DOI: 10.2478/auom-2022-0013
An. Şt. Univ. Ovidius Constanţa **Vol. 30(1),2022, 231–246**

On characterization of finite modules by hypergraphs

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Abstract

With a finite R-module M we associate a hypergraph $\mathcal{CIH}_R(M)$ having the set V of vertices being the set of all nontrivial submodules of M. Moreover, a subset E_i of V with at least two elements is a hyperedge if for K, L in E_i there is $K \cap L \neq 0$ and E_i is maximal with respect to this property. We investigate some general properties of $\mathcal{CIH}_R(M)$, providing condition under which $\mathcal{CIH}_R(M)$ is connected, and find its diameter. Besides, we study the form of the hypergraph $\mathcal{CIH}_R(M)$ when M is semisimple, uniform module and it is a direct sum of its each two nontrivial submodules. Moreover, we characterize finite modules with three nontrivial submodules according to their co-intersection hypergraphs. Finally, we present some illustrative examples for $\mathcal{CIH}_R(M)$.

1 Preliminaries

In [4, 5], Berge introduced hypergraphs as a generalization of the graph approach. A hypergraph $\mathcal{H} = (V; E)$ on a finite set of vertices (or nodes) $V = \{v_1, \ldots, v_n\}$ is defined as a family of hyperedges $E = \{e_j \mid 1 \leq j \leq m\}$ where each hyperedge is a non-empty subset of V and such that $\cup_{j=1}^{m} e_j = V$. It means that in a hypergraph, a hyperedge links one or more vertices. In [8], the definition of hypergraphs includes also hyperedges that are empty sets as hyperedges are defined as a family of subsets of a finite vertex set and it is not

Key Words: Hyperedge, hypergraph, co-intersection hypergraph.
2010 Mathematics Subject Classification: Primary 05C65 05C20, 05C25, 05C69.
Received: 09.05.2021
Accepted: 31.07.2021

necessary that the union covers the vertex set. Both the vertex set and the family of hyperedges can be empty; if they are both empty in the same time, the hypergraph is then designated as the empty hypergraph. This definition of hypergraph opens their use in various collaboration networks.

In a hypergraph, an edge can join any number of vertices. In contrast, in an ordinary graph, an edge connects exactly two vertices. Formally, a hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (V, E)$ where V is a set of elements called nodes or vertices, and E is a set of non-empty subsets of V called hyperedges or edges. Therefore, E is a subset of $P(V) \setminus \{\emptyset\}$. The size of the vertex set is called the order of the hypergraph, and the size of edges set is the size of the hypergraph. In this work, we apply the general definition of hypergraphs. A k-uniform hypergraph is a hypergraph such that all its hyperedges have size k (in other words, one such hypergraph is a collection of sets such that every hyperedge connects k nodes). So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. Some papers in this context can be seen in [6] and [7].

Let $\mathcal{H} = (V, E)$ be a hypergraph. A *path* P in $\mathcal{H}$ from x to y, is a vertex-hyperedge alternative sequence $x = x_1e_1x_2e_2, \dots, x_se_sx_{s+1} = y$ such that $x_1, x_2, \dots, x_s, x_{s+1}$ are distinct vertices (with the possibility that $x_1 = x_{s+1}$), $e_1, e_2, \dots, e_s$ are distinct hyperedges, and $\{x_i, x_{i+1}\} \subseteq e_i$, for all $i \in \{1, 2, \dots, s\}$. If $x = x_1 = x_{s+1} = y$, the path is called a *cycle*. The integer s is the length of the path P. Notice that if there is a path from x to y there is also a path from y to x. In this case we say that P connects x and y. A hypergraph is connected if for any pair of vertices, there is a path which connects these vertices. The distance $d(x, y)$ between two vertices x and y is the minimum length of a path which connects x and y. If there is a pair of vertices x, y with no path from x to y (or from y to x), we define $d(x, y) = \infty$ ($\mathcal{H}$ is not connected). The diameter $d(\mathcal{H})$ of H is defined by $d(\mathcal{H}) = \max\{d(x, y) \mid x, y \in V\}$.

In [1], the authors introduced intersection graph on submodules of a module. Let R be a ring with identity and M be a unitary right R-module. The intersection graph of M, denoted by $G(M)$, is defined to be the undirected simple graph whose vertices are in one-to-one correspondence with all nontrivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have nonzero intersection. The complement of $G(M)$ was also introduced in [2]. This graph is denoted by $\Gamma(M)$, is defined to be a graph whose vertices are in one-to-one correspondence with all nontrivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have zero intersection.

In this paper, motivated by [1] and works done about hypergraphs, we introduce a new hypergraph assigned to a right R-module M. We define

$\mathcal{CIH}_R(M)$ as a hypergraph where the vertices are all nontrivial submodules of M and a subset E_i with at least two elements of the set all nontrivial submodules of M is a hyperedge of $\mathcal{CIH}_R(M)$ provided for each two $N, K \in E_i$, $N \cap K \neq 0$ and E_i is maximal with respect to this property. Connecting to $G(M)$, we can say E_i is a hyperedge in $\mathcal{CIH}_R(M)$ if and only if E_i is a maximal subset of V with respect to the property that the elements of E_i form a complete subgraph of $G(M)$.

In Section 2, we investigate some general properties of $\mathcal{CIH}_R(M)$. We provides a condition which ensure us $\mathcal{CIH}_R(M)$ is connected. We also prove that the diameter of $\mathcal{CIH}_R(M)$ is at least 2. It is shown that a module M is a direct sum of its each two nontrivial submodules if and only if $\mathcal{CIH}_R(M)$ is null. We also characterize finite modules with exactly three nontrivial submodules via their co-intersection hypergraphs. According to the lattices of submodules of a finite module, we provide all co-intersection hypergraphs of order four.

In Section 3, we present some examples of various co-intersection hypergraphs for finite modules.

All rings considered in this paper will be associative with an identity element and all modules will be unitary finite right modules unless otherwise stated. Let R be a ring and M an R-module. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \subsetneq M, L + N \neq M$). Dually, a submodule K of M is essential in M (denoted by $K \leq_e M$) provided $K \cap L \neq 0$ for each nonzero submodule L of M. $Rad(M)$ and $Soc(M)$ stand for the radical of M and the socle of M, respectively.

Any unexplained terminologies related to modules and rings can be found in [10] and we refer the readers to [9] for more information about graphs and related concepts.

2 Properties of co-intersection hypergraphs of submodules of modules

In this section we shall investigate some properties of the co-intersection hypergraph of a module $\mathcal{CIH}_R(M)$. We prove that a module M is a direct sum of its each two nontrivial submodules if and only if $\mathcal{CIH}_R(M)$ is null. We also characterize finite modules with exactly three nontrivial submodules via their co-intersection hypergraphs. According to the lattices of submodules of a finite module, we provide all co-intersection hypergraphs of order four.

Definition 2.1. Let M be a finite right R-module. We define a co-intersection hypergraph $\mathcal{CIH}_R(M)$ on M where the vertices are all nontrivial submodules of M namely V and a subset E_i of V with at least two elements, is a hyperedge of $\mathcal{CIH}_R(M)$ provided for each two $N, K \in E_i$, $N \cap K \neq 0$ and E_i is maximal with respect to this property.

We may view our definition as follows: Let M be a finite module and $V = \{K < M \mid K \neq 0\}$. If we consider $G(M)$, then any maximal subset E_i of V where the elements of E_i form a complete subgraph of $G(M)$, is a hyperedge in $\mathcal{CIH}_R(M)$. So we can say, the number of maximal subsets E_i of V, where the elements of E_i form a complete subgraph of $G(M)$ is equal to hyperedges of $\mathcal{CIH}_R(M)$.

The following provides an important characterization of modules such that their corresponding co-intersection hypergraphs are null.

Theorem 2.2. Let M be a finite R-module and N a nontrivial submodule of M. Then $deg_{\mathcal{CIH}_R(M)}N = 0$ if and only if M is a direct sum of its each two nontrivial submodules.

Proof. Let N be a nontrivial submodule of M such that N belongs to no hyperedge of $\mathcal{CIH}_R(M)$. Suppose that K is a nontrivial submodule of M different from N. Now, consider the submodule $N + K$ of M. If $N + K$ is nontrivial, then $N \cap (N + K) \neq 0$ which is a contradiction. It follows that $N+K = M$. Note that N does not belong to a hyperedge of $\mathcal{CIH}_R(M)$, implies that $N \oplus K = M$. Therefore, $N \oplus K = M$ for each nontrivial submodule K of M. Let the intersection of N with any other nontrivial submodule of M be zero. Hence, we conclude that N can not be contained in any other nontrivial submodule. So that, N is maximal submodule of M. Consider a submodule L of M such that $L \neq N$ and $L \neq K$. Then $N \oplus L = M = N \oplus K$. As N is maximal, then K and L are simple submodules of M. Since $L \neq K$ and both of them are simple, we conclude that $L \cap K = 0$. If $L \oplus K$ is nontrivial, then $N \oplus (L \oplus K) = M$. As $M = N \oplus L$, we have $L = L \oplus K$ which causes a contradiction. Therefore, $L \oplus K = M$. For the converse, suppose that N is an arbitrary nontrivial submodule of M. As $N \oplus K = M$ for each nontrivial submodule of M, we have $N \cap K = 0$. Therefore, $deg_{\mathcal{CIH}_R(M)}N = 0$. □

Corollary 2.3. *Let M be a finite R-module. Then $\mathcal{CIH}_R(M)$ is null if and only if M is a direct sum of its each two nontrivial submodules.*

Proof. Let M be finite such that $\mathcal{CIH}_R(M)$ is null. Then for any nontrivial submodule N of M, we have $deg_{\mathcal{CIH}_R(M)}(N) = 0$. The rest follows from Theorem 2.2. Conversely, suppose that M is a direct sum of its each two nontrivial submodules. Consider a nontrivial submodule N of M. Since $N \oplus K = M$ for each nontrivial submodule K of M different from N, we conclude that N does not belong to any hyperedge of $\mathcal{CIH}_R(M)$. Therefore, $\mathcal{CIH}_R(M)$ is null. □

Corollary 2.4. Let M be a finite right R-module. Then for every nontrivial submodule of N of M, we have $deg_{\mathcal{CIH}_R(M)}(N) \neq 0$ if and only if M is not a direct sum of its each two nontrivial submodules.

We next discuss about the connectivity of $\mathcal{CIH}_R(M)$ and its diameter.

Theorem 2.5. Let M be a finite right R-module. If $\delta(\mathcal{CIH}_R(M)) \geq 1$, then $\mathcal{CIH}_R(M)$ is connected and $d(\mathcal{CIH}_R(M)) \leq 2$.

Proof. Case 1: Suppose that M is not semisimple. Let N and K be two nontrivial distinct submodules of M. If N and K belong to a same hyperedge, then they are obviously connected. Otherwise, there are two distinct hyperedges E_i and E_j of $\mathcal{CIH}_R(M)$ such that $N \in E_i$ and $K \in E_j$. Since, M is not semisimple and M is finite, $Soc(M)$ is an essential submodule of M. That means that $Soc(M)$ is contained in both E_i and E_j. It follows that there is a path N E_i $Soc(M)$ E_j K from N to K.

Case 2: Let M be a semisimple finite right R-module. Since $\mathcal{CIH}_R(M)$ is not null, we conclude that $l_R(M) > 2$. Let N and K be two nontrivial distinct submodules of M. If $N \cap K \neq 0$, then both belong to a same hyperedge. Otherwise, $N \cap K = 0$. Consider the submodule $N + K$ of M. If $N + K$ is proper, then N and $N + K$ are included in a hyperedge E_i and K and $N + K$ belong to another hyperedge E_j. Then $N - E_i - (N + K) - E_j - K$ is a path. If $M = N + K$, then $M = N \oplus K$. As $l_R(M) > 2$, we conclude that either N is not a maximal submodule or K is not a maximal submodule. Suppose that K is not maximal. So that K is contained properly in a nontrivial submodule L of M. If $N \cap L = 0$, we show that $L \subseteq K$. Suppose that $l \in L$ is arbitrary. Then $l = n + k$ where $n \in N$ and $k \in K$. Then $l - k = n$ which is an element of $N \cap L = 0$. It follows that $l = n$ which implies that $L \subseteq N$, that will be a contradiction. Hence, $N \cap L \neq 0$. If we suppose $N, L \in E_t$ and $L, K \in E_s$, then N E_t L E_s K is a path from N to K. It is clear that in both cases $d(\mathcal{CIH}_R(M)) \leq 2$. □

The following is an easy characterization for a module such that its co-intersection hypergraph has just one hyperedge containing all nontrivial submodules of that module.

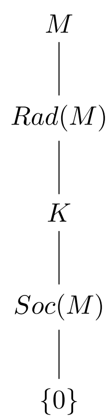
Proposition 2.6. Let M be a finite right R-module with at least two nontrivial submodules. Then $\mathcal{CIH}_R(M)$ has only a hyperedge containing all nontrivial submodules of M if and only if M is uniform.

Proof. Suppose that $\mathcal{CIH}_R(M)$ has only a hyperedge $E = \{M_i \mid 0 \neq M_i \leq M\}$. Let N be a nontrivial submodule of M. Then $N \in E$, implies that $N \cap K \neq 0$ for each nontrivial submodule K of M different from N. Therefore, N is essential in M. For the converse, let M be uniform. It follows that, the intersection of each two nontrivial submodules of M is nonzero. Hence the only hyperedge of $\mathcal{CIH}_R(M)$ is the set of all nontrivial submodules of M. $\square$

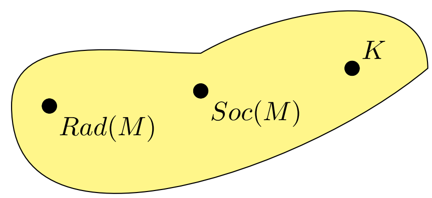
We next characterize finite modules with three nontrivial submodules via their co-intersection hypergraphs.

Theorem 2.7. Let M be a finite right R-module with three nontrivial submodules. Then one of the following holds for M:

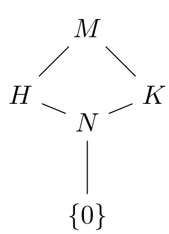
(1) M is linearly ordered. It means that submodules of M provide a chain. In this case, $\mathcal{CIH}_R(M)$ has a unique hyperedge $E = \{Soc(M), K, Rad(M)\}$:



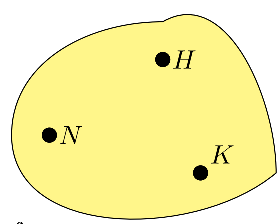
$\mathcal{CIH}_R(M)$ in this case is of the form:



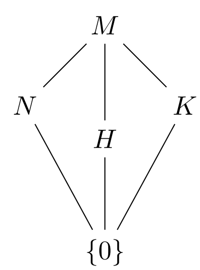
(2) M is a uniform right R-module which is not linearly ordered. The submodules of M satisfy in $H \cap K = N = Soc(M) = Rad(M)$ and $\mathcal{CIH}_R(M)$ has a unique hyperedge $E = \{H, K, Rad(M) = Soc(N)\}$.



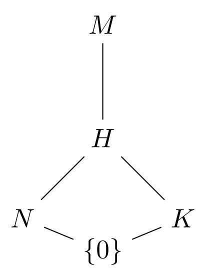
$\mathcal{CIH}_R(M)$ in this case is of the form:



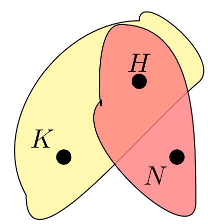
(3) The module M is semisimple and all nontrivial submodules of M are simple. In this case M can be written as a direct sum of each two nontrivial submodules. Accordingly $\mathcal{CIH}_R(M)$ is null.



(4) M is neither semisimple nor uniform. In this case, $Soc(M) = N \oplus K = H = Rad(M) \ll M$, $N \cap K = 0$, and the lattice of submodules of M is:



$\mathcal{CIH}_R(M)$ in this case is of the form:



Proof. (1) Let M be a linearly ordered module. Then it is clear that submodules of M form a chain, M must be uniform and $\mathcal{CIH}_R(M)$ has a unique hyperedge $E = \{Soc(M), K, Rad(M)\}$.

(2) Suppose that M is uniform but it is not a linearly ordered module. Then, there are two nontrivial submodules of M namely H and K such that $H \nsubseteq K$ and $K \nsubseteq H$. Being M uniform implies $H \cap K$ must be the third nontrivial submodule namely N. Therefore, H and K are both maximal in M and N is the only simple submodule of M, so that $Rad(M) = Soc(M) = N$. As M is uniform, $\mathcal{CIH}_R(M)$ has a unique hyperedge $E = \{Soc(M), H, K\}$.

(3) Let M be semisimple. Consider the nontrivial submodules N, H and K of M. Suppose that $N \oplus H = M$. Then $K \oplus N = M$ or $K \oplus H = M$. Suppose that $K \oplus N = M$. We show that $K \cap H = 0$. Otherwise, let $H \cap K \neq 0$. If $H \cap K = N$, then N must be contained in H, a contradiction. If $H \cap K = H$, then H is a direct summand of K as M is semisimple. Therefore, $H \oplus N = K$ which causes a contradiction. The case $H \cap K = K$ is the same. Now, consider the submodule $H + K$ of M. If $H + K = N$, then $H \cap N \neq 0$, a contradiction. The cases $H + K = H$ and $H + K = K$ imply contradictions. Hence $H \oplus K = M$. Note that, if $K \oplus H = M$ then by a same argument as above we can conclude that $K \oplus N = M$. Altogether we come to a conclusion that $\mathcal{CIH}_R(M)$ is null.

(4) Suppose that M is neither uniform nor semisimple. Since M is finite and is not a semisimple module, then $Soc(M)$ is an essential submodule of M. It follows that $Soc(N) = N \cap Soc(M) \neq 0$ and $Soc(K) = K \cap Soc(M) \neq 0$. As M is not uniform, then $N \cap K = 0$. Having M just three nontrivial submodules, we conclude that $N, K \subseteq Soc(M)$. Therefore, $H = Soc(M) = N \oplus K$, since $N \oplus K$ is a proper submodule of M. It is clear that M has just one maximal submodule that must be $Soc(M)$. Hence, $Rad(M) = Soc(M)$. Note that in this case, $\mathcal{CIH}_R(M)$ has just two hyperedges $\{H, N\}$ and $\{H, K\}$. $\square$

Corollary 2.8. Let M be a finite R-module with exactly three nontrivial submodules H_1, H_2 and H_3. Then $\mathcal{CIH}_R(M)$ either has just one hyperedge of the form $\{H_1, H_2, H_3\}$ or $\mathcal{CIH}_R(M)$ has two hyperedges $\{H_1, H_3\}$ and $\{H_2, H_3\}$ or $\mathcal{CIH}_R(M)$ is a null hypergraph.

The following includes examples of modules satisfying conditions in Theorem 2.7.

Example 2.9. (1) *Let K be a field. Consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in K\}$. Then the bimodule ${}_RR_R = M$ has exactly three nontrivial submodules $H = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in K\}$, $K = \{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in K\}$ and $N = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in K\}$. Then H and K are maximal submodules of M and $N = H \cap K = Rad(M) = Soc(M)$. Also, M is uniform and $\mathcal{CIH}_R(M)$ has a unique hyperedge $\{Soc(M), H, K\}$ (case 2 of Theorem 2.7).*
(2) *Let $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{Z}_4, b, c \in \mathbb{Z}_2\}$. Consider the R-module $M = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_4, b \in \mathbb{Z}_2\}$. Then M has just three nontrivial submodules $N = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a = 0, 2\}$, $K = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b = 0, 1\}$ and $H = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a = 0, 2,\ b = 0, 1\}$. Then $N \cap K = 0$, $N, K \subseteq H$, $N \oplus K = H = Rad(M) = Soc(M)$(case 4 of Theorem 2.7).*

Let M be a finite module with exactly four nontrivial submodules. We next show that $\mathcal{CIH}_R(M)$ can not have just two hyperedges $\{N, K\}$ and $\{H, L\}$.

Proposition 2.10. Let M be a finite right R-module with four nontrivial submodules N, K, L, H. Then $\mathcal{CIH}_R(M)$ can not have two hyperedges $\{N, K\}$ and $\{H, L\}$.

Proof. Let M be finite right R-module with four nontrivial submodules N, K, L, H. On the contrary, suppose that $\mathcal{CIH}_R(M)$ has just two hyperedges $\{N, K\}$ and $\{H, L\}$.

Case 1: Let M be semisimple. Since $\mathcal{CIH}_R(M)$ is not null, we conclude that $l_R(M) \geq 2$. As $\{N, K\}$ is a hyperedge in $\mathcal{CIH}_R(M)$, $N \cap K \neq 0$. As M is semisimple, $N \cap K$ is a direct summand of N. It follows that $(N \cap K) \oplus L = N$ or $(N \cap K) \oplus H = N$. Note that either $N \oplus H = M$ or $N \oplus L = M$. If $(N \cap K) \oplus L = N$, then $N \oplus H = M$ that is a contradiction. Otherwise, if $(N \cap K) \oplus H = N$ implies $N \oplus L = M$ which contradicts $L \cap H \neq 0$.

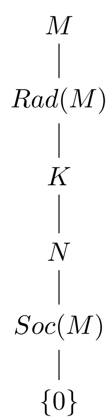
Case 2: Suppose that M is not semisimple. It follows that $Soc(M)$ is an essential submodule of M and hence $Soc(M)$ must be included in each of two hyperedges of $\mathcal{CIH}_R(M)$. It will be a contradiction. □

Proposition 2.11. Let M be a finite right R-module with exactly four nontrivial submodules. Then M is semisimple if and only if M can be written as a direct sum of each two nontrivial distinct submodules. In this case, $\mathcal{CIH}_R(M)$ is null.

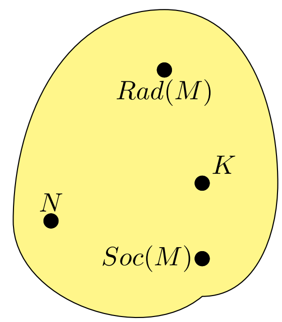
Proof. Let N, L, H, K be only nontrivial submodules of M. Since M is semisimple, then $M = N \oplus T$ for $T \in \{L, H, K\}$. Suppose that $T = L$. If N is not simple, there exists $P \lneq N$, and so $N = P \oplus Q$ since N is semisimple. Clearly, $P \neq Q$ and $P, Q \notin \{L, N\}$. Hence, it can be assumed that $P = H$ and $Q = K$. Since M has just four nontrivial submodules, then L must be simple. Thus, $M = H \oplus K \oplus L$ where $K \oplus L$ is a submodule of M such that $K \oplus L \notin \{N, H, K, L\}$, which is a contradiction. Hence, N is simple. By a similar argument, we can show L is simple. Moreover, similarly, it can be shown that $M = H \oplus T$ for $T \in \{N, K, L\}$. To sum up, $\mathcal{CIH}_R(M)$ is null. The converse is straightforward. □

Remark 2.12. According to lattice of submodules of a finite module M with four nontrivial submodules, we have the following cases:

(1) M is a linearly ordered module:

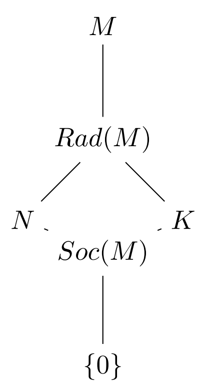


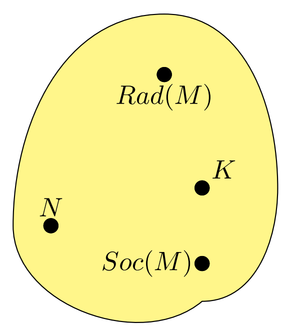
$\mathcal{CIH}_R(M)$ in this case is of the form:



In this case $\mathcal{CIH}_R(M)$ has a unique hyperedge containing all nontrivial submodules. For example consider the $\mathbb{Z}$-module $\mathbb{Z}_{32}$.

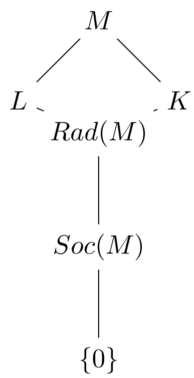
(2) M is uniform while it is not linearly ordered. By the lattice of submodules, $Rad(M)$ is the only maximal submodule of M and $Soc(M)$ is the only simple submodule of M:

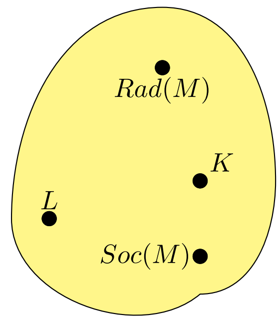




In this case $\mathcal{CIH}_R(M)$ has a unique hyperedge containing all nontrivial submodules.

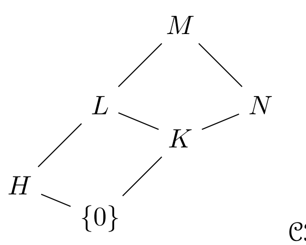
(3) M is uniform and it is not linearly ordered. According to the lattice, L and K are two maximal submodules of M:



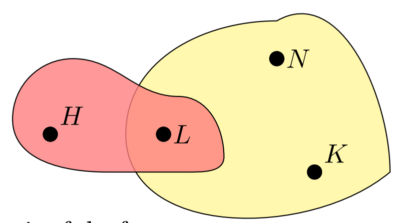


In this case $\mathcal{CIH}_R(M)$ has a unique hyperedge containing all nontrivial submodules.

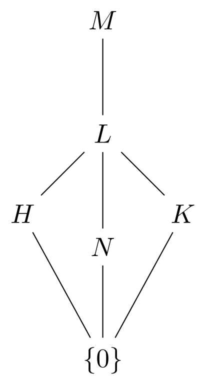
(4) L and N are two maximal submodules of M and $Rad(M) = L \cap N = K$. Also, H and K are simple submodules of M and $Soc(M) = H \oplus K = L$. As an example we can consider the $\mathbb{Z}$-module $\mathbb{Z}_{12}$.



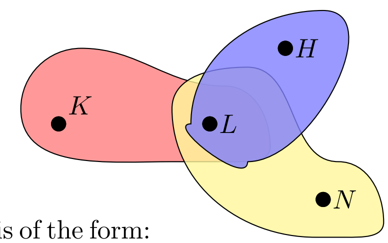
$\mathcal{CIH}_R(M)$ in this case is of the form:



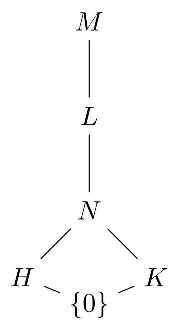
(5) The module M is neither semisimple nor uniform. According to the lattice, $Rad(M) = L$ is a maximal submodule of M, $Soc(M) = H \oplus N \oplus K = L$, and $H \cap N = H \cap K = N \cap K = 0$ such that H, N and K are simple submodules of M:



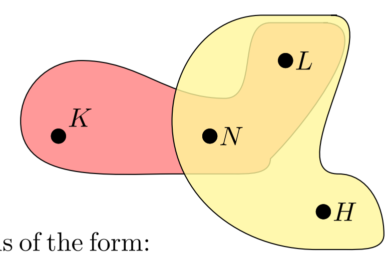
$\mathcal{CIH}_R(M)$ in this case is of the form:



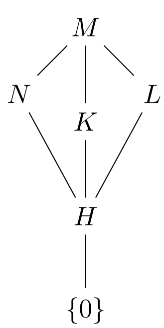
(6) Applying the lattice of submodules of M, we conclude that $Rad(M) = L$ is a maximal submodule of M, $Soc(M) = H \oplus K = N$, and $H \cap K = 0$ such that H and K are simple submodules of M:

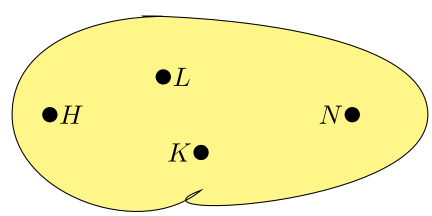


$\mathcal{CIH}_R(M)$ in this case is of the form:



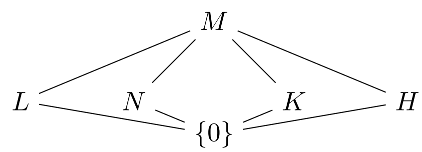
(7) According to the lattice, M is uniform, $Soc(M) = H$ is a simple submodule of M and $Rad(M) = N \cap K \cap L = H$ such that N, K and L are maximal submodules of M:





In this case, $\mathcal{CIH}_R(M)$ has a unique hyperedge containing all nontrivial submodules. For example consider $M = R = \frac{\mathbb{Z}_2[x,y]}{<x^2,y^2>}$.

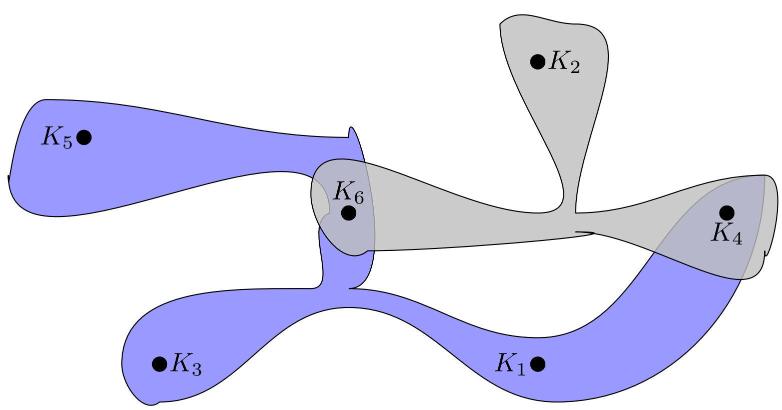
(8) M is semisimple. $M = H \oplus N = H \oplus K = H \oplus L = N \oplus K = N \oplus L = K \oplus L$, for maximal submodules H, N, K and L of M (we can consider the $\mathbb{Z}$-module $\mathbb{Z}_3 \oplus \mathbb{Z}_3$):



$\mathcal{CIH}_R(M)$ in this case is null.

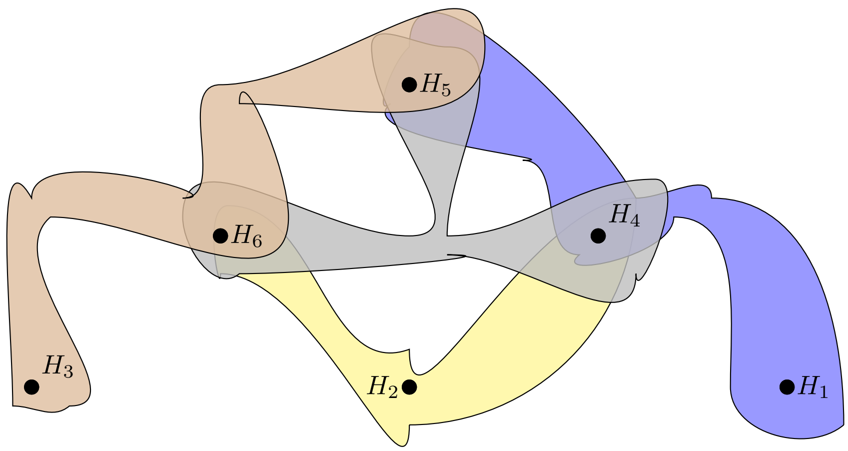
3 Examples of co-intersection hypergraphs of submodules of a module

Example 3.1. *Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^3q}$ where p and q are distinct prime numbers such that $p < q$. Then all nontrivial submodules are $K_1 =< p^2q >$, $K_2 =< p^3 >$, $K_3 =< pq >$, $K_4 =< p^2 >$, $K_5 =< q >$ and $K_6 =< p >$. Then we have $E = \{\{K_1, K_3, K_4, K_5, K_6\}, \{K_2, K_4, K_6\}\}$ and the hypergraph $\mathcal{CIH}_{\mathbb{Z}}(\mathbb{Z}_{p^3q})$ has the form:*



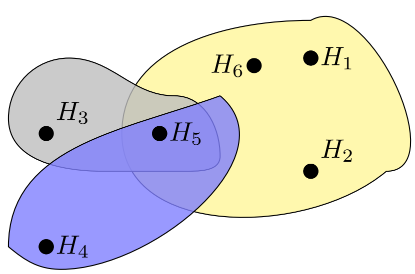
Example 3.2. *All nontrivial submodules of the $\mathbb{Z}$-module $\mathbb{Z}_{pqr}$ (where p, q, r are prime numbers and $p < q < r$) are $H_1 =< qr >$, $H_2 =< pr >$, $H_3 =< pq >$, $H_4 =< r >$, $H_5 =< q >$ and $H_6 =< p >$. Then we have*

$E = \{\{H_1, H_4, H_5\}, \{H_2, H_4, H_6\}, \{H_3, H_5, H_6\}, \{H_4, H_5, H_6\}\}$ and the hypergraph $\mathcal{CIH}_{\mathbb{Z}}(\mathbb{Z}_{pqr})$ has the form:

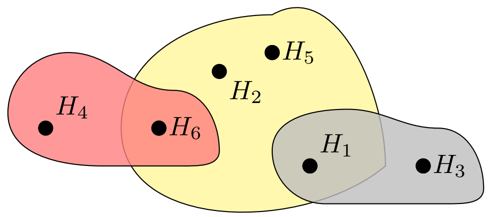


Example 3.3. *Consider the semisimple $\mathbb{Z}$-module $M = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Then $H_1 = \{(0,0), (0,1), (0,2)\}$, $H_2 = \{(0,0), (1,0), (2,0)\}$, $H_3 = \{(0,0), (1,1), (2,2)\}$ and $H_4 = \{(0,0), (1,2), (2,1)\}$ are all nontrivial submodules of M. Hence, the hypergraph $\mathcal{CIH}_{\mathbb{Z}}(M)$ is null.*

Example 3.4. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ where the nontrivial submodules are:*
$H_1 = \{(0,0), (0,1), (0,2), (0,3)\}$, $H_2 = \{(0,0), (1,1), (0,2), (1,3)\}$, $H_3 = \{(0,0), (1,0)\}$, $H_4 = \{(0,0), (1,2)\}$, $H_5 = \{(0,0), (0,2), (1,0), (1,2)\}$ and $H_6 = \{(0,0), (0,2)\}$. Hence, $E = \{\{H_1, H_2, H_5, H_6\}, \{H_3, H_5\}, \{H_4, H_5\}\}$, and the hypergraph $\mathcal{CIH}_{\mathbb{Z}}(M)$ has the form:



Example 3.5. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_6$ where all nontrivial submodules are:*
$H_1 = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\}$, $H_2 = \{(0,0),(0,2),(0,4)\}$, $H_3 = \{(0,0),(0,3)\}$, $H_4 = \{(0,0),(1,0)\}$, $H_5 = \{(0,0),(0,2),(1,1),(1,3),(0,4),(1,5)\}$ *and* $H_6 = \{(0,0),(1,2),(0,4),(1,0),(0,2),(1,4)\}$. *Hence,*
$E = \{\{H_1,H_3\},\{H_4,H_6\},\{H_1,H_2,H_5,H_6\}\}$, *and the hypergraph* $\mathcal{CIH}_{\mathbb{Z}}(M)$ *has the form:*



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