and $F_i \in DKT(X, X_i)$. In addition assume for each $i \in$*

$\{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$ (here x_i is the projection of x on X_i).

Proof: For $i \in \{1, ..., N\}$ let $S_i : X \to X_i$ with $S_i(x) \neq \emptyset$ for $x \in X$, $co\,(S_i(x)) \subseteq F_i(x)$ for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Let $K = \prod_{i=1}^N K_i$ and note K is compact. Let $F_i^\star$ denote the restriction of F_i to K. We claim $F_i^\star \in DKT(K, X_i)$ for each $i \in \{1, ..., N\}$. To see this let $S_i^\star$ denote the restriction of S_i to K. Note trivially $S_i^\star(x) \neq \emptyset$ and $co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$ (since $S_i(x) \neq \emptyset$ and $co\,(S_i(x)) \subseteq F_i(x)$ for $x \in X$). Also note if $y \in X_i$ then

$$\begin{aligned}(S_i^\star)^{-1}(y) &= \{z \in K : y \in S_i^\star(z)\} = \{z \in K : y \in S_i(z)\} \\ &= K \cap \{z \in X : y \in S_i(z)\} = K \cap S_i^{-1}(y)\end{aligned}$$

which is open in $K \cap X = K$. Thus for each $i \in \{1, ..., N\}$ we have $F_i^\star \in DKT(K, X_i)$ so since $F_i(X) \subseteq K_i$ we have $F_i^\star \in DKT(K, K_i)$; note for $y \in K_i$ that $(S_i^\star)^{-1}(y) = K \cap S_i^{-1}(y)$ which is open in K. Now for each $i \in \{1, ..., N\}$ from [6] there exists a continuous (single valued) selection $f_i : K \to K_i$ of $F_i^\star$ with $f_i(x) \in co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of K_i with $f_i(K) \subseteq co\,(C_i) \equiv D_i$; note $co\,(C_i) \subseteq co\,(K_i) = K_i$ i.e. $D_i \subseteq K_i$. Let

$$D = \prod_{i=1}^N D_i \quad \text{and} \quad f(x) = \prod_{i=1}^N f_i(x),\ x \in K.$$

Now $f : K \to K$ is continuous with $f(K) \subseteq D$. Since $D = \prod_{i=1}^N D_i \subseteq \prod_{i=1}^N K_i = K$ we have $f : D \to D$ and $f(D)$ lies in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$. Note $D_i = co\,(C_i) \subseteq K_i$ is compact and D is compact and convex. Brouwer's fixed point theorem guarantees that there exists a $x \in D\,(\subseteq K)$ with $x = f(x)$. Thus $x_j = f_j(x) \in co\,(S_j^\star(x)) \subseteq F_j^\star(x)$ for each $j \in \{1, ..., N\}$ i.e. $x_j \in F_j^\star(x)$ for each $j \in \{1, ..., N\}$. $\square$

We now consider Theorem 2.1 in a more general setting.

Theorem 2.2. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and $F_i \in DKT(X, X_i)$. In addition assume for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i \in I} E_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Proof: For $i \in I$ let S_i be as in Theorem 2.1, $K = \prod_{i \in I} K_i$ and $F_i^\star$ the restriction of F_i to K. The same reasoning as in Theorem 2.1 guarantees that $F_i^\star \in DKT(K,K_i)$ for $i \in I$. Now for each $i \in I$ from [6] there exists a continuous (single valued) selection $f_i : K \to K_i$ of $F_i^\star$ with $f_i(x) \in co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of K_i with $f_i(K) \subseteq co\,(C_i) \equiv D_i$; note $co\,(C_i) \subseteq co\,(K_i) = K_i$ i.e. $D_i \subseteq K_i$. Let

$$D = \prod_{i \in I} D_i \text{ and } f(x) = \prod_{i \in I} f_i(x),\, x \in K.$$

Now $f : K \to K$ is continuous with $f(K) \subseteq D$. Since $D = \prod_{i \in I} D_i \subseteq \prod_{i \in I} K_i = K$ we have $f : D \to D$ with D Schauder admissible (since X is q–Schauder admissible). Theorem 1.1 guarantees a $x \in D\,(\subseteq K)$ with $x = f(x)$ and as in Theorem 2.1 we immediately have $x_j \in F_j^\star(x)$ for each $j \in I$. $\square$

Remark 2.3. (i). Note in the statement of Theorem 2.1 and Theorem 2.2 we could replace $F_i \in DKT(X,X_i)$ with $F_i \in DKT(X,K_i)$. To see this let $S_i : X \to K_i$ with $S_i(x) \neq \emptyset$ for $x \in X$, $co\,(S_i(x)) \subseteq F_i(x)$ for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in K_i$. Let $F_i^\star$ (respectively, $S_i^\star$) denote the restriction of F_i (respectively, S_i) to K. Now $F_i^\star \in DKT(K,K_i)$; note for $y \in K_i$ that $(S_i^\star)^{-1}(y) = K \cap S_i^{-1}(y)$ which is open in K. Apply now the result in [6] and follow the proof in Theorem 2.1 and Theorem 2.2.
(ii). In Theorem 2.2 we could replace "for each $i \in I$ suppose there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$" with "for each $i \in I$ suppose there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$" provided X is a q–Schauder admissible subset of E is replaced by X is a p–Schauder admissible subset of E (X is a p–Schauder admissible subset of E if for any nonempty compact subset Ω_0 of X the set $co\,(\Omega_0)$ is Schauder admissible). To see this let $K = \prod_{i \in I} K_i$ and note [6] that $co\,(K)$ is paracompact. Let $F_i^\star$ (respectively, $S_i^\star$) denote the restriction of F_i (respectively, S_i) to $\Omega \equiv co\,(K)$. We claim $F_i^\star \in DKT(\Omega, X_i)$ for $i \in I$ since if $y \in X_i$ then

$$\begin{aligned}(S_i^\star)^{-1}(y) &= \{z \in \Omega : y \in S_i^\star(z)\} = \{z \in \Omega : y \in S_i(z)\} \\ &= \Omega \cap \{z \in X : y \in S_i(z)\} = \Omega \cap S_i^{-1}(y)\end{aligned}$$

which is open in $\Omega \cap X = \Omega$. Now for each $i \in I$ from [6] (recall Ω is paracompact) there exists a continuous (single valued) selection $f_i : \Omega \to X_i$ of $F_i^\star$ with $f_i(\Omega) \subseteq F_i^\star(\Omega) \subseteq F_i(X) \subseteq K_i$ so $f_i : \Omega \to K_i$. Let $f(x) = \prod_{i \in I} f_i(x)$ for $x \in \Omega$ and note $f : \Omega \to \Omega$ is continuous (note for each $i \in I$ we have $f_i(\Omega) \subseteq K_i$ so $f(\Omega) \subseteq K \subseteq co\,(K) = \Omega$). Now since Ω is a Schauder admissible subset of E then Theorem 1.1 guarantees a $x \in \Omega$ with $x = f(x)$, so for each $i \in I$ we have $x_i = f_i(x) \in F_i^\star(x)$.

We can apply this idea to many other classes of maps. We will supply one more result to the reader.

Theorem 2.4. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Also assume for $i \in \{1, ..., N\}$ that $F_i \in Ad(X, X_i)$. In addition assume X is a Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i=1}^N E_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$.*

Proof: Let

$$K = \prod_{i=1}^N K_i \quad \text{and} \quad F(x) = \prod_{i=1}^N F_i(x),\ x \in K.$$

Since a finite product of admissible maps of Gorniewicz is an admissible map of Gorniewicz [8] then $F \in Ad(X, X)$ with $F(X) \subseteq K$. Now Theorem 1.1 guarantees a $x \in K$ with $x \in F(x)$. $\square$

In our next two results we will replace the compactness condition on F_i with a coercive type condition [4, 5]. We will now also consider a subclass of the $DKT(Z, W)$ maps (see [4, 6]). Let G be a multifunction and we say $G \in \Phi^\star(Z, W)$ [4] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Theorem 2.5. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \neq \emptyset$ and has convex values for $x \in X$, $S_i(x) \subseteq F_i(x)$ for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact subset K of X and for each $i \in \{1, .., N\}$ a convex compact subset Y_i of X_i with $S_i(x) \cap Y_i \neq \emptyset$ for $x \in X \backslash K$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$.*

Proof: With K given in the statement of Theorem 2.5 let $F_i^\star$ (respectively, $S_i^\star$) denote the restriction of F_i (respectively, S_i) to K. The same reasoning as in Theorem 2.1 guarantees that $F_i^\star \in \Phi^\star(K, X_i)$ for $i \in \{1, ..., N\}$; note for $y \in X_i$ that $(S_i^\star)^{-1}(y) = K \cap S_i^{-1}(y)$ which is open in K. Now for each $i \in \{1, ..., N\}$ from [4, 6] there exists a continuous (single valued) selection $f_i : K \to X_i$ of $F_i^\star$ with $f_i(x) \in S_i^\star(x) \subseteq F_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of X_i with $f_i(K) \subseteq co\,(C_i)$. Let

$$\Omega_i = co\ (co\,(C_i) \cup Y_i) \quad \text{for} \quad i \in \{1, ..., N\}$$

which is a convex compact [3, pp.125] subset of X_i. Let $F_i^{\star\star}(x) = F_i(x) \cap \Omega_i$ for $x \in X$ and $i \in \{1, ..., N\}$. We claim $F_i^{\star\star} \in \Phi^\star(X, \Omega_i)$ for $i \in \{1, ..., N\}$. Let $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i$ for $x \in X$ and $i \in \{1, ..., N\}$. If $x \in X \backslash K$ then $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i \neq \emptyset$ since $S_i(x) \cap Y_i \neq \emptyset$ and $Y_i \subseteq \Omega_i$ whereas if $x \in K$ then since

$f_i(K) \subseteq S_i(K)$ and $f_i(K) \subseteq co\,(C_i) \subseteq \Omega_i$ we have $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i \neq \emptyset$. Next if $x \in K$ then $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i \subseteq F_i(x) \cap \Omega_i = F_i^{\star\star}(x)$. Also note if $y \in \Omega_i$ then

$$\begin{aligned} (S_i^{\star\star})^{-1}(y) &= \{z \in X : y \in S_i^{\star\star}(z)\} = \{z \in X : y \in S_i(z) \cap \Omega_i\} \\ &= \{z \in X : y \in S_i(z)\} = S_i^{-1}(y) \end{aligned}$$

which is open in X. Thus $F_i^{\star\star} \in \Phi^\star(X, \Omega_i)$ for $i \in \{1, ..., N\}$. Now apply Theorem 2.1 (with K_i replaced by Ω_i and F_i replaced by $F_i^{\star\star}$) and Remark 2.3 and we see that there exists a $x \in X$ with $x_i \in F_i^{\star\star}(x) = F_i(x) \cap \Omega_i$ for $i \in \{1, ..., N\}$. $\square$

Theorem 2.6. *Let I be an index set and $\{X_i\}_{\in I}$ a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \neq \emptyset$ and has convex values for $x \in X$, $S_i(x) \subseteq F_i(x)$ for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i with $S_i(x) \cap Y_i \neq \emptyset$ for $x \in X \backslash K$. In addition suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i \in I} E_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Proof: For $i \in I$ let $F_i^\star$ and $S^\star$ be as in Theorem 2.5 and the argument in Theorem 2.5 guarantees that $F_i^\star \in \Phi^\star(K, X_i)$ for $i \in I$. Also for $i \in I$ let f_i, C_i, Ω_i and $F_i^{\star\star}$ be as in Theorem 2.5 and the argument in Theorem 2.5 guarantees that $F_i^{\star\star} \in \Phi^\star(X, \Omega_i)$. Now apply Theorem 2.2 (with K_i replaced by Ω_i and F_i replaced by $F_i^{\star\star}$) and Remark 2.3 and we see that there exists a $x \in X$ with $x_i \in F_i^{\star\star}(x) = F_i(x) \cap \Omega_i$ for $i \in I$. $\square$

One of the conditions in say Theorem 2.1 and Theorem 2.5 is that we assume for each $x \in X$ that $S_i(x) \neq \emptyset$ for $i \in \{1, ..., N\}$. We will relax this condition in our next results.

Theorem 2.7. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, ..., N\}$ suppose $F_i : X \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume for each $i \in \{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Finally suppose for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $S_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in \{1, .., N\}$ with $x_i \in F_i(x)$ (in fact we will show $x \in K = \prod_{i=1}^N K_i$).*

Proof: Note $A_i = \{x \in X : S_i(x) \neq \emptyset\}, i \in \{1, .., N\}$ is an open covering of X (recall the fibres of S_i are open). Now from [7, Lemma 5.1.6, pp301] there exists a covering $\{B_i\}_{i=1}^N$ of X where B_i is closed and $B_i \subset A_i$ for all $i \in \{1, ..., N\}$. For each $i \in \{1, .., N\}$ let $G_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$G_i(x) = \begin{cases} F_i(x), \; x \in B_i \\ X_i, \; x \in X \backslash B_i \end{cases}$$

and

$$T_i(x) = \begin{cases} S_i(x), \; x \in B_i \\ X_i, \; x \in X \backslash B_i. \end{cases}$$

We claim for $i \in \{1, ..., N\}$ that $G_i \in \Phi^\star(X, X_i)$. Note first for $i \in \{1, ..., N\}$ that $T_i(x) \neq \emptyset$ for $x \in X$ since if $x \in B_i$ then $T_i(x) = S_i(x) \neq \emptyset$ since $B_i \subset A_i$ whereas if $x \in X \backslash B_i$ then $T_i(x) = X_i$. Also for $x \in X$ and $i \in \{1, ..., N\}$ then if $x \in B_i$ we have $T_i(x) = S_i(x) \subseteq F_i(x) = G_i(x)$ whereas if $x \in X \backslash B_i$ we have $T_i(x) = X_i = G_i(x)$. Also note if $y \in X_i$ then

$$\begin{aligned} T_i^{-1}(y) &= \{z \in X : y \in T_i(z)\} \\ &= \{z \in X \backslash B_i : y \in T_i(z) = X_i\} \cup \{z \in B_i : y \in T_i(z)\} \\ &= (X \backslash B_i) \cup \{z \in B_i : y \in S_i(z)\} \\ &= (X \backslash B_i) \cup [B_i \cap \{z \in X : y \in S_i(z)\}] \\ &= (X \backslash B_i) \cup \left[B_i \cap S_i^{-1}(y)\right] \\ &= X \cap \left[(X \backslash B_i) \cup S_i^{-1}(y)\right] = (X \backslash B_i) \cup S_i^{-1}(y) \end{aligned}$$

which is open in X (note $S_i^{-1}(y)$ is open in X and B_i is closed in X). Thus for $i \in \{1, ..., N\}$ we have $G_i \in \Phi^\star(X, X_i)$.

Let $K = \prod_{i=1}^N K_i$ (note K is compact) and let $G_i^\star$ denote the restriction of G_i to K. We claim for $i \in \{1, ..., N\}$ that $G_i^\star \in \Phi^\star(K, X_i)$. To see this let $T_i^\star$ denote the restriction of T_i to K. Note $T_i^\star(x) \neq \emptyset$ for $x \in K$ (since $T_i(x) \neq \emptyset$ for $x \in X$) and $T_i^\star(x) \subseteq G_i^\star(x)$ for $x \in K$ (since $T_i(x) \subseteq G_i(x)$ for $x \in X$). Also if $y \in X_i$ then

$$\begin{aligned} (T_i^\star)^{-1}(y) &= \{z \in K : y \in T_i^\star(z)\} = \{z \in K : y \in T_i(z)\} \\ &= K \cap \{z \in X : y \in T_i(z)\} = K \cap T_i^{-1}(y) \end{aligned}$$

which is open in $K \cap X = K$. Thus for $i \in \{1, ..., N\}$ we have $G_i^\star \in \Phi^\star(K, X_i)$ with K compact. Now for $i \in \{1, ..., N\}$ let $G_i^{\star\star}$ be given by $G_i^{\star\star}(x) = G_i^\star(x) \cap K_i$ for $x \in K$. We claim for $i \in \{1, ..., N\}$ that $G_i^{\star\star} \in \Phi^\star(K, K_i)$. To see this let $T_i^{\star\star}$ be given by $T_i^{\star\star}(x) = T_i^\star(x) \cap K_i$ for $x \in K$. Note first for $i \in \{1, ..., N\}$ that $T_i^{\star\star}(x) \neq \emptyset$ for $x \in K$ since if $x \in B_i \cap K$ then $T_i^{\star\star}(x) = S_i(x) \cap K_i \neq \emptyset$ since $B_i \subset A_i$ and $S_i(x) \subseteq F_i(x) \subseteq K_i$ whereas if $x \in K \backslash B_i$ then $T_i^{\star\star}(x) =$

$X_i \cap K_i \neq \emptyset$. Next note if $x \in K$ then $T_i^{\star\star}(x) = T_i^\star(x) \cap K_i \subseteq G_i^\star(x) \cap K_i = G_i^{\star\star}(x)$. Also note if $y \in K_i$ then

$$\begin{aligned}(T_i^{\star\star})^{-1}(y) &= \{z \in K : y \in T_i^{\star\star}(z)\} = \{z \in K : y \in T_i^\star(z) \cap K_i\} \\ &= K \cap \{z \in X : y \in T_i(z) \cap K_i\} \\ &= K \cap \{z \in X : y \in T_i(z)\} = K \cap T_i^{-1}(y)\end{aligned}$$

which is open in $K \cap X = K$. Thus for $i \in \{1, ..., N\}$ we have $G_i^{\star\star} \in \Phi^\star(K, K_i)$ with K compact. Now for each $i \in \{1, ..., N\}$ from [4] there exists a continuous (single valued) selection $f_i : K \to K_i$ of $G_i^{\star\star}$ with $f_i(x) \in T_i^{\star\star}(x) \subseteq G_i^{\star\star}(x)$ for $x \in K$ and also there exists a finite set C_i of K_i with $f_i(K) \subseteq co\,(C_i) \equiv D_i$; note $co\,(C_i) \subseteq co\,(K_i) = K_i$ i.e. $D_i \subseteq K_i$. Let

$$D = \prod_{i=1}^N D_i \quad \text{and} \quad f(x) = \prod_{i=1}^N f_i(x),\ x \in K.$$

Now $f : K \to K$ is continuous with $f(K) \subseteq D$. Since $D = \prod_{i=1}^N D_i \subseteq \prod_{i=1}^N K_i = K$ we have $f : D \to D$ and $f(D)$ lies in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$. Note $D_i \subseteq co\,(C_i) \subseteq K_i$ is compact and D is compact and convex. Brouwer's fixed point theorem guarantees that there exists a $x \in D\,(\subseteq K)$ with $x = f(x)$ i.e. $x_j = f_j(x) \in T_j^{\star\star}(x) \subseteq G_j^{\star\star}(x)$ for each $j \in \{1, .., N\}$. Thus $x_j \in G_j^\star(x) \cap K_j = G_j(x) \cap K_j$ for each $j \in \{1, .., N\}$ i.e. $x_j \in G_j(x)$ for each $j \in \{1, ..., N\}$. Since $\{B_i\}_{i=1}^N$ is a covering of X there exists a $j_0 \in \{1, .., N\}$ with $x \in B_{j_0}$ so $x_{j_0} \in G_{j_0}(x) = F_{j_0}(x)$. $\square$

Remark 2.8. In Theorem 2.7 we showed there exists a $x \in K$ and a $j_0 \in \{1, .., N\}$ with $x_{j_0} \in F_{j_0}(x)$ and from our proof note $x \in B_{j_0} \subset A_{j_0}$. Also we showed $x_j \in G_j(x)$ for each $j \in \{1, ..., N\}$ where

$$G_j(x) = \begin{cases} F_j(x),\ x \in B_j \\ X_j,\ x \in X \backslash B_j. \end{cases}$$

Remark 2.9. In the statement of Theorem 2.7 we could replace "there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$" with "there exists a map $S_i : X \to K_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in K_i$". Here we define $G_i : X \to K_i$ and $T_i : X \to K_i$ by

$$G_i(x) = \begin{cases} F_i(x),\ x \in B_i \\ K_i,\ x \in X \backslash B_i \end{cases}$$

and

$$T_i(x) = \begin{cases} S_i(x),\ x \in B_i \\ K_i,\ x \in X \backslash B_i. \end{cases}$$

The argument in Theorem 2.7 guarantees for each $i \in \{1,..,N\}$ that $G_i \in \Phi^\star(X, K_i)$ and if $G_i^\star$ is the restriction of G_i to K then $G_i^\star \in \Phi^\star(K, K_i)$; note if $T_i^\star$ is the restriction of T_i to K and if $y \in K_i$ then $(T_i^\star)^{-1}(y) = K \cap T_i^{-1}(y)$ which is open in K. Next we can immediately apply the result in [4] to guarantee a continuous selection $f_i : K \to K_i$ of $G_i^\star$ and follow the reasoning in Theorem 2.7 (note the introduction of $G_i^{\star\star}$ in Theorem 2.7 is not needed here). Thus there exist a $x \in D\,(\subseteq K)$ with $x_j \in G_j(x)$ for each $j \in \{1,...,N\}$ so there exists a $j_0 \in \{1,..,N\}$ with $x \in B_{j_0}$ and so $x_{j_0} \in G_{j_0}(x) = F_{j_0}(x)$.

The same reasoning in Theorem 2.7 except Theorem 1.1 is used instead of Brouwer's fixed point theorem immediately yields our next result.

Theorem 2.10. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i \in I} X_i$ paracompact. Also assume X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i \in I} E_i$. For each $i \in I$ suppose $F_i : X \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Finally suppose for each $x \in X$ there exists a $i \in I$ with $S_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (in fact $x \in K = \prod_{i \in I} K_i$).*

Remark 2.11. (i). Note there is an analogue Remark 2.9 for Theorem 2.10.
(ii). If in Theorem 2.10 we replace "suppose for each $x \in X$ there exists a $i \in I$ with $S_i(x) \neq \emptyset$" with "suppose there exists a finite subset I_0 of I such that for each $x \in X$ there exists a $i \in I_0$ with $S_i(x) \neq \emptyset$" then X being a q–Schauder admissible subset of the Hausdorff topological vector space E can be removed. Note we use the Brouwer fixed point theorem instead of Theorem 1.1 since $A_i = \{x \in X : S_i(x) \neq \emptyset\}, i \in I$ is an open covering of X can be replaced by $A_i = \{x \in X : S_i(x) \neq \emptyset\}, i \in I_0$ is an open covering of X and proceed as in Theorem 2.7 with $\{1,...,N\}$ replaced by I_0. An example of the above situation is if $X \equiv \prod_{i \in I} X_i$ is compact (of course no reference to paracompactness is needed in this situation and one could restate Theorem 2.10).

Theorem 2.12. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1,...,N\}$ suppose $F_i : X \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact subset K of X and for each $i \in \{1,..,N\}$ a convex compact subset Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1,..,N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Suppose for each $x \in X$ there exists a $i \in \{1,...,N\}$ with $S_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in \{1,..,N\}$ with $x_i \in F_i(x)$.*

Proof: Let A_i, B_i, G_i and T_i be as in Theorem 2.7. The same reasoning as in Theorem 2.7 guarantees that for $i \in \{1, ..., N\}$ we have $G_i \in \Phi^\star(X, X_i)$. Let K be as in the statement of Theorem 2.12 and let $G_i^\star$ (respectively, $T_i^\star$) denote the restriction of G_i (respectively, T_i) to K. The same reasoning as in Theorem 2.7 guarantees that for $i \in \{1, ..., N\}$ we have $G_i^\star \in \Phi^\star(K, X_i)$. Now for each $i \in \{1, ..., N\}$ from [4] there exists a continuous (single valued) selection $f_i : K \to X_i$ of $G_i^\star$ with $f_i(x) \in T_i^\star(x) \subseteq G_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of X_i with $f_i(K) \subseteq co\,(C_i)$. Let

$$\Omega_i = co\ (co\,(C_i) \cup Y_i) \quad \text{for} \quad i \in \{1, ..., N\}$$

which is a convex compact [3] subset of X_i. For each $x \in X$ and $i \in \{1, ..., N\}$ let $F_i^{\star\star}(x) = F_i(x) \cap \Omega_i$ and $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i$. Note if $x \in X$ then $S_i^{\star\star}(x) = S_i(x) \cap \Omega_i \subseteq F_i(x) \cap \Omega_i = F_i^{\star\star}(x)$. Also note if $y \in \Omega_i$ then

$$\begin{aligned} (S_i^{\star\star})^{-1}(y) &= \{z \in X : y \in S_i^{\star\star}(z)\} = \{z \in X : y \in S_i(z) \cap \Omega_i\} \\ &= \{z \in X : y \in S_i(z)\} = S_i^{-1}(y) \end{aligned}$$

which is open in X.

Let $x \in X$. We claim there exists a $i \in \{1, .., N\}$ with $S_i^{\star\star}(x) \neq \emptyset$. This is immediate if $x \in X \backslash K$ since from one of our assumptions in the statement of Theorem 2.12 there exists a $j \in \{1, .., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$ so $S_j^{\star\star}(x) = S_j(x) \cap \Omega_j \neq \emptyset$ since $Y_j \subseteq \Omega_j$. It remains to consider $x \in K$. Since $\{B_i\}_{i=1}^N$ is a covering of X there exists a $j_0 \in \{1, .., N\}$ with $x \in B_{j_0}$. Note $f_{j_0}(x) \in T_{j_0}^\star(x) = T_{j_0}(x) = S_{j_0}(x)$ since $x \in B_{j_0}$ and $f_{j_0}(x) \in co\,(C_{j_0}) \subseteq \Omega_{j_0}$. Thus $S_{j_0}^{\star\star}(x) = S_{j_0}(x) \cap \Omega_{j_0} \neq \emptyset$. Combining all the above we see that there exists a $i \in \{1, .., N\}$ with $S_i^{\star\star}(x) \neq \emptyset$.

Next note for $i \in \{1, ..., N\}$ that $F_i^{\star\star}(X) \subseteq \Omega_i$ (since $F_i^{\star\star}(x) = F_i(x) \cap \Omega_i \subseteq \Omega_i$) and Ω_i is a convex compact subset of X_i. Now apply Theorem 2.7 (with K_i replaced by Ω_i, F_i replaced by $F_i^{\star\star}$ and S_i replaced by $S_i^{\star\star}$) and Remark 2.9 so there exists a $x \in X$ and a $i \in \{1, ..., N\}$ with $x_i \in F_i^{\star\star}(x) = F_i(x) \cap \Omega_i$ i.e. $x_i \in F_i(x)$. $\square$

The same reasoning in Theorem 2.12 except Theorem 2.10 replaces Theorem 2.7 immediately yields our next result.

Theorem 2.13. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i \in I} X_i$ paracompact. Also assume X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i \in I} E_i$. For each $i \in I$ suppose $F_i : X \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact subset K of X and for each $i \in I$ a convex* *compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$. Suppose for each $x \in X$ there exists a $i \in I$ with $S_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Now we present an analytic alternative.

Theorem 2.14. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact. For $i \in \{1, ..., N\}$ let f_i, $g_i : X \times X_i \to \mathbf{R}$ with $g_i(x, y) \leq f_i(x, y)$ for all $(x, y) \in X \times X_i$, let $\lambda_i \in \mathbf{R}$ and let for $x \in X$,*

$$F_i(x) = \{z_i \in X_i : \ f_i(x, z_i) > \lambda_i\} \quad \text{and} \quad S_i(x) = \{z_i \in X_i : \ g_i(x, z_i) > \lambda_i\}.$$

Assume for each $i \in \{1, ..., N\}$ that $S_i(x)$ is convex valued for each $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. In addition suppose either

(1). for each $i \in \{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$,

or

(2). there is a compact subset K of X and for each $i \in \{1, .., N\}$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in \{1, .., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$,

hold. Then either

(A1). there exists a $x \in X$ and a $i \in \{1, ..., N\}$ with $x_i \in F_i(x)$ (i.e. $f_i(x, x_i) > \lambda_i$),

or

(A2). there exists a $x \in X$ with $\sup_{z_i \in X_i} g_i(x, z_i) \leq \lambda_i$ for all $i \in \{1, .., N\}$

occurs.

Proof: Note either (a). there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ or (b). for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $S_i(x) \neq \emptyset$.

Suppose (a) holds. Then for this x we have $S_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ so for all $i \in \{1, ..., N\}$ we have $g_i(x, z_i) \leq \lambda_i$ for $z_i \in X_i$ (so $\sup_{z_i \in X_i} g_i(x, z_i) \leq \lambda_i$).

Suppose (b) holds. Note S_i is a selection of F_i so Theorem 2.7 (if (1) occurs)) or Theorem 2.12 (if (2) occurs) guarantees a $x \in X$ and a $i \in \{1, ..., N\}$ with $x_i \in F_i(x)$ so $f_i(x, x_i) > \lambda_i$ (i.e. (A1) occurs). $\square$

The same reasoning in Theorem 2.14 except Theorem 2.10 (respectively, Theorem 2.12) replaces Theorem 2.7 (respectively, Theorem 2.13) immediately yields our next result.

Theorem 2.15. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i\in I} X_i$ paracompact. Also assume X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. For $i \in I$ let $f_i,\, g_i : X \times X_i \to \mathbf{R}$ with $g_i(x, y) \leq f_i(x, y)$ for all $(x, y) \in X \times X_i$, let $\lambda_i \in \mathbf{R}$ and let for $x \in X$,*

$$F_i(x) = \{z_i \in X_i :\ f_i(x, z_i) > \lambda_i\} \quad \text{and} \quad S_i(x) = \{z_i \in X_i :\ g_i(x, z_i) > \lambda_i\}.$$

Assume for each $i \in I$ that $S_i(x)$ is convex valued for each $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. In addition suppose either

(1). for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$,

or

(2). there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$,

hold. Then either

(A1). there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (i.e. $f_i(x, x_i) > \lambda_i$),

or

(A2). there exists a $x \in X$ with $\sup_{z_i\in X_i}\, g_i(x, z_i) \leq \lambda_i$ for all $i \in I$

occurs.

Note Theorem 2.14 (and Theorem 2.15) immediately guranatees a minimax inequality. Suppose f_i and g_i are as in Theorem 2.14 and let $\lambda_i = \sup_{x\in X}\, [f_i(x, x_i)]$. Assume $\lambda_i < \infty$ for all $i \in \{1, ..., N\}$. Now suppose the assumptions in Theorem 2.14 hold with these λ_i, $i \in \{1, .., N\}$. First note that $(A1)$ cannot occur since if there a $x \in X$ and a $i \in \{1, ..., N\}$ with $f_i(x, x_i) > \lambda_i$ i.e. $f_i(x, x_i) > \sup_{x\in X}\, [f_i(x, x_i)]$ we have a contradiction. Thus there exists a $x \in X$ with $\sup_{z_i\in X_i}\, g_i(x, z_i) \leq \lambda_i$ for all $i \in \{1, ..., N\}$ i.e.

$$\sup_{z_i\in X_i} g_i(x, z_i) \leq \sup_{x\in X} f_i(x, x_i) \quad \text{for all} \quad i \in \{1, ..., N\}.$$

Remark 2.16. A game has N players and each player must select a strategy in a set determined by the strategies chosen by the other players. Here X_i denotes the set of strategies of the i^{th} player and each element of $X = \prod_{i=1}^{N} X_i$ determines an outcome. The payoff to the i^{th} player is h_i (which is defined on X). Let x^i be given in X^i (the strategies of all the others). For $x \in X$, $i \in \{1, ..., N\}$, $y_i \in X_i$ we write (x^i, y_i) as a point in X having the same components as x except the i^{th} component is replaced by y_i; note any $x \in X$ can be written as (x^i, x_i) for any $i \in \{1, ..., N\}$ where x^i denotes the projection of x onto X^i. The i^{th} player chooses $y_i \in X_i$ so as to maximize $h_i(x^i, y_i)$. An equilibrium point is a strategy point $x \in X$ such that for all $i \in \{1,..,N\}$ we have

$$x_i \in X_i \quad \text{and} \quad h_i(x) = \max_{y_i \in X_i} h_i(x^i, y_i).$$

Suppose for $i \in \{1,..,N\}$ we have $f_i : X \times X_i \to \mathbf{R}$ given by

$$f_i(x, y_i) = h_i(x^i, y_i) - h_i(x) \quad \text{and} \quad g_i(x, y_i) = f_i(x, y_i)$$

for all $x \in X$ and $y_i \in X_i$. Let $\lambda_i = \sup_{x \in X} \, [f_i(x, x_i)]$ and assume $\lambda_i < \infty$ for all $i \in \{1, ..., N\}$. Suppose the assumptions in Theorem 2.14 hold with these λ_i, $i \in \{1, ..., N\}$; here $F_i = S_i$ for $i \in \{1,..,N\}$ in this case. From the above minimax inequality we deduce that there exists a $x \in X$ with

$$\sup_{y_i \in X_i} f_i(x, y_i) \leq \sup_{x \in X} f_i(x, x_i) \quad \text{for all} \quad i \in \{1, ..., N\}$$

i.e.

$$\sup_{y_i \in X_i} [h_i(x^i, y_i) - h_i(x)] \leq \sup_{x \in X} [h_i(x^i, x_i) - h_i(x)] = 0 \quad \text{for all} \quad i \in \{1, ..., N\}.$$

Thus $\sup_{y_i \in X_i} h_i(x^i, y_i) \leq h_i(x)$ for all $i \in \{1, ..., N\}$. Note also for $i \in \{1, ..., N\}$ that $h_i(x) = h_i(x^i, x_i) \leq \sup_{y_i \in X_i} h_i(x^i, y_i)$ so

$$h_i(x) = \sup_{y_i \in X_i} h_i(x^i, y_i) \quad \text{for all} \quad i \in \{1, ..., N\}.$$

To guarantee an equilibrium point conditions are put on h_i to guarantee that the above achieves its maximum on X_i.

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