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Complete parts and subhypergroups in reversible regular hypergroups

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Abstract

In this paper we analyse the center and centralizer of an element in the context of reversible regular hypergroups, in order to obtain the class equation in regular reversible hypergroups, by using complete parts. After an introduction in which basic notions and results of hypergroup theory are presented, particularly complete parts, then we give several properties, characterisations and also examples for the center and centralizer of an element for two classes of hypergroups. The next paragraph is dedicated to hypergroups associated with binary relations. We establish a connection between several types of equivalence relations, introduced by J.Jantosciak, such as the operational relation, the inseparability and the essential indistin-guishability and the conjugacy relation for complete hypergroups. Finally, we analyse Rosenberg hypergroup associated with a conjugacy relation.

1 Introduction

We present some basic notins and results in Hypergroup theory, which was introduced in 1934 by F. Mary and is developing nowadays especially for its applications, see [8] by P. Corsini and V. Leoreanu. Also the theoretical point of view is analysed in hundreds of papers of hyperstructures and several books, such as [4] by P. Corsini or [12] by B. Davvaz.

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In this paper we analyse some important subhypergroups, such as the centralizer of an element, the center of a hypergroup in the context of reversible re-gular hypergroups, by using complete parts. The center of a reversible regular hypergroup was already introduced in [15], but we propose here an easier equivalent definition and obtain some new results using it. The heart of re-gular reversible hypergroups was studied in [16]. The center and centralizer of an element were studied in poligroups by B. Davvaz, see [12] and analysed in other papers by A. Hokmabadi et al, see[13]. The last part of the paper presents new results on hypergroups associated with binary relations.

Complete hypergroups, introduced in [9] are reversible regular hypergroups. An important class of complete hypergroups are hypergroups of associativity, introduced in [2]. They are the fist kind of complete hypergroups ever introduced.

Complete parts were introduced by M. Koskas and subsequently analysed by P. Corsini, I. Sureau, M. de Salvo, R. Migliorato. The following results can be found in [4] or [8].

A subset A of a hypergroup H is called *complete* if the following implication holds:

$$\forall n \in \mathbf{N}^*, \forall x_1, x_2, ..., x_n \in H, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

If A is a nonempty subset of H, then the intersection of all complete parts of H, which contain A is called the *complete closure* of A in H and it is denoted by $\mathcal{C}(A)$.

The *heart* ω_H of a hypergroup $(H, \cdot)$ is the kernel of the canonical projection $p : H \to H/\beta,\ p(x) = \beta(x)$, where β is an equivalence relation defined in H as follows:

$x \beta y \iff \exists n \in \mathbf{N}^*, \exists z_1, z_2, ..., z_n \in H :\ \{x, y\} \subseteq \prod_{i=1}^{n} z_i$.

Notice that the quotient H/β is a group and ω_H is the smallest subhypergroup, complete part of H, with respect to inclusion.

If A is a nonempty subset of H, one has $\mathcal{C}(A) = \cup_{a \in A} \mathcal{C}(a)$ and moreover, $\mathcal{C}(A) = \omega_H \cdot A = A \cdot \omega_H$. Hence if $a, b \in H$ then $\mathcal{C}(ab) = \mathcal{C}(a)\mathcal{C}(b)$, that is $ab\omega_H = a\omega_H b\omega_H$.

A regular hypergroup $(H, \cdot)$ is a hypergroup with identities and such that all elements have inverses. Denote by $i(a)$ the set of inverses of an element a of H. Since ω_H is a closed subhypergroup, it follows that $i(u) \subseteq \omega_H$ for $u \in \omega_H$.

A regular hypergroup $(H, \cdot)$ is called reversible if for all $x, y, z \in H$ such that $x \in y \cdot z$ it follows that $y \in x \cdot z'$ and $z \in y' \cdot x$ for some $y' \in i(y), z' \in i(z)$.

A nonempty set A of H is *normal* if for all $a \in H$, we have $a \cdot A = A \cdot a$.

This paper is organized as follows: the first section containes results about complete parts in reversible regular hypergroups, in the second section we introduce the center of a reversible regular hypergroup and the centralizer of an element and we give some examples. The last section presents some new results about hypergroups associated with binary relations, more exactly we analyse Rosenberg hypergroup for the conjugacy relation.

2 Complete parts in reversible regular hypergroups

In what follows, we consider $(H, \cdot)$ a reversible regular hypergroup.

First of all, in [11] (Theorem 4.2.7 and 4.2.11) it is proved that $\mathcal{C}(a) = \beta(a)$. This fact can be proved independently as follows.

Lemma 1. *Let H be a hypergroup. Then $\mathcal{C}(a) = \beta(a)$, for all $a \in H$.*

Proof. Let $(H/\beta, \otimes)$ be the fundamental group of H and $x \in \mathcal{C}(a)$. So there exists $w_1 \in \omega_H$ such that $x \in w_1 a$. Thus $\beta(x) = \beta(w_1) \otimes \beta(a) = \beta(a)$ and hence $x \in \beta(a)$.

Conversely, let $x \in \beta(a)$. Then there exist $z_1, z_2, \ldots, z_n$ such that the next relation holds on $\{a, x\} \subseteq \prod_{i=1}^n z_i$. Thus $a \in \prod_{i=1}^n z_i \cap \mathcal{C}(a)$ implies $x \in \mathcal{C}(a)$. □

According to the previous lemma we have $\mathcal{C}(ab) = \beta(x)$ for every $x \in ab$.

From the above Lemma we obtain the next result as a corollary:

Lemma 2. *Let a, b, c, d be elements of H.*

i) If $\mathcal{C}(a) \cap \mathcal{C}(b) \neq \emptyset$ then $\mathcal{C}(a) = \mathcal{C}(b)$;

ii) If $\mathcal{C}(a) \cap \mathcal{C}(bcd) \neq \emptyset$ then $a \in \mathcal{C}(bcd)$;

iii) If $\mathcal{C}(ab) \cap \mathcal{C}(cd) \neq \emptyset$ then $\mathcal{C}(ab) = \mathcal{C}(cd)$;

iv) If $\mathcal{C}(\prod_{i=1}^n a_i) \cap \mathcal{C}(\prod_{i=j}^m b_j) \neq \emptyset$ then $\mathcal{C}(\prod_{i=1}^n (a_i) = \mathcal{C}(\prod_{i=j}^m b_j)$.

Remark 3. *$\{\mathcal{C}(a) \mid a \in H\} = \{\mathcal{C}(a_i)\}_{i=1}^r$ is a partition of H.*

Let us define now the following relation in H :

$$a \sim b \iff \exists c \in H : \mathcal{C}(ca) = \mathcal{C}(bc).$$

Theorem 4. *The relation " $\sim$ " is an equivalence relation.*

Proof. For $c = a$ we obtain the reflexivity. For symmetry, let $c \in H$ be such that $\mathcal{C}(ca) = \mathcal{C}(bc)$. We check that there is $d \in H$ such that $\mathcal{C}(db) = \mathcal{C}(ad)$. We have $cac'\omega_H = b\omega_H$, where $c' \in i(c)$. Hence, $c'cac'\omega_H = c'b\omega_H$, which means that $ac'\omega_H = c'b\omega_H$. Hence take $d = c'$ and we get the conclusion.

Let us check now the transitivity. Let $d, e \in H$ be such that $da\omega_H = bd\omega_H$ and $eb\omega_H = ce\omega_H$. We find an element $f \in H$ such that $fa\omega_H = cf\omega_H$. Set $d' \in i(d)$ and $e' \in i(e)$. We have $dad'\omega_H = b\omega_H$ and $e'eb\omega_H = e'ce\omega_H$. Hence $dad'\omega_H = e'ce\omega_H$, whence $da\omega_H = e'ced\omega_H$, so $eda\omega_H = ced\omega_H$. There are $f, f_1 \in ed$ such that $f_1a\omega_H \cap cf\omega_H \neq \emptyset$. But $f\beta f_1$, so $f\omega_H = f_1\omega_H$. Hence $fa\omega_H \cap cf\omega_H \neq \emptyset$ and according to previous Lemma, *iii*) we have $fa\omega_H = cf\omega_H$.

Thus " $\sim$ " is an equivalence relation in H. □

Denote by $[a]$ the conjugacy class of an element a of H. We have

$$[a] = \{b \in H \mid \exists c \in H : \mathcal{C}(ca) = \mathcal{C}(bc)\},$$

or equivalently, by Lemma 2, *ii*)

$$[a] = \{b \in H \mid \exists c \in H : b \in \mathcal{C}(cac'), \text{where } c' \in i(c)\},$$

whence

$$[a] = \bigcup_{c \in H, c' \in i(c)} \mathcal{C}(cac').$$

Theorem 5. *Let $u \in cac'$, where $c' \in i(c)$. Then $\mathcal{C}(cac') = \mathcal{C}(u)$.*

Proof. It follows from Lemma 1. □

3 The center of a reversible regular hypergroup and the centralizer of an element

Definition 6. *Let $(H, \cdot)$ be a reversible regular hypergroup. The following set*

$$Z(H) = \{a \in H \mid \forall b \in H,\ \mathcal{C}(ab) = \mathcal{C}(ba)\}$$

is called the center of H.

Theorem 7. *$Z(H)$ is a complete part and a normal subhypergroup in a reversible regular hypergroup H.*

Proof. Let $a_1, a_2 \in Z(H)$ and $a \in a_1a_2$. So, $\mathcal{C}(a_1b) = \mathcal{C}(ba_1)$, $\mathcal{C}(a_2b) = \mathcal{C}(ba_2)$. We have $\mathcal{C}(ab) \subseteq \mathcal{C}(a_1a_2b) = \mathcal{C}(ba_1a_2)$ whence there exists $t \in a_1a_2$, such that $\mathcal{C}(ab) \cap \mathcal{C}(bt) \neq \emptyset$ and according to Lemma 2, *iii)* we have the following equality $\mathcal{C}(ab) = \mathcal{C}(bt) = \mathcal{C}(ba)$ since $a, t \in a_1a_2$ so $a\beta t$, that is $\mathcal{C}(a) = \mathcal{C}(t)$. Hence $Z(H)$ is a subsemihypergroup of H.

Let us check now that $Z(H) \subseteq a_1Z(H)$. Moreover, we show that if $a_2 \in a_1c$, then $c \in Z(H)$, which means that $Z(H)$ is a closed subhypergroup. From $a_2 \in a_1c$ it follows that $c \in a_1'a_2$ for some $a_1' \in i(a_1)$. Clearly, $a_1' \in Z(H)$, whence $c \in Z(H)$.

Notice that $\omega_H \subseteq Z(H)$, hence $Z(H)$ is a complete part subhypergroup of H.

Finally, check that $xZ(H) = Z(H)x$, for all $x \in H$. Let $t \in xa$, where a is an whichever element of $Z(H)$. Then $t \in \mathcal{C}(xa) = \mathcal{C}(ax)$, whence there exists $u \in \omega_H$ such that $t \in uax$. But $ua \subseteq Z(H)$, which means that $t \in Z(H)x$. Similarly, we obtain the inverse inclusion, hence $Z(H)$ is normal. □

Corollary 8. *The quotient $H/Z(H)$ is a group.*

Remark 9. *If $Z(H) = H$, then the quotient group H/β is commutative. This does not mean that H is commutative, as we can see in the first example of Section 4.*

The following example satisfies the above Remark.

Example 10. *Consider $H = (\{1, 2\}, \circ)$ where $\circ$ defines as*

$\circ$	1	2
1	1	2
2	$\{1, 2\}$	$\{1, 2\}$

Then H is the smallest non-commutative hypergroup such that $Z(H) = H$.

Remark 11. *For all $a \in Z(H)$, $[a] = \mathcal{C}(a)$.*

Indeed, we have $Z(H) = \{a \in H \mid \forall b \in H, \forall b' \in i(b),\ \mathcal{C}(a) = \mathcal{C}(bab')\}$.

Definition 12. *Let a be an element of a reversible regular hypergroup. The following set*

$$C_H(a) = \{b \in H \mid \mathcal{C}(ab) = \mathcal{C}(ba)\}$$

is called the centralizer of a in H.

In a similar way as for Theorem 4, it can be checked the next result:

Theorem 13. *For all $a \in H$, $C_H(a)$ is a subhypergroup of H.*

The centralizer of an element was introduced in [15]. We propose here an easier definition, which is equivalent to that one given in [15].

Definition 14. *(see [15]) Let $(H, \cdot)$ be a reversible regular hypergroup and $x \in H$. The set*

$$C_H(x) = \{t \in H \mid \exists n \in \mathbf{N}, \forall i \in I_n = \{1, 2, ..., n\}, \exists g_i \in H,$$

$$t \in \prod_{i=1}^{n} g_i : \forall i \in I_n, \mathcal{C}(g_i x) \cap \mathcal{C}(x g_i) \neq \emptyset\}$$

is called the centralizer of x in H.

Theorem 15. *Definitions 12 and 14 are equivalent.*

Proof. First of all, we notice that the condition $\mathcal{C}(g_i x) \cap \mathcal{C}(x g_i) \neq \emptyset$ is equivalent to $\mathcal{C}(g_i x) = \mathcal{C}(x g_i)$, according to Lemma 2, *iii*). So $\mathcal{C}(\prod_{i=1}^{n} g_i x) = \mathcal{C}(x \prod_{i=1}^{n} g_i)$, hence $\mathcal{C}(tx) \subseteq \mathcal{C}(\prod_{i=1}^{n} g_i x) = \cup_{v \in \prod_{i=1}^{n} g_i} \mathcal{C}(xv)$, which means there exist $v \in \prod_{i=1}^{n} g_i$ for which $\mathcal{C}(tx) \cap \mathcal{C}(xv) \neq \emptyset$. By Lemma 2, *iii*) we get $\mathcal{C}(tx) = \mathcal{C}(xv) = \mathcal{C}(xt)$, since $v \beta t$. Thus, Definition 12 implies Definition 14.

Conversely, let $t \in H$ be such that $\mathcal{C}(tx) = \mathcal{C}(xt)$. Consider $g_1 = t$ and so $t \in C_H(x)$, defined by Definition 14. Therefore, Definitions 12 and 14 are equivalent. $\square$

Let us see now what the class equation becomes for reversible reugular hypergroups. For all $a \in H$, the conjugacy class of a is $[a] = \bigcup_{c \in H, c' \in i(c)} \mathcal{C}(cac')$.

Theorem 16. *Let $c, d \in H$, $c' \in i(c), d' \in i(d)$. Then $\mathcal{C}(cac') = \mathcal{C}(dad')$ if and only if $cC_H(a) = dC_H(a)$.*

Proof. Indeed, we have $\mathcal{C}(cac') = \mathcal{C}(dad')$ if and only if $\mathcal{C}(d'ca) = \mathcal{C}(ad'c)$, which is equivalent to $\mathcal{C}(va) = \mathcal{C}(av)$, for all $v \in d'c$, according to Lemma 2,*iii*), which means that $d'c \subseteq C_H(a)$. So, $c \in dd'c \subseteq dC_H(a)$. Similarly, $d \in cC_H(a)$, hence $cC_H(a) = dC_H(a)$. $\square$

Notice that, if $\mathcal{C}(cac') \cap \mathcal{C}(dad') \neq \emptyset$, then $\mathcal{C}(cac') = \mathcal{C}(dad')$, according to Lemma 2, *iv*).

If H be a finite reversible regular hypergroup, then we have

$$|H| = |\{[a] \mid a \in H\}|.$$

If $a \in Z(H)$, then $[a] = \mathcal{C}(a) = a\omega_H$.
If $a \in H - Z(H)$, then $[a] = \bigcup_{c \in H, c' \in i(c), u \in cac'} \mathcal{C}(u)$.

Hence we obtain again the partition $\{\mathcal{C}(a_i)\}_{i=1}^r$ of H.

Therefore, the class equation in reversible regular hypergroups becomes at the end

$$|H| = \sum_{i=1}^{r} |\mathcal{C}(a_i)| = \sum_{i=1}^{r} |a_i\omega_H|.$$

Now, we analyse two noncommutative reversible regular hypergroups, given in [4] and we find the center and the centralizer of an element for each of them.

Example 17. *Let $(H, \cdot)$ be a hypergroup. Let G be a group and $\{A_i\}_{i\in G}$ be a family of nonempty sets such that $A_1 = H$, where 1 is the identity of G and $A_i \cap A_j = \emptyset$ for different indexes $i, j \in G$.*

Set $K = \cup_{i\in G} A_i$ and consider the following hyperoperation in K :

$$\forall (x, y) \in H^2,\ x \circ y = xy, \textit{the hyperproduct in } H$$

$$\forall (x, y) \in A_i \times A_j, \textit{if } (i, j) \neq (1, 1) \textit{and } ij = k,\ x \circ y = A_k.$$

Then $(K, \circ)$ is a hypergroup, called (H, G)-hypergroup.

We have $\omega_K = H$. If H is a reversible regular hypergroup, then K is regular and reversible, too. The identities of K are exactly the identities of H and if $a \in A_i,\ i \neq 1$, then $i(a) = A_{i^{-1}}$.

Clearly, $\omega_K = H \subseteq Z(K)$. If $a \in Z(K)$ then for all $b \in K$ we have $abH = baH$. Suppose $a \in A_i, i \neq 1$ and take $b \in A_j$. This means that $ij = ji$, for all $j \in G$ if and only if $i \in Z(G)$.

We obtain the following result:

Proposition 18. *If $(K, \circ)$ is a (H, G)-hypergroup, then*

i) $Z(K) = \cup\{A_i \mid i \in Z(G)\}$.

ii) If $a \in H$, then $C_K(a) = K$. If $a \in A_i,\ i \neq 1$ then $C_K(a) = \cup_{j\in C_G(i)} A_j$.

Notice that if G is commutative, then $Z(K) = K$, which means that K/β_K is a commutative group, but K is not necessary a commutative hypergroup. Indeed, K is commutative if and only if G and H are both commutative.

Also, notice that $C_K(a) = K$ means that $\beta_K(a)$ commutes with all elements of K/β_K. It is possible that a does not commute with all elements of H.

Example 19. *Let $(H, \cdot)$ be a hypergroup and $\{A(x)\}_{x \in H}$ be a family of nonempty sets, such that $\forall (x, y) \in H^2,\ x \neq y$ we have $A(x) \cap A(y) = \emptyset$.*

Set $K_H = \cup_{x \in H} A(x)$ and define $\forall a \in K_H,\ g(a) = x \iff a \in A(x)$. Consider the following hyperoperation in K_H :

$$\forall (a, b) \in K_H^2,\ a \circ b = \cup_{z \in g(a) g(b)} A(z).$$

Then $(K_H, \circ)$ is a hypergroup.

If H is a reversible regular hypergroup, then K_H is also regular and reversible.

If P is a nonempty subset of H, denote $K(P) = \cup_{x \in P} A(x)$.

Then the set of identities $E(K_H)$ of K_H is $K(E(H))$, while the heart $\omega_{K_H} = K(\omega_H)$.

For all $a \in K_H$, we have $i(a) = K(i(g(a))) = g^{-1}(i(g(a))$.

By a direct check, we obtain:

Proposition 20. *If $(K_H, \circ)$ is the hypergroup defined above, then*

i) $Z(K_H) = K(Z(H))$.

ii) *If $a \in K_H$, then $C_{K_H}(a) = K(C_H(g(a)))$.*

4 Hypergroups associated with binary relations

First, we recall a representation theorem for complete hypergroups. In a complete hypergroup all hyperproducts are complete parts.

Theorem 21. *A hypergroup H is complete if and only if $H = \bigcup\limits_{g \in G} A_g$, where G and A_g satisfy the conditions:*

1) *$(G, \cdot)$ is a group;*
2) *for all $(g_1, g_2) \in G^2$, $g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \varnothing$;*
3) *if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.*

If G is a commutative group, then H is a complete commutative hypergroup, that is a join space.

J. Jantosciak [14] associated three equivalence relations with an arbitrary hypergroup $(H, \circ)$. These equivalence relations were analysed in [10]. Let us recall them.

The operational relation, denoted by " $\sim_o$ " , is defined as follows:

$$x \sim_o y \Leftrightarrow a \circ x = a \circ y;\ x \circ a = y \circ a;\ \forall\ a \in H.$$

The inseparability, denoted by " $\sim_i$ ", as follows:

$$x \sim_i y \Leftrightarrow \text{ for } a,\, b \in H,\, x \in a \circ b \Leftrightarrow y \in a \circ b.$$

The essential indistinguishability, denoted by " $\sim_e$ "

$$x \sim_e y \Leftrightarrow x \sim_o y \text{ and } x \sim_i y.$$

Denote by $\hat{x}_o$, $\hat{x}_i$, $\hat{x}_e$ the equivalence class of x with respect to " $\sim_o$ ", " $\sim_i$ ", " $\sim_e$ ", respectively.

We intend to establish a connection between these equivalence relations and the conjugacy relation, defined in section 2, in the context of complete hypergroups.

We give first some results.

Proposition 22. *If $(H, \circ)$ is a complete hypergroup and G is the associated group of H, according Theorem 21, then :*

$$\hat{x}_o = \hat{x}_i = \hat{x}_e = A_g^x, \text{ where } g \in G \text{ and } x \in A_g.$$

Proof. Let H be a complete hypergroup and G be the associated group, such that $|G| = n$, $n \in \mathbb{N}^*$ and A_{g_i}, $i = \overline{1, n}$ are the associated nonempty subsets, by Theorem 21. Let $(x, y) \in H^2$ be such that $x \circ a = y \circ a$ and $a \circ x = a \circ y$, for all $a \in H$. Since H is a complete hypergroup, it follows that for $x \in H$ there is a unique $g_1 \in G$ such that $x \in A_{g_1}$ and for $y \in H$ there is a unique $g_2 \in G$ such that $y \in A_{g_2}$. Finally, for $a \in H$ there is a unique $g_a \in G$ such that $a \in A_{g_a}$ Hence

$$A_{g_1 g_a} = A_{g_2 g_a} \Rightarrow g_1 = g_2 \Rightarrow \{x, y\} \subseteq A_{g_1}.$$

Therefore, $x \sim_o y$ if x and y belong to the same set.

Clearly, for $x,\, y \in A_g$ it follows that $x \sim_o y$. Hence, $\hat{x}_o = A_g^x$, where $g \in G$ and $x \in A_g$. In a similar way, we reason for relation " $\sim_i$ " $\square$

In what follows, we associate the Rosenberg hypergroup to the complete hypergroup H, with respect to conjugacy relation" $\sim_H$ ", redenoted by ρ .

Let $(H_\rho, \circ_\rho)$ be the Rosenberg hypergroup, for which the hyperoperation " $\circ_\rho$ " is defined as follows:

$$x \circ_\rho y = \{z \in H|\ (x, z) \in \rho \text{ or } (y, z) \in \rho\}. \tag{1}$$

We intend to analyse what relations " $\sim_o$ ", " $\sim_i$ ", " $\sim_e$ " become in the context of Rosenberg hypergoup and to establish a connection with ρ, defined in a complete hypergroup H. Rosenberg found conditions on R, such that H_R is a hypergroup or a join space.

Theorem 23. *H_R is a hypergroup if and only if:*
1) *$D(R) = H$;*
2) *$R(R) = H$;*
3) *$R \subseteq R^2$;*
4) *$(a, x) \in R^2 \Rightarrow (a, x) \in R$, for any x an outer element of R.*

Let

$$\begin{aligned} P &= \{x \in H | \ x \notin x \circ x\}; \\ K &= \{e \in H | \ P \subset e \circ e\}. \end{aligned} \tag{2}$$

Theorem 24. *H_R is regular if and only if $K \neq \varnothing$.*

Theorem 25. *If $K \neq \varnothing$ and R is symmetric, then H_R is a regular reversible hypergroup.*

Remark 26. *The hypergroup $(H_\rho, \circ_\rho)$ is regular and reversible according to Theorems 24 and 25.*

Therefore,

$$x \sim_{o_\rho} y \Leftrightarrow a \circ_\rho x = a \circ_\rho y; \ x \circ_\rho a = y \circ_\rho a; \ \forall \ a \in H_\rho.$$

Proposition 27. *Let $(H_\rho, \circ_\rho)$ be the hypergroup associated with the complete hypergroup H. We have*

$$x \sim_{o_\rho} y \text{ if and only if } (x, y) \in \rho.$$

Proof. Denoting the he ρ-equivalence class of x by $\hat{x}_\rho$, we have $x \circ_\rho y = \hat{x}_\rho \cup \hat{y}_\rho$. So, we have $x \sim_{o_\rho} y$ iff $\hat{a}_\rho \cup \ \hat{x}_\rho = \hat{a}_\rho \cup \hat{y}_\rho$, for every $a \in H$. This is clearly equivalent to $\hat{x}_\rho = \hat{y}_\rho$. In conclusion $(x, y) \in \rho$. □

Proposition 28. *Let $(H_\rho, \circ_\rho)$ be the hypergroup associated with a complete hypergroup H. Then*

$$x \sim_i y, \text{ if and only if } (x, y) \in \rho.$$

Proof. $x \sim_i y$ if and only if for $a, \ b \in H$, $x \in a \circ_\rho b \Leftrightarrow y \in a \circ_\rho b$. So, $x \in \hat{a}_\rho \cup \hat{b}_\rho \Leftrightarrow y \in \hat{a}_\rho \cup \hat{b}_\rho$ for all a, $b \in H$. This implies it is equivalent to the condition $x \in \hat{a}_\rho \Leftrightarrow y \in \hat{a}_\rho$ for all $a \in H$, which holds if and only if $x \rho y$. □

Remark 29. *Propositions 27 and 28 hold for all equivalence relations.*

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