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On the Upper Bound of the Third Hankel Determinant for Certain Class of Analytic Functions Related with Exponential Function

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Abstract

In the present paper we introduce a new class of analytic functions f in the open unit disk normalized by $f(0) = f'(0)-1 = 0$, associated with exponential functions. The aim of the present paper is to investigate the third-order Hankel determinant $H_3(1)$ for this function class and obtain the upper bound of the determinant $H_3(1)$.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$. Also we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions f in Δ. The familiar coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) was first presented by the Bieberbach [1] in 1916 and proved by de-Branges [2] in 1985. During 1916-1985 many mathematicians struggled to prove or disprove this conjecture. As result they defined several subfamilies of

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the set $\mathcal{S}$ connected with different image domains. Further, we recall some of them. Let the notations $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ indicate the families of starlike, convex and close-to-convex functions respectively with the following Taylor-Maclaurin series representations:

$$\mathcal{S}^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in \Delta \right\}; \tag{2}$$

$$\mathcal{C} = \left\{ f \in S : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, z \in \Delta \right\}; \tag{3}$$

$$\mathcal{K} = \left\{ f \in S : \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}, \text{ for } g \in C,\ z \in \Delta \right\}, \tag{4}$$

where the symbol " $\prec$ " denotes the familiar concept of differential subordination between analytic functions. Now, we recall here the definition of subordination.

Suppose that f and g are two analytic functions in Δ. We say that the function f is subordinate to g and we write $f(z) \prec g(z)$, if there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that (see [14]) $f(z) = g(w(z))$. Thus, $f(z) \prec g(z)$ implies $f(\Delta) \subset g(\Delta)$. In case of univalency of f in Δ, the function f is subordinate to g if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Assume that $\mathcal{P}$ denote the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \tag{5}$$

and satisfying the condition $\Re p(z) > 0$, $z \in \Delta$. It is easy to see that if $p \in \mathcal{P}$, then there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that (see [25])

$$p(z) = \frac{1 + w(z)}{1 - w(z)}. \tag{6}$$

Padmanabhan and Parvatham introduced in the paper [20] a unified families of starlike and convex functions using familiar notion of convolution with the function $z/(1-z)^a$, for all $a \in \mathbb{R}$. Later on Shanmugam [23] generalized the idea of paper [20] and introduced the set

$$\mathcal{S}_h^*(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f*h)'}{(f*h)} \prec \phi(z), \quad z \in \Delta \right\}, \tag{7}$$

where the symbol " $*$ " stands for the familiar notion of convolution, ϕ is convex and h is a fixed function in $\mathcal{A}$. We obtain the families $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ when taking $z/1-z$ and $z/(1-z)^2$ instead of h in $\mathcal{S}_h^*(\phi)$ respectively.

In 1992, Ma and Minda [12] reduced the above restriction to a weaker supposition that ϕ is a function, with $\Re\phi > 0$ in Δ with, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and discussed some properties. Here are these classes:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Delta \right\};$$

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad z \in \Delta \right\}.$$

The classes $\mathcal{S}^*(\phi)$ and $C(\phi)$ unify various subclasses of starlike $\mathcal{S}^*$ or convex $\mathcal{C}$ functions in Δ. For example, the class $\mathcal{S}^*(\phi)$ generalizes various subfamilies of the set $\mathcal{A}$ as follows:

1. If the function $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then $\mathcal{S}^*[A, B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions defined in [8]. Further, if $A = 1 - 2\alpha$ and $B = -1$ with $0 \leq \alpha < 1$, then we get the set $\mathcal{S}^*(\phi)$ of starlike function of order α.

2. The family $\mathcal{S}^*_L := \mathcal{S}^*\left(\sqrt{1+z}\right)$ was introduced by Sokol and Stankiewicz in [24], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\left|w^2 - 1\right| < 1$.

3. For the function $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi)$ leads to the class $\mathcal{S}^*_{\sin}$, introduced in [3].

4. The family $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ was introduced by Mediratta et al. in [13] given as:

$$\mathcal{S}^*_e = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^z, \quad z \in \Delta \right\}, \tag{8}$$

or equivalently

$$\mathcal{S}^*_e = \left\{ f \in \mathcal{S} : \left| \log \frac{zf'(z)}{f(z)} \right| \prec e^z, \quad z \in \Delta \right\}. \tag{9}$$

By using Alexander type relation, we also recall [13] by the following set:

$$\mathcal{C}_e = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec e^z, \quad z \in \Delta \right\}.$$

The above mentioned families $\mathcal{S}^*_e$ and $\mathcal{C}_e$ are symmetric about the real axis.

In [15], Noonan and Thomas studied the q^{th} Hankel determinants $H_q(n)$ of functions $f \in \mathcal{A}$ of the form (1) for $q \geq 1$ and $n \geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1). \tag{10}$$

In particular

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad (a_1 = 1).$$

Since $f \in \mathcal{S}$, $a_1 = 1$, thus

$$H_3(1) = a_3 \left(a_2 a_4 - a_3^2\right) - a_4 \left(a_4 - a_2 a_3\right) + a_5 \left(a_3 - a_2^2\right).$$

The concept of Hankel determinant is very useful in the theory of singularities [4] and in the study of power series with integral coefficients. The Hankel determinant $H_q(n)$ have been investigated by several authors to study its rate of growth as $n \to \infty$ and to determine bounds on it for specific values of q and n. For example, Pommerenke [22] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < kn^{-\left(\frac{1}{2}+\beta\right)q+\frac{3}{2}}$, $(n = 1, 2, \dots, q = 2, 3, \dots)$ where $\beta > 1/1400$ and k depends only on q. Note that the Hankel determinant $H_2(1) = a_3 - a_2^2$ is related to the well-known Fekete-Szegő functional [7] for univalent functions. Although we know many sharp bounds of $H_2(2)$ and significantly less sharp bounds of $H_3(1)$ for some proper subfamilies of $\mathcal{S}$, the sharp results for the whole class $\mathcal{S}$ are not known. Moreover, we are even unable to formulate a reasonable conjecture about it. Ehrenborg studied Hankel determinant of the exponential polynomials [6] and Noor studied Hankel determinant for Bazilevic functions in [18] and for functions with bounded boundary rotations in [17] and [16]; also for close-to-convex functions in [19]. Until now, very few researches have studied the above determinants for the function class, subordinate to e^z. Thus, in this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ for a certain class defined below, which is associated with exponential function and obtain the upper bound of the determinant. To derive our results, we shall need the following results.

2 Preliminary results

Some preliminary results required in the following section are now listed.

Lemma 1. *([5]) If $p \in \mathcal{P}$ and has the form (5) then*

$$|c_n| \leq 2, \ n = 1, 2, \dots \tag{11}$$

and the inequality is sharp.

Lemma 2. *([21], [9]) If $p \in \mathcal{P}$ and has the form (5) then*

$$|c_{n+k} - \mu c_n c_k| < 2 \quad \text{for } 0 \leq \mu \leq 1; \tag{12}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad \text{for } m + n = k + l; \tag{13}$$

$$\left|c_{n+2k} - \mu c_n c_k^2\right| \leq 2\left(1 + 2\mu\right) \quad \text{for } \mu < -\frac{1}{2}; \tag{14}$$

$$\left|c_2 - \tfrac{c_1^2}{2}\right| < 2 - \tfrac{|c_1|^2}{2}; \tag{15}$$

and for the complex number λ, we have

$$c_2 - \lambda c_1^2 \leq 2 \max\{1, |2\lambda - 1|\}. \tag{16}$$

For the inequalities (12) , (13) , (14) , (15) see [21] and (16) is given in [9].

Lemma 3. *([10], [11]) If the function $p \in \mathcal{P}$ is given by (5) , then exists some x, z with $|x| \leq 1, |z| \leq 1$ such that*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right); \tag{17}$$

$$4c_3 = c_1^3 + 2c_1 x\left(4 - c_1^2\right) - \left(4 - c_1^2\right) c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right) z. \tag{18}$$

3 Main results

Definition 1. *A function $f \in \mathcal{S}$ is said to be in the class SC_α^*, $\alpha \in [0, 1]$, if satisfies the following condition:*

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z.$$

Remark. *For $\alpha = 0$, the family $SC_0^* := \mathcal{S}_e^* = \mathcal{S}^*(e^z)$ was introduced by Mediratta et al. in [13] and for $\alpha = 1$, we reobtain the set $SC_1^* := C_e$.*

Theorem 1. *If the function $f \in SC_\alpha^*$, where f is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{C}$ then we have*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2(1 + 2\alpha)}. \tag{19}$$

Proof. Because $f \in SC_{\alpha}^{*}$, from the definition of subordination, we know that exists a Schwartz function $w(z)$, with $w(0)=0$ and $|w(z)|<1$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=e^{w(z)}.$$

But

$$(1-\alpha)\frac{zf'(z)}{f(z)}=$$

$$=(1-\alpha)\left[1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3+\ldots\right]. \tag{20}$$

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\alpha\left(1+\frac{\sum_{n=2}^{\infty}na_n(n-1)z^{n-1}}{1+\sum_{n=2}^{\infty}na_nz^{n-1}}\right)= \tag{21}$$

$$=\alpha\left[1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3+\ldots\right].$$

From the relations (20) and (21) we obtain

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)= \tag{22}$$

$$=1+za_2(1+\alpha)+z^2\left[2a_3(1+2\alpha)-a_2^2(1+3\alpha)\right]+$$

$$+z^3\left[a_2^3(1+7\alpha)-3a_2a_3(1+5\alpha)+3a_4(1+3\alpha)\right]+\ldots$$

We define a function

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\ldots,$$

$p(z)\in\mathcal{P}$ and

$$w(z)=\frac{p(z)-1}{p(z)+1}=\frac{c_1z+c_2z^2+c_3z^3+\ldots}{2+c_1z+c_2z^2+c_3z^3+\ldots}.$$

But

$$e^{w(z)}=1+w(z)+\frac{[w(z)]^2}{2!}+\frac{[w(z)]^3}{3!}+\ldots=$$

$$=1+\frac{c_1z+c_2z^2+c_3z^3+\ldots}{2+c_1z+c_2z^2+c_3z^3+\ldots}+\frac{1}{2}\left(\frac{c_1z+c_2z^2+c_3z^3+\ldots}{2+c_1z+c_2z^2+c_3z^3+\ldots}\right)^2+$$

$$+\frac{1}{6}\left(\frac{c_1z+c_2z^2+c_3z^3+\ldots}{2+c_1z+c_2z^2+c_3z^3+\ldots}\right)^3+\ldots= \tag{23}$$

$$= 1 + \frac{1}{2}\left(c_1 z + c_2 z^2 + \ldots\right)\left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2}\right)z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right)z^3 + \ldots\right]$$

$$+\frac{1}{2}\left(c_1 z + c_2 z^2 + \ldots\right)^2\left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2}\right)z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right)z^3 + \ldots\right]^2$$

$$+\frac{1}{48}\left(c_1 z + c_2 z^2 + \ldots\right)^3\left[1 - \frac{c_1 z}{2} + \left(\frac{c_1^2}{4} - \frac{c_2}{2}\right)z^2 - \left(\frac{c_1^3}{8} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right)z^3 + \ldots\right]^3 + \ldots$$

$$= 1 + \frac{1}{2}c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right)z^2 + \left(\frac{c_1^3}{48} - \frac{c_1 c_2}{4} + \frac{c_3}{2}\right)z^3 + \ldots$$

On comparing the coefficients of z, z^2 and z^3 between the equations (22) and (23) we obtain

$$a_2 = \frac{c_1}{2(1+\alpha)}; \tag{24}$$

$$a_3 = \frac{c_2}{4(1+2\alpha)} + \frac{c_1^2\left(1+4\alpha-\alpha^2\right)}{16(1+2\alpha)(1+\alpha)^2}; \tag{25}$$

It can be written,

$$\left|a_3 - a_2^2\right| = \left|\frac{c_2}{4(1+2\alpha)} - \frac{c_1^2(\alpha+3)}{16(1+2\alpha)(1+\alpha)}\right|.$$

Using Lemma 3, we thus know that

$$\left|a_3 - a_2^2\right| = \left|\frac{x\left(4-c_1^2\right)}{8(1+2\alpha)} - \frac{c_1^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)}\right|.$$

Letting $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$ and applying the triangle inequality, the above equation reduces to

$$\left|a_3 - a_2^2\right| \leq \frac{t\left(4-c^2\right)}{8(1+2\alpha)} + \frac{c^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)}.$$

Suppose that

$$F(c,t) := \frac{t\left(4-c^2\right)}{8(1+2\alpha)} + \frac{c^2(1-\alpha)}{16(1+2\alpha)(1+\alpha)},$$

then we get

$$\frac{\partial F}{\partial t} = \frac{4-c^2}{8(1+2\alpha)} \geq 0,$$

which shows that $F(c,t)$ is an increasing function on the closed interval $[0,1]$ about t. Therefore the function $F(c,t)$ can get the maximum value at $t=1$, that is

$$\max F(c,t) = F(c,1) = \frac{4-c^2}{8\,(1+2\alpha)} + \frac{c^2\,(1-\alpha)}{16\,(1+2\alpha)\,(1+\alpha)}.$$

Next, let

$$G\,(c) = \frac{4-c^2}{8\,(1+2\alpha)} + \frac{c^2\,(1-\alpha)}{16\,(1+2\alpha)\,(1+\alpha)} = \frac{1}{2\,(1+2\alpha)} - \frac{c^2\,(1+3\alpha)}{16\,(1+2\alpha)\,(1+\alpha)}.$$

The function $G\,(c)$ has a maximum value at $c=0$, which is

$$\left|a_3 - a_2^2\right| \le G\,(0) = \frac{1}{2\,(1+2\alpha)}$$

and the proof is done. □

Theorem 2. *If the function $f \in SC_\alpha^*$, where f is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \mathbb{C}$ then we have*

$$|a_2a_3 - a_4| \le \frac{\left(4-\widetilde{c}^2\right)\widetilde{c}}{24} + \frac{3\left(4-\widetilde{c}^2\right)\widetilde{c}}{8} + \frac{4-\widetilde{c}^2}{12} + \frac{\widetilde{c}^3\varepsilon(\rho)}{72} \tag{26}$$

where

$$\widetilde{c} = -\frac{9\sqrt{\frac{499\,312}{3} - \frac{23\,840}{3}\sqrt{298}} - 108}{298\sqrt{298} - 6236} \in [0,2]$$

and $\varepsilon(\rho) = -2\alpha^3 - 14\alpha^2 + 17\alpha + 5,\ \rho = \frac{1}{6}\sqrt{2}\sqrt{149} - \frac{7}{3}$.

Proof. Knowing that

$$\begin{aligned} a_4 \quad = \quad & \frac{c_3}{6\,(1+3\alpha)} - \frac{c_1c_2\left(4\alpha^2 - 9\alpha - 1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} + \\ & + \frac{c_1^3\left(4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1\right)}{288\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)}; \end{aligned} \tag{27}$$

we have

$$|a_2a_3 - a_4| =$$

$$\left|\frac{c_1c_2\left(2\alpha^2+1\right)}{12\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} - \frac{c_3}{6\,(1+3\alpha)} - \frac{c_1^3\left(2\alpha^4 - 2\alpha^3 - 39\alpha^2 - 40\alpha - 5\right)}{144\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)}\right|.$$

Again, by applying Lemma 3, we get

$$|a_2a_3 - a_4| = \left| \frac{c_1^3\left(-2\alpha^4 - 16\alpha^3 + 3\alpha^2 + 22\alpha + 5\right)}{144\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)} + \right.$$

$$\left. + \frac{\left(4-c_1^2\right)c_1x^2}{24\,(1+3\alpha)} - \frac{\left(4-c_1^2\right)\left(1-|x|^2\right)z}{12\,(1+3\alpha)} - \frac{c_1x\left(4-c_1^2\right)\left(2\alpha^2+6\alpha+1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} \right|.$$

Assume that $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. Then, using the triangle inequality, we deduce that

$$|a_2a_3 - a_4| \leq \frac{\left(4-c^2\right)ct^2}{24\,(1+3\alpha)} + \frac{\left(4-c^2\right)ct\left(2\alpha^2+6\alpha+1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} +$$

$$+ \frac{\left(4-c^2\right)}{12\,(1+3\alpha)} + \frac{c^3\left(-2\alpha^4 - 16\alpha^3 + 3\alpha^2 + 22\alpha + 5\right)}{144\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)}.$$

Setting

$$F(c,t) := \frac{\left(4-c^2\right)ct^2}{24\,(1+3\alpha)} + \frac{\left(4-c^2\right)ct\left(2\alpha^2+6\alpha+1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} +$$

$$+ \frac{\left(4-c^2\right)}{12\,(1+3\alpha)} + \frac{c^3\left(-2\alpha^4 - 16\alpha^3 + 3\alpha^2 + 22\alpha + 5\right)}{144\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)}.$$

Hence, we have

$$\frac{\partial F}{\partial t} = \frac{\left(4-c^2\right)ct}{12\,(1+3\alpha)} + \frac{\left(4-c^2\right)c\left(2\alpha^2+6\alpha+1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} \geq 0,$$

namely, $F(c,t)$ is an increasing function on the closed interval $[0,1]$ about t. This implies that the maximum value of $F(c,t)$ occurs at $t = 1$, which is

$$\max F(c,t) = F(c,1) = \frac{\left(4-c^2\right)c}{24\,(1+3\alpha)} + \frac{\left(4-c^2\right)c\left(2\alpha^2+6\alpha+1\right)}{24\,(1+3\alpha)\,(1+2\alpha)\,(1+\alpha)} +$$

$$+ \frac{\left(4-c^2\right)}{12\,(1+3\alpha)} + \frac{c^3\left(-2\alpha^4 - 16\alpha^3 + 3\alpha^2 + 22\alpha + 5\right)}{144\,(1+\alpha)^3\,(1+2\alpha)\,(1+3\alpha)}.$$

Then

$$\max F(c,t) \leq \frac{\left(4-c^2\right)c}{24} + \frac{\left(4-c^2\right)c\left(2\alpha^2+6\alpha+1\right)}{24}$$

$$+ \frac{4-c^2}{12} + \frac{c^3\,(\alpha+1)\,\varepsilon(\alpha)}{144}$$

$$\leq \frac{\left(4-c^2\right)c}{24}+\frac{9\left(4-c^2\right)c}{24}+\frac{4-c^2}{12}+\frac{2c^3\varepsilon(\rho)}{144}$$

where $\varepsilon(\alpha)=-2\alpha^3-14\alpha^2+17\alpha+5$ which is a positive function, $\varepsilon(\alpha)\leq\varepsilon(\rho)$ and $\rho=\frac{1}{6}\sqrt{2}\sqrt{149}-\frac{7}{3}$.

Now, we define

$$E(c)=\frac{\left(4-c^2\right)c}{24}+\frac{9\left(4-c^2\right)c}{24}+\frac{4-c^2}{12}+\frac{2c^3\varepsilon(\rho)}{144}.$$

Equation $E'(c)=0$ which is $3\left(-30+\varepsilon(\rho)\right)c^2-12c+120=0$ implies

$$c^*=\frac{9\sqrt{\frac{499\,312}{3}-\frac{23\,840}{3}\sqrt{298}}+108}{298\sqrt{298}-6236}\notin[0,2],$$

$$\widetilde{c}=-\frac{9\sqrt{\frac{499\,312}{3}-\frac{23\,840}{3}\sqrt{298}}-108}{298\sqrt{298}-6236}\in[0,2].$$

The function $E(c)$ is an increasing function on the closed interval $[0,\widetilde{c}]$ and also an decreasing function on the interval $[\widetilde{c},2]$ and the maximum value of $E(c)$ occurs at $c=\widetilde{c}$. Thus $|a_2a_3-a_4|\leq\max\limits_{c\in[0,2]}E(c)=E(\widetilde{c})\approx 1.74$ and the proof of the Theorem 2 is completed. □

Theorem 3. *If the function $f\in SC^*_\alpha$, where f is given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, $z\in\mathbb{C}$ then we have*

$$\left|a_2a_4-a_3^2\right|\leq 2. \tag{28}$$

Proof. Suppose that $f\in SC^*_\alpha$, then from the equation (25) we have

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1c_3}{12\left(1+\alpha\right)\left(1+3\alpha\right)}-\frac{c_2^2}{16\left(1+2\alpha\right)^2}-\frac{c_1^2c_2}{96}\cdot\frac{7\alpha^3+5\alpha^2-\alpha+1}{\left(1+\alpha\right)^2\left(1+2\alpha\right)^2\left(1+3\alpha\right)}\right.$$
$$\left.+\frac{c_1^4\left(5\alpha^5-25\alpha^4-262\alpha^3-394\alpha^2-175\alpha-13\right)}{2304\left(1+\alpha\right)^4\left(1+2\alpha\right)^2\left(1+3\alpha\right)}\right|.$$

In view of Lemma 3, we thus obtain

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1^4\left(5\alpha^5+47\alpha^4-46\alpha^3-178\alpha^2-103\alpha-13\right)}{2304\left(1+\alpha\right)^4\left(1+2\alpha\right)^2\left(1+3\alpha\right)}\right.$$
$$+\frac{c_1^2x\left(4-c_1^2\right)\left(7\alpha^3+17\alpha^2+11\alpha+1\right)}{192\left(1+\alpha\right)^2\left(1+2\alpha\right)^2\left(1+3\alpha\right)}-\frac{\left(4-c_1^2\right)c_1^2x^2}{48\left(1+\alpha\right)\left(1+3\alpha\right)}$$

$$+\frac{c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z}{24\left(1+\alpha\right)\left(1+3\alpha\right)}-\frac{x^2\left(4-c_1^2\right)^2}{64\left(1+2\alpha\right)^2}\Bigg|.$$

Also let $|x|=t\in[0,1]$, $c_1=c\in[0,2]$. Then, using the triangle inequality we get

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\le \frac{c^4\left|5\alpha^5+47\alpha^4-46\alpha^3-178\alpha^2-103\alpha-13\right|}{2304\left(1+\alpha\right)^4\left(1+2\alpha\right)^2\left(1+3\alpha\right)}\\ &+\frac{c^2t\left(4-c^2\right)\left(7\alpha^3+17\alpha^2+11\alpha+1\right)}{192\left(1+\alpha\right)^2\left(1+2\alpha\right)^2\left(1+3\alpha\right)}+\frac{\left(4-c^2\right)c^2t^2}{48\left(1+\alpha\right)\left(1+3\alpha\right)}\\ &+\frac{c\left(4-c^2\right)}{24\left(1+\alpha\right)\left(1+3\alpha\right)}+\frac{t^2\left(4-c^2\right)^2}{64\left(1+2\alpha\right)^2}.\end{aligned}$$

Knowing that $\max\limits_{\alpha\in[0,1]}\left|5\alpha^5+47\alpha^4-46\alpha^3-178\alpha^2-103\alpha-13\right|=288$ the above inequality can be rewritten

$$\left|a_2a_4-a_3^2\right|\le\frac{288c^4}{2304}+\frac{36c^2\left(4-c^2\right)}{192}+\frac{\left(4-c^2\right)c^2}{48}+\frac{c\left(4-c^2\right)}{24}+\frac{\left(4-c^2\right)^2}{64}$$

or equivalent

$$\left|a_2a_4-a_3^2\right|\le\frac{-39c^4-24c^3+408c^2+96c+144}{576}.$$

Next, let

$$G(c):=\frac{-39c^4-24c^3+408c^2+96c+144}{576}.$$

Now it easily to derive that $G(c)$ is an increasing function on the interval $[0,2]$ therefore we have a maximum value at $c=2$, also which is $\left|a_2a_4-a_3^2\right|\le G(c)=2$. The proof of the Theorem 3 is thus completed. □

Theorem 4. *If the function $f\in SC_\alpha^*$, then we have*

$$|H_3(1)|\le 18,001. \tag{29}$$

Proof. In order to establish the upper bound for $H_3(1)$ we proceed to compute certain inequalities. Using the form of a_5 posted below

$$\begin{aligned}a_5 &= \frac{c_1c_3\left(1+7\alpha\right)}{12\left(1+\alpha\right)\left(1+3\alpha\right)\left(1+4\alpha\right)}-\frac{c_1^2c_2\left(-188\alpha^5+184\alpha^4+340\alpha^3-2\alpha^2-44\alpha-2\right)}{192\left(1+4\alpha\right)\left(1+3\alpha\right)\left(1+2\alpha\right)^2\left(1+\alpha\right)^2}\\ &+\frac{c_1^4\left(484\alpha^7-2328\alpha^6+520\alpha^5+616\alpha^4-1424\alpha^3-308\alpha^2+132\alpha+4\right)}{4608\left(1+4\alpha\right)\left(1+\alpha\right)^4\left(1+2\alpha\right)^2\left(1+3\alpha\right)}\\ &+\frac{c_2^24\alpha\left(1-\alpha\right)}{32\left(1+4\alpha\right)\left(1+2\alpha\right)^2}+\frac{1}{8\left(1+4\alpha\right)}\left(c_4-\frac{1}{2}c_1c_3\right).\end{aligned} \tag{30}$$

and the equalities from Lemma 3 we get

$$a_5 = \left(\frac{M(\alpha)}{4} + \frac{N(\alpha)}{2} + P(\alpha) + \frac{Q(\alpha)}{4}\right) c_1^4$$

$$+\left(\frac{M(\alpha)}{2} + \frac{N(\alpha)}{2} + \frac{Q(\alpha)}{2}\right) x c_1^2 \left(4 - c_1^2\right) - \frac{M(\alpha)}{4} x^2 c_1^2 \left(4 - c_1^2\right)$$

$$+\frac{M(\alpha)}{2} c_1 \left(4 - c_1^2\right) \left(1 - |x|^2\right) z + \frac{Q(\alpha)}{4} x^2 \left(4 - c_1^2\right)^2 + \frac{1}{8(1+4\alpha)} \left(c_4 - \frac{1}{2} c_1 c_3\right)$$

where $M(\alpha) = \frac{1+7\alpha}{12(1+\alpha)(1+3\alpha)(1+4\alpha)}$, $N(\alpha) = \frac{-188\alpha^5+184\alpha^4+340\alpha^3-2\alpha^2-44\alpha-2}{192(1+4\alpha)(1+3\alpha)(1+2\alpha)^2(1+\alpha)^2}$, $P(\alpha) = \frac{484\alpha^7-2328\alpha^6+520\alpha^5+616\alpha^4-1424\alpha^3-308\alpha^2+132\alpha+4}{4608(1+4\alpha)(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)}$ and $Q(\alpha) = \frac{4\alpha(1-\alpha)}{32(1+4\alpha)(1+2\alpha)^2}$. According to inequality (12)

$$\left|c_4 - \frac{1}{2} c_1 c_3\right| \leq 2.$$

Let $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. By making use of the triangle inequality and the maximum values of the functions $M(\alpha), N(\alpha), P(\alpha)$ and $Q(\alpha)$ on the interval $[0,1]$ for the argument α we have

$$|a_5| \leq \frac{547}{384} c^4 + \frac{211}{192} c^2 \left(4 - c^2\right) + \frac{1}{6} c^2 \left(4 - c^2\right) + \frac{1}{3} c \left(4 - c^2\right) + \frac{1}{128} \left(4 - c^2\right)^2 + \frac{1}{4}$$

or equivalent

$$|a_5| \leq \frac{1}{384} \left(64c^4 - 128c^3 + 1920c^2 + 512c + 144\right) \leq \frac{8848}{384}.$$

since the function $\varphi(c) = 64c^4 - 128c^3 + 1920c^2 + 512c + 144$ gets its maximum at $c = 2$.

For the coefficient a_3 , we deduce using triangle inequality

$$|a_3| \leq \frac{|c_2|}{4(1+2\alpha)} + \frac{|c_1|^2 \left|1 + 4\alpha - \alpha^2\right|}{16(1+2\alpha)(1+\alpha)^2} \leq \frac{2}{4} + \frac{4 \cdot 4}{16} = 1,5.$$

It follows the upper bound for the coefficient a_4.

$$|a_4| = \left|\frac{c_3}{6(1+3\alpha)} - \frac{c_1 c_2 \left(4\alpha^2 - 9\alpha - 1\right)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} + \frac{c_1^3 \left(4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1\right)}{288(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\right|$$

$$\leq \frac{2}{6} + \frac{2 \cdot 2}{24} \left|4\alpha^2 - 9\alpha - 1\right| + \frac{1}{128} \cdot 8 \cdot \left|4\alpha^4 - 31\alpha^3 + 21\alpha^2 - 17\alpha - 1\right| \leq 2.$$

Because

$$H_3(1) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right)$$

by applying the triangle inequality, we obtain the Hankel determinant of order three

$$|H_3(1)| \leq |a_3| \cdot \left|a_2a_4 - a_3^2\right| + |a_4| \cdot |a_4 - a_2a_3| + |a_5| \cdot \left|a_3 - a_2^2\right|. \tag{31}$$

Next, substituting relations (26), (28), (19) in (31) we get the inequality (29). Thus, the proof is completed. □

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