



On 1-absorbing δ -primary ideals

Abdelhaq El Khalfi, Najib Mahdou, Ünsal Tekir and Suat Koç

Abstract

Let R be a commutative ring with nonzero identity. Let $\mathcal{J}(R)$ be the set of all ideals of R and let $\delta : \mathcal{J}(R) \rightarrow \mathcal{J}(R)$ be a function. Then δ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let δ be an expansion function of ideals of R . In this paper, we introduce and investigate a new class of ideals that is closely related to the class of δ -primary ideals. A proper ideal I of R is said to be a 1-absorbing δ -primary ideal if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \delta(I)$. Moreover, we give some basic properties of this class of ideals and we study the 1-absorbing δ -primary ideals of the localization of rings, the direct product of rings and the trivial ring extensions.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. If R is a ring, then \sqrt{I} denotes the radical of an ideal I of R , in the sense of [12, page 17]. Let also $\text{Spec}(R)$ denotes the set of all prime ideals of R .

The prime ideal, which is an important subject of ideal theory, has been widely studied by various authors. Among the many recent generalizations of the notion of prime ideals in the literature, we find the following, due to Badawi [2]. A proper ideal I of a ring R is said to be a 2-absorbing ideal if whenever

Key Words: 1-absorbing prime ideal, 1-absorbing δ -primary ideal, δ -primary ideal, trivial ring extension.

2010 Mathematics Subject Classification: 13A99, 13C13.

Received: 09.03.2021

Accepted: 30.04.2021

$a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this case $\sqrt{I} = P$ is a prime ideal with $P^2 \subseteq I$ or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are incomparable prime ideals with $\sqrt{I}^2 \subseteq I$, cf. [2, Theorem 2.4]. Recently, Badawi and Yetkin [4] consider a new class of ideals called the class of 1-absorbing primary ideals. A proper ideal I of a ring R is called a 1-absorbing primary ideal of R if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. In [14], A. Yassine et. al introduced the concept of 1-absorbing prime ideals which is a generalization of prime ideals. A proper ideal I of a ring R is a 1-absorbing prime ideal if whenever we take nonunit elements $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in I$. In this case $\sqrt{I} = P$ is a prime ideal, cf. [14, Theorem 2.3]. If R is a ring in which exists a 1-absorbing prime ideal that is not prime, then R is a local ring, that is a ring with one maximal ideal.

Let $\mathcal{J}(R)$ be the set of all ideals of a ring R . Zhao [15] introduced the concept of expansion of ideals of R . We recall from [15] that a function $\delta : \mathcal{J}(R) \rightarrow \mathcal{J}(R)$ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Note that there are explanatory examples of expansion functions included in [15, Example 1.2] and [3, Example 1]. In addition, recall from [15] that a proper ideal I of R is said to be a δ -primary ideal of R if whenever $a, b \in R$ with $ab \in I$, we have $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R . Also, recall from [5] that a proper ideal I of R is called a δ -semiprimary ideal of R if $ab \in I$ implies $a \in \delta(I)$ or $b \in \delta(I)$. In this paper, we introduce and investigate a new concept of ideals that is closely related to the class of δ -primary ideals. A proper ideal I of R is said to be a 1-absorbing δ -primary ideal if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \delta(I)$. For example, let $\delta_1 : \mathcal{J}(R) \rightarrow \mathcal{J}(R)$ such that $\delta_1(I) = \sqrt{I}$ for each ideal I of R . Then δ_1 is an expansion function of ideals of R , and hence a proper ideal I of R is a 1-absorbing δ_1 -primary ideal of R if and only if I is a 1-absorbing primary ideal of R . Among many results in this paper are given to disclose the relations between this new class and others that already exist. The reader may find it helpful to keep in mind the implications noted in the following figure.

prime ideal \longrightarrow 1-absorbing prime ideal \longrightarrow 1-absorbing δ -primary ideal

Among other things, we give an example of 1-absorbing δ -primary ideal that is not 1-absorbing prime ideal (Example 2.3). Also, we show (Theorem 2.6) that if a ring R admits a 1-absorbing δ -primary ideal of R that is not a δ -primary ideal, then R is a local ring. Moreover, we prove that if R is a chained ring with maximal ideal M , then the only 1-absorbing δ -primary

ideals of R are M^2 and the δ -primary ideals of R (Theorem 2.11). Finally, we give an idea about 1-absorbing δ -primary ideals of the localization of rings, the direct product of rings and the trivial ring extensions.

2 Main Results

We start this section by the following definition.

Definition 2.1. *A proper ideal I of a ring R is called a 1-absorbing δ -primary ideal if whenever $abc \in I$ for some nonunit elements $a, b, c \in R$, then $ab \in I$ or $c \in \delta(I)$.*

Remark 2.2. *Let R be a ring, I a proper ideal of R and δ be an expansion function of $\mathcal{J}(\mathcal{R})$.*

- (1) *If $\delta(I) = I$, then I is a 1-absorbing δ -primary ideal of R if and only if it is a 1-absorbing prime ideal.*
- (2) *If $\delta(I) = \sqrt{I}$, then I is a 1-absorbing δ -primary ideal of R if and only if it is a 1-absorbing primary ideal.*
- (3) *Every 1-absorbing prime ideal is a 1-absorbing δ -primary ideal.*
- (4) *Every δ -primary ideal is a 1-absorbing δ -primary ideal.*
- (5) *Let γ be an expansion function of $\mathcal{J}(\mathcal{R})$ such that $\delta(I) \subseteq \gamma(I)$. If I is a 1-absorbing δ -primary ideal of R , then I is a 1-absorbing γ -primary ideal of R .*

Next, we give an example of a 1-absorbing δ -primary ideal that is not a 1-absorbing prime ideal.

Example 2.3. *Let $R := K[[X_1, X_2, X_3]]$ be a ring of formal power series where K is a field. Consider the expansion function $\delta : \mathcal{J}(\mathcal{R}) \rightarrow \mathcal{J}(\mathcal{R})$ defined by $\delta(I) = I + M$ where $M = (X_1, X_2, X_3)$ is the maximal ideal of R . Consider the ideal $I = (X_1X_2X_3)$ of R . Thus, I is not a 1-absorbing prime ideal of R since $X_1X_2X_3 \in I$ but neither $X_1X_2 \in I$ nor $X_3 \in I$. Now, let x, y, z be nonunit elements of R such that $xyz \in I$. Clearly I is a 1-absorbing δ -primary because $z \in \delta(I) = M$.*

Proposition 2.4. *(i) Every 1-absorbing δ -primary ideal is also a 2-absorbing δ -primary ideal of R .*

(ii) Let I be a 1-absorbing δ -primary ideal of R and $\delta(I)$ be a radical ideal, that is, $\sqrt{\delta(I)} = \delta(I)$. Then I is a δ -semiprimary ideal of R .

Proof. (i) Let $abc \in I$ for some $a, b, c \in R$. If at least one of a, b, c is a unit of R , then we are done. So assume that a, b, c are nonunits of R . Since I is a 1-absorbing δ -primary ideal of R , we get $ab \in I$ or $c \in \delta(I)$, which implies that $ab \in I$ or $ac \in \delta(I)$ or $bc \in \delta(I)$. Therefore, I is a 2-absorbing δ -primary ideal of R .

(ii) Suppose that $ab \in I$ for some $a, b \in R$. Then we may assume that a, b are nonunits. Thus $a^2b \in I$ implies that $a^2 \in I$ or $b \in \delta(I)$. Then we have $a \in \sqrt{I} \subseteq \sqrt{\delta(I)} = \delta(I)$ or $b \in \delta(I)$. Hence, I is a δ -semiprimary ideal of R . \square

The converse of previous proposition (i) is not true in general. See the following example.

Example 2.5. (*2-absorbing δ -primary ideal that is not 1-absorbing δ -primary ideal*)

Let $R = \mathbb{Z}$, $I = pq\mathbb{Z}$, where $p \neq q$ are prime numbers, and $\delta(I) = I + p\mathbb{Z}$. Since I is a 2-absorbing ideal, so is 2-absorbing δ -primary. However, it is easy to see that $ppq \in I$, $p^2 \notin I$ and $q \notin \delta(I)$. Thus, I is not a 1-absorbing δ -primary ideal of R .

In the next result, we show that if a ring R admits a 1-absorbing δ -primary ideal that is not a δ -primary ideal, then R is a local ring.

Theorem 2.6. *Let δ be an ideal expansion. Suppose that a ring R admits a 1-absorbing δ -primary ideal that is not a δ -primary ideal. Then R is a local ring.*

Proof. Assume that I is a 1-absorbing δ -primary ideal that is not a δ -primary ideal of R . Hence there exist nonunit elements $a, b \in R$ such that $ab \in I$, $a \notin I$ and $b \notin \delta(I)$. Let d be a nonunit element of R . As $dab \in I$, I is a 1-absorbing δ -primary ideal of R and $b \notin \delta(I)$, we conclude that $da \in I$. Let c be a unit element of R . Suppose that $d + c$ is a nonunit element of R . Since $(d + c)ab \in I$, I is a 1-absorbing δ -primary ideal of R and $b \notin \delta(I)$, we get that $(d + c)a = da + ca \in I$. Since $da \in I$, we conclude that $a \in I$, which gives a contradiction. Hence, $d + c$ is a unit element of R . Now, the result follows from [4, Lemma 1]. \square

Next, we give a method to construct 1-absorbing δ -primary ideals that are not δ -primary ideals.

Theorem 2.7. *Let R be a local ring with maximal ideal M and δ be an ideal expansion. Let x be a nonzero prime element of R such that $\delta(xM) \subsetneq M$. If $x \in \delta(xM)$, then xM is a 1-absorbing δ -primary ideal of R that is not a δ -primary ideal of R .*

Proof. First, we will show that xM is a 1-absorbing δ -primary ideal of R . Assume that $abc \in xM$ for some nonunit elements $a, b, c \in R$. If $ab \notin xM$, then $a \notin xR$ and $b \notin xR$, so $ab \notin xR$ because x is a prime element of R . Moreover, the fact that $abc \in xR$ and $ab \notin xR$ implies that $c \in xR \subseteq \delta(xM)$. Now, we prove that xM is not a δ -primary ideal of R . By hypothesis, we can pick an element $a \in M \setminus \delta(xM)$, hence $xa \in xM$. However, $x \notin xM$ since x is an irreducible element of R by [4, Lemma 2]. Which implies that xM is not a δ -primary ideal, this completes the proof. \square

Theorem 2.8. *Suppose that I is a 1-absorbing δ -primary ideal of R that is not a δ -primary ideal of R . Then there exists an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I$, but neither $x \in I$ nor $y \in \delta(I)$. Furthermore, if $ab \in I$ for some nonunit elements $a, b \in R$ such that neither $a \in I$ nor $b \in \delta(I)$, then a is an irreducible element of R .*

Proof. Suppose that I is not a δ -primary ideal of R . Hence neither $x \in I$ nor $y \in \delta(I)$ for some nonunit elements $x, y \in R$ with $xy \in I$. Assume on the contrary that x is not an irreducible element of R . Then $x = cd$ for some nonunit elements $c, d \in R$. Since $xy = cdy \in I$ and I is a 1-absorbing δ -primary ideal of R and $y \notin \delta(I)$, we conclude that $cd = x \in I$, a contradiction. Hence, x is an irreducible element of R . \square

Theorem 2.9. *Let I be a 1-absorbing δ -primary ideal of a ring R where δ is an ideal expansion and let $d \in R \setminus I$ be a nonunit element of R . Then $(I : d) = \{x \in R \mid dx \in I\}$ is a δ -primary ideal of R . In particular, for every proper ideal J of R with $J \not\subseteq I$, $(I : J)$ is a δ -primary ideal of R .*

Proof. Suppose that $ab \in (I : d)$ for some elements $a, b \in R$. Without loss of generality, we may assume that a and b are nonunit elements of R . Suppose that $a \notin (I : d)$. Since $dab \in I$ and I is a 1-absorbing δ -primary ideal of R , we conclude that $b \in \delta(I)$. So, $b \in \delta((I : d))$ and this completes the proof. The rest is similar. \square

Theorem 2.1. *Let δ be an expansion function of $\mathcal{J}(\mathcal{R})$, I a proper ideal of R and $(I : x) = (I : x^2)$ for each $x \in R - \delta(I)$. If I is an irreducible ideal, then I is a 1-absorbing δ -primary ideal of R .*

Proof. Assume on the contrary that I is not a 1-absorbing δ -primary ideal of R . Then there exist nonunit elements $a, b, c \in R$ with $abc \in I$ and neither $ab \in I$ nor $c \in \delta(I)$. Consider $(I + Rab) \cap (I + Rc)$. It is clear that $I \subseteq (I + Rab) \cap (I + Rc)$. Let $z \in (I + Rab) \cap (I + Rc)$. Then there are $i_1, i_2 \in I$ and $r_1, r_2 \in R$ such that $z = i_1 + r_1ab = i_2 + r_2c$. Then $zc = i_1c + r_1abc = i_2c + r_2c^2 \in I$. Hence $r_2c^2 \in I$; and so we have $r_2 \in (I : c^2) = (I : c)$ by the assumption. So

$z = i_2 + r_2c \in I$. Thus we conclude that $I = (I + Rab) \cap (I + Rc)$ which is a contradiction. Therefore, I is a 1-absorbing δ -primary ideal of R . \square

Proposition 2.10. *Let R be a ring, δ an ideal expansion and I be a proper ideal of R . If I is a 1-absorbing δ -primary ideal of R , then either I is a δ -semiprimary ideal of R or R is local, say with maximal ideal M , such that $M^2 \subseteq I$.*

Proof. If R is not local, then Theorem 2.6 implies that I is δ -primary and so I is a δ -semiprimary ideal of R . Now, assume that R is local with maximal ideal M such that I is not a δ -semiprimary ideal of R . Since I is proper, we infer that $I \subseteq M$. Observe that $\delta(I) \neq M$. Indeed, otherwise from Proposition 2.4 (ii), we conclude that I is a δ -semiprimary ideal which contradicts with our assumption. Moreover, there are $a, b \in M \setminus \delta(I)$ such that $ab \in I$. To prove that $M^2 \subseteq I$, it suffices to show that $xy \in I$ for all $x, y \in M$. Let $x, y \in M$. Then $xyab \in I$. Since $xy, a, b \in M$, $b \notin \delta(I)$ and I is a 1-absorbing δ -primary ideal, we conclude that $xya \in I$. Again, since $x, y, a \in M$, $a \notin \delta(I)$ and I is a 1-absorbing δ -primary ideal, we have that $xy \in I$. \square

Recall that a ring R is a chained ring if the set of all ideals of R is linearly ordered by inclusion. Moreover, R is said to be an arithmetical ring if R_M is a chained ring for each maximal ideal M of R . We next determine the 1-absorbing δ -primary ideals of a chained ring.

Theorem 2.11. *Let R be a chained ring with maximal ideal M , δ an ideal expansion and I be a proper ideal of R such that $I \neq M^2$. Then I is a 1-absorbing δ -primary ideal of R if and only if I is a δ -primary ideal of R .*

Proof. We need only prove the “only if” assertion. Let I be a 1-absorbing δ -primary ideal. Thus, Proposition 2.10 gives that either I is a δ -semiprimary ideal of R or $M^2 \subseteq I$. First, assume that I is a δ -semiprimary ideal of R and $ab \in I$ for some nonunit elements $a, b \in R$ such that $b \notin \delta(I)$. Hence, $a \in \delta(I)$. Now, since R is a chained ring, we conclude that $a \in bR$ and thus $a = br$ for some nonunit element $r \in R$. As $brb \in I$, $b \notin \delta(I)$ and I is a 1-absorbing δ -primary ideal of R , we conclude that $a = br \in I$. Which gives that I is a δ -primary ideal of R . Now, we suppose that $M^2 \subseteq I$. We may assume that $M \neq I$. Thus, we can pick $a \in M \setminus I$ and $b \in I \setminus M^2$. Then $b \in aR$ since R is a chained ring. So, $b = ar$ for some nonunit element $r \in R$ and thus $b \in M^2$, a contradiction. This completes the proof. \square

In view of Theorem 2.11, we have the following result.

Corollary 2.12. *Let R be an arithmetical ring with Jacobson radical M and I be a proper ideal of R such that $I \neq M^2$. Then I is a 1-absorbing δ -primary ideal of R if and only if I is a δ -primary ideal of R .*

Proof. Assume that R is local with maximal ideal M . Since R is an arithmetical ring, we conclude that $R = R_M$ is a chained ring and thus the claim follows from Theorem 2.11. In the remaining case, suppose that R is not a local ring. Then the result follows by Theorem 2.6. \square

Proposition 2.13. *Let R be a local ring with principal maximal ideal M , δ an ideal expansion and I be a proper ideal of R . Then I is a 1-absorbing δ -primary ideal of R if and only if either I is a δ -primary ideal of R or $M^2 \subseteq I$. In addition, if $\sqrt{I} \subseteq \delta(I)$ then I is a 1-absorbing δ -primary ideal of R if and only if I is a δ -primary ideal of R .*

Proof. By Remark 2.2(4) and Proposition 2.10, we need only prove that if I is a 1-absorbing δ -primary ideal of R which is a δ -semiprimary ideal then I is a δ -primary ideal (along with the hypothesis that R be a local ring with principal maximal ideal M). Also, we may assume that $\delta(I) \neq R$. Set $M = xR$ and let a and b be nonunit elements of R such that $b \notin \delta(I)$ and $ab \in I$. Since I is a δ -semiprimary ideal of R , we get that $a \in \delta(I)$. Moreover, $a = rx$ for some $r \in R$. If r is a unit element of R then $M = \delta(I)$ and thus I is a δ -primary ideal. If r is a nonunit element of R then $rx = ab \in I$. That implies $a = rx \in I$ since I is a 1-absorbing δ -primary ideal. This completes the proof. The in addition statement is clear. \square

Proposition 2.14. *Let $\{J_i \mid i \in D\}$ be a directed set of 1-absorbing δ -primary ideals of R , where δ is an ideal expansion. Then the ideal $J = \cup_{i \in D} J_i$ is a 1-absorbing δ -primary ideal of R .*

Proof. Let $abc \in J$ for some nonunits $a, b, c \in R$, then $abc \in J_i$ for some $i \in D$. Since J_i is a 1-absorbing δ -primary ideal of R , $ab \in J_i$ or $c \in \delta(J_i) \subseteq \delta(J)$. Hence, J is a 1-absorbing δ -primary ideal of R . \square

Proposition 2.15. *Let I be a 1-absorbing δ -primary ideal of R such that $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, where δ is an ideal expansion. Then, \sqrt{I} is a δ -primary ideal of R .*

Proof. Let $ab \in \sqrt{I}$ such that $a \notin \sqrt{I}$. Hence, there exists a positive integer n such that $(ab)^n \in I$. So, $a^m a^n b^n \in I$ for some positive integer m . Since I is a 1-absorbing δ -primary ideal of R and $a^{2m} \notin I$, we conclude that $b^n \in \delta(I)$. That implies $b \in \sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$ and so \sqrt{I} is a δ -primary ideal of R . \square

Proposition 2.16. *Let I be a proper ideal of a ring R and δ be an ideal expansion such that $\delta(\delta(I)) = \delta(I)$. Then the following statements are satisfied.*

(1) *If I is a 1-absorbing δ -primary ideal and a, b are nonunit elements with $ab \notin I$, then $\delta(I : ab) = \delta(I)$.*

(2) *$\delta(I)$ is a 1-absorbing δ -primary ideal of R if and only if $\delta(I)$ is a 1-absorbing prime ideal of R .*

Proof. (1) Let I be a 1-absorbing δ -primary ideal and $ab \notin I$. Note that $I \subseteq (I : ab)$ and so $\delta(I) \subseteq \delta(I : ab)$. Let $c \in (I : ab)$. Then $c \in \delta(I)$ since $abc \in I$ and $ab \notin I$. Thus $(I : ab) \subseteq \delta(I)$. We get $\delta(I : ab) \subseteq \delta(\delta(I)) = \delta(I)$. Hence we conclude the equality.

(2) By Remark 2.2(3), we need only prove the “only if” assertion. Let $abc \in \delta(I)$ for some nonunit elements $a, b, c \in R$. Hence $ab \in \delta(I)$ or $c \in \delta(\delta(I)) = \delta(I)$. Thus $\delta(I)$ is a 1-absorbing prime ideal of R . \square

Proposition 2.17. *Let R be a ring, I a proper ideal of R and δ be an ideal expansion. Then I is a 1-absorbing δ -primary ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2 and I_3 of R , then $I_1 I_2 \subseteq I$ or $I_3 \subseteq \delta(I)$.*

Proof. It suffices to prove the “if” assertion. Suppose that I is a 1-absorbing δ -primary ideal and let I_1, I_2 and I_3 be proper ideals of R such that $I_1 I_2 I_3 \subseteq I$ and $I_3 \not\subseteq \delta(I)$. Thus $abc \in I$ for every $a \in I_1, b \in I_2$ and $c \in I_3 \setminus \delta(I)$. Since I is a 1-absorbing δ -primary ideal, we then have $I_1 I_2 \subseteq I$, as desired. \square

Recall from [15] that an ideal expansion δ is said to be intersection preserving if $\delta(I_1 \cap I_2 \cap \dots \cap I_n) = \delta(I_1) \cap \delta(I_2) \cap \dots \cap \delta(I_n)$ for any ideals I_1, \dots, I_n of R .

Proposition 2.18. *Let δ be an intersection preserving ideal expansion. If I_1, I_2, \dots, I_n are 1-absorbing δ -primary ideals of R , and $\delta(I_i) = P$ for some ideal P of R and all $i \in \{1, 2, \dots, n\}$, then $I_1 \cap I_2 \cap \dots \cap I_n$ is a 1-absorbing δ -primary ideal of R .*

Proof. Let $abc \in J = I_1 \cap I_2 \cap \dots \cap I_n$ such that $ab \notin J$. Let $i \in \{1, 2, \dots, n\}$ such that $ab \notin I_i$. Since $abc \in I_i$ and I_i is a 1-absorbing δ -primary ideal, we conclude that $c \in \delta(I_i) = \delta(J)$. Therefore, J is a 1-absorbing δ -primary ideal of R . \square

Proposition 2.19. *Let R be a ring and δ be an expansion function of $\mathcal{J}(R)$. Then the following statements are equivalent.*

- (1) *Every proper principal ideal is a 1-absorbing δ -primary ideal of R .*
- (2) *Every proper ideal is a 1-absorbing δ -primary ideal of R .*

Proof. Assume that (1) holds and let I be a proper ideal of R . Let a, b, c be nonunit elements of R such that $abc \in I$. Hence $abc \in abcR$ which implies that $ab \in abcR \subseteq I$ or $c \in \delta(abcR) \subseteq \delta(I)$. Therefore I is a 1-absorbing δ -primary ideal of R . The converse is clear. \square

An expansion function δ of $\mathcal{J}(R)$ is said to satisfy *condition (*)* if $\delta(I) \neq R$ for each proper ideal I of R . Note that the identity function and the radical operation are examples of expansion functions satisfying condition (*).

Theorem 2.20. *Let R be a ring and δ an expansion function of $\mathcal{J}(R)$ satisfying condition (*) and $\delta(\text{Jac}(R)) = \text{Jac}(R)$. Suppose that $\delta(xI) = x\delta(I)$ for every proper ideal I of R and every $x \in R$. The following statements are equivalent.*

- (i) *Every proper principal ideal is a 1-absorbing δ -primary ideal of R .*
- (ii) *Every proper ideal is a 1-absorbing δ -primary ideal of R .*
- (iii) *R is local with $\text{Jac}(R)^2 = (0)$.*

Proof. (i) \Leftrightarrow (ii) Follows from Proposition 2.19.

(ii) \Rightarrow (iii) Assume that every proper ideal is a 1-absorbing δ -primary ideal R . Choose $x, y \in \text{Jac}(R)$. Now, we will show that $xy = 0$. If x or y is zero, then we are done. Assume that $x, y \neq 0$. Since $x^2y \in (x^2y)$ and (x^2y) is a 1-absorbing δ -primary ideal, we conclude that $x^2 \in (x^2y)$ or $y \in \delta((x^2y)) = y\delta((x^2))$. Suppose that $y \in y\delta((x^2))$. Then there exists $a \in \delta((x^2)) \subseteq \delta(\text{Jac}(R)) = \text{Jac}(R)$ such that $y = ya$. Which implies that $y(1 - a) = 0$. Since $1 - a$ is unit, we have $y = 0$, which is a contradiction. Thus we have, $x^2 \in (x^2y)$. Then we can write $x^2 = rx^2y$ for some $r \in R$. This implies that $x^2(1 - ry) = 0$. Since $1 - ry$ is unit, we have $x^2 = 0$. Likewise, we get $y^2 = 0$. Now, choose another $z \in \text{Jac}(R)$. Since $xyz \in (xyz)$ and (xyz) is a 1-absorbing δ -primary, we get $xy \in (xyz)$ or $z \in \delta((xyz)) = z\delta((xy))$. First, assume that $xy \in (xyz)$. Then there exists $r \in R$ such that $xy = rxyz$, which implies that $xy(1 - rz) = 0$. Since $1 - rz$ is unit, we have $xy = 0$ which completes the proof. Now, assume that $xy \notin (xyz)$, that is, $z \in \delta((xyz)) = z\delta((xy))$. Then there exists $a \in \delta((xy)) \subseteq \text{Jac}(R)$ such that $z = za$. This implies that $z(1 - a) = 0$ so that $z = 0$. Now, choose $z = x + y$. Then by above argument, we have either $xy = 0$ or $z = x + y = 0$. If $z = x + y = 0$, then we have $x = -y$ and so $xy = -y^2 = 0$ which completes the proof. Therefore, $\text{Jac}(R)^2 = (0)$.

Now, we will show that R is a local ring. Choose maximal ideals M_1, M_2 of R . Now, put $I = M_1 \cap M_2$. Since $M_1^2 M_2 \subseteq I$ and I is a 1-absorbing δ -primary ideal, we have either $M_1^2 \subseteq I \subseteq M_2$ or $M_2 \subseteq \delta(I) \subseteq \delta(M_1)$. **Case 1:** Suppose that $M_1^2 \subseteq M_2$. Since M_2 is prime, clearly we have $M_1 \subseteq M_2$ which implies that $M_1 = M_2$. **Case 2:** Suppose that $M_2 \subseteq \delta(M_1)$. Since δ satisfies condition (*), $\delta(M_1)$ is proper. As $M_1 \subseteq \delta(M_1)$ and M_1 is a maximal ideal, we have $M_1 = \delta(M_1)$. Then we get $M_2 \subseteq M_1$, which implies that $M_1 = M_2$. Therefore, R is a local ring.

(iii) \Rightarrow (i) Suppose that R is a local ring with $\text{Jac}(R)^2 = (0)$. Let I be a proper ideal of R and $abc \in I$ for some nonunits $a, b, c \in R$. Then $a, b, c \in \text{Jac}(R)$ since R is local. As $\text{Jac}(R)^2 = (0)$, we have $ab = 0 \in I$.

Therefore, I is a 1-absorbing δ -primary ideal of R . \square

It can be easily seen that, in Theorem 2.20, (iii) always implies (i) without any assumption on δ . But we give some examples showing that the converse is not true if we drop the aforementioned assumptions on δ .

Example 2.21. Let $R = \mathbb{Z}_{p^3}$, where p is a prime number and $\delta(I) = R$ for every proper ideal I of R . Note that δ does not satisfy condition (*) and note that every ideal I of R is 1-absorbing δ -primary. Thus $\text{Jac}(R)^2 \neq (0)$, while R is a local ring.

Example 2.22. Let k be a field and consider the formal power series ring $R = k[[X]]$. Then R is a local ring with unique maximal ideal $m = (X)$. Define expansion function δ as $\delta(I) = \sqrt{I}$ for every ideal I of R . Then it is easy to see that every ideal of R is a 1-absorbing δ -primary ideal. Also, it is clear that δ satisfies condition (*) and $\delta(\text{Jac}(R)) = \text{Jac}(R)$ but not satisfy the condition $\delta(xI) = x\delta(I)$. Furthermore, $\text{Jac}(R)^2 \neq (0)$. Thus Theorem 2.20 fails without assumption $\delta(xI) = x\delta(I)$.

Corollary 2.23. Let R be a ring. The following statements are equivalent.

- (i) Every proper ideal is a 1-absorbing prime ideal of R .
- (ii) Every proper principal ideal is a 1-absorbing prime ideal of R .
- (iii) R is local with $\text{Jac}(R)^2 = (0)$.

Proof. (i) \Leftrightarrow (ii) Follows from Proposition 2.19.

(ii) \Rightarrow (iii) Let δ be the identity expansion function, that is, $\delta(I) = I$ for every ideal I of R . Note that δ satisfies all axioms in Theorem 2.20. Then R is a local ring with $\text{Jac}(R)^2 = (0)$.

(iii) \Rightarrow (i) It is similar to Theorem 2.20 (iii) \Rightarrow (i). \square

An ideal expansion δ is called a prime expansion if for any 1-absorbing primary ideal I of R , $\delta(I)$ is a prime ideal of R .

Proposition 2.24. Let R be a local ring with maximal ideal M and δ be a prime expansion function of $\mathcal{J}(R)$. Assume that one of the following conditions holds:

- (1) $\text{Spec}(R) = \{\delta(0)\}$.
- (2) $\text{Spec}(R) = \{\delta(0), M\}$ and $\delta(0)M = 0$.

Then every proper ideal of R is 1-absorbing δ -primary.

Proof. Let I be a proper ideal of R and assume that (1) holds. Since $\delta(0)$ is maximal ideal and $\delta(I) \supseteq \delta(0)$, we conclude that $\delta(I) = R$ or $\delta(I) = \delta(0)$. Clearly I is a 1-absorbing δ -primary ideal of R . Now, assume that $\text{Spec}(R) = \{\delta(0), M\}$ and $\delta(0)M = 0$. Since $\delta(0) \subseteq M$, we have $\sqrt{I} = M$ or $\sqrt{I} = \delta(0)$. First assume that $\sqrt{I} = M$. Then I is 1-absorbing primary ideal so that $\delta(I) = M$ or $\delta(I) = \delta(0)$. If $\delta(I) = M$, we have then I is 1-absorbing δ -primary. So assume that $\delta(I) = \delta(0)$. Let $abc \in I$ for some nonunit elements $a, b, c \in R$ such that $c \notin \delta(0)$. As $I \subseteq \delta(0)$, we get that either $a \in \delta(0)$ or $b \in \delta(0)$. Thus $ab = 0 \in I$ which gives that I is a 1-absorbing δ -primary ideal of R . In the remaining case, $I \subseteq \sqrt{I} = \delta(0) \subseteq \delta(I)$. A similar argument shows that I is a 1-absorbing δ -primary ideal of R . \square

Let $f : R \rightarrow S$ be a ring homomorphism and δ, γ expansion functions of $\mathcal{J}(R)$ and $\mathcal{J}(S)$ respectively. Recall from [3] that f is called a $\delta\gamma$ -homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for each ideal I of S . Also note that if f is a $\delta\gamma$ -epimorphism and I is an ideal of R containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$.

Theorem 2.25. *Let $f : R \rightarrow S$ be a ring $\delta\gamma$ -homomorphism where δ, γ are expansion functions of $\mathcal{J}(R)$ and $\mathcal{J}(S)$ respectively. Suppose that $f(a)$ is nonunit in S for every nonunit element a in R . Then the following statements hold.*

- (1) *If J is a 1-absorbing γ -primary ideal of S , then $f^{-1}(J)$ is a 1-absorbing δ -primary ideal of R .*
- (2) *If f is an epimorphism and I is a proper ideal of R containing $\ker(f)$, then I is a 1-absorbing δ -primary ideal of R if and only if $f(I)$ is a 1-absorbing γ -primary ideal of S .*

Proof. (1) Assume that $abc \in f^{-1}(J)$, for some nonunit elements $a, b, c \in R$. Then $f(a)f(b)f(c) \in J$. Thus $f(a)f(b) \in J$ or $f(c) \in \gamma(J)$, which implies that $ab \in f^{-1}(J)$ or $c \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$. Therefore, $f^{-1}(J)$ is a 1-absorbing δ -primary ideal of R .

(2) Suppose that $f(I)$ is 1-absorbing γ -primary ideal of S . Since $I = f^{-1}(f(I))$, we conclude that I is a 1-absorbing δ -primary ideal of R by (1). Conversely, let x, y, z be nonunit elements of S with $xyz \in f(I)$. Then there exist $a, b, c \in R$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$ with $f(abc) = xyz \in f(I)$. Since $\ker(f) \subseteq I$, we then have $abc \in I$. Since I is a 1-absorbing δ -primary ideal of R and $abc \in I$, we conclude that $ab \in I$ or $c \in \delta(I)$ which gives that $xy \in f(I)$ or $z \in f(\delta(I)) = \gamma(f(I))$. Thus $f(I)$ is a 1-absorbing δ -primary ideal of S . \square

Let δ be an expansion function of $\mathcal{J}(\mathcal{R})$ and I an ideal of R . Then the function $\bar{\delta} : \frac{R}{I} \rightarrow \frac{R}{I}$ defined by $\bar{\delta}(\frac{J}{I}) = \frac{\delta(J)}{I}$ for all ideals $I \subseteq J$, becomes an expansion function of $\frac{R}{I}$. Then, we have the following result.

Corollary 2.26. *Let R be a ring, δ an expansion function of $\mathcal{J}(\mathcal{R})$ and $I \subseteq J$ be proper ideals of R . Assume that $a + I$ is a nonunit element of $\frac{R}{I}$ for every nonunit element $a \in R$. Then J is a 1-absorbing δ -primary ideal of R if and only if $\frac{J}{I}$ is a 1-absorbing $\bar{\delta}$ -primary ideal of $\frac{R}{I}$.*

Proposition 2.27. *Let S be a multiplicatively closed subset of a ring R and δ_S an expansion function of $\mathcal{J}(S^{-1}R)$ such that $\delta_S(S^{-1}I) = S^{-1}(\delta(I))$ for each ideal I of R . If I is a 1-absorbing δ -primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a 1-absorbing δ_S -primary ideal of $S^{-1}R$.*

Proof. Let I be a 1-absorbing δ -primary ideal of R such that $I \cap S = \emptyset$ and $\frac{a}{s} \frac{b}{t} \frac{c}{r} \in S^{-1}I$ for some nonunit elements $a, b, c \in R$ and $s, t, r \in S$ such that $\frac{a}{s} \frac{b}{t} \notin S^{-1}I$. Then $xabc \in I$ for some $x \in S$. Since I is a 1-absorbing δ -primary ideal and $xab \notin I$, we conclude that $c \in \delta(I)$. Thus $\frac{c}{r} \in S^{-1}(\delta(I)) = \delta_S(S^{-1}I)$ which completes the proof. \square

Let S be a multiplicatively closed subset of a ring R and I an ideal of R . The next example shows that if $S^{-1}I$ is a 1-absorbing δ_S -primary ideal of $S^{-1}R$, then I need not to be a 1-absorbing δ -primary ideal of R .

Example 2.28. *Let $p \neq q$ be two prime numbers. Set $I = pq\mathbb{Z}$ and δ be an ideal expansion such that $\delta(I) = I + q\mathbb{Z}$ for each ideal I of \mathbb{Z} . Clearly, I is not a 1-absorbing δ -primary ideal of \mathbb{Z} as it is shown in Example 2.5. Now, let $S = \mathbb{Z} \setminus p\mathbb{Z}$ and note that $S^{-1}I = S^{-1}(p\mathbb{Z})$. Let $\frac{a}{r_1} \frac{b}{r_2} \frac{c}{r_3} \in S^{-1}I$ for some nonunit elements $\frac{a}{r_1}, \frac{b}{r_2}, \frac{c}{r_3} \in S^{-1}\mathbb{Z}$. Note that $\frac{x}{r} \in S^{-1}\mathbb{Z}$ is nonunit if and only if $x \in p\mathbb{Z}$. Thus $a \in p\mathbb{Z}$ and $b \in p\mathbb{Z}$. Which gives that $\frac{a}{r_1} \frac{b}{r_2} \in S^{-1}I$ and hence $S^{-1}I$ is a 1-absorbing δ_S -primary ideal.*

Let R_1 and R_2 be two rings, let δ_i be an expansion function of $\mathcal{J}(\mathcal{R}_i)$ for each $i \in \{1, 2\}$ and $R = R_1 \times R_2$. For a proper ideal $I_1 \times I_2$, the function δ_{\times} defined by $\delta_{\times}(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ is an expansion function of $\mathcal{J}(\mathcal{R})$. The following result characterizes the 1-absorbing δ -primary ideals of the direct product of rings.

Theorem 2.29. *Let R_1 and R_2 be rings, $R = R_1 \times R_2$ and let δ_i be an expansion function of $\mathcal{J}(\mathcal{R}_i)$ for $i = 1, 2$. Then the following statements are equivalent:*

- (1) I is a 1-absorbing δ_{\times} -primary ideal of R .
- (2) I is a δ_{\times} -primary ideal of R .
- (3) Either $I = I_1 \times R_2$, where I_1 is a δ_1 -primary ideal of R_1 or $I = R_1 \times I_2$, where I_2 is a δ_2 -primary ideal of R_2 or $I = I_1 \times I_2$, where I_1 and I_2 are proper ideals of R_1, R_2 , respectively with $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$.

Proof. (1) \Leftrightarrow (2). This follows from Theorem 2.6.

(2) \Leftrightarrow (3) Let I be a δ_\times -primary ideal of R . Hence I has the form $I = I_1 \times I_2$ where I_1 and I_2 are ideals of R_1 and R_2 respectively. Without loss of generality, we may assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1 . We show that I_1 is a δ_1 -primary ideal of R_1 . Deny. Then there are $a, b \in R_1$ such that $ab \in I_1$, $a \notin I_1$ and $b \notin \delta_1(I_1)$. Hence $(a, 1)(b, 1) \in I_1 \times R_2$. Which implies that $(a, 1) \in I_1 \times R_2$ or $(b, 1) \in \delta_\times(I_1 \times R_2)$ and so $a \in I_1$ or $b \in \delta_1(I_1)$, which gives a contradiction. Now suppose that both I_1 and I_2 are proper. As $(1, 0)(0, 1) \in I_1 \times I_2$ and $(1, 0), (0, 1) \notin I_1 \times I_2$, we have $(1, 0), (0, 1) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$. Therefore $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$. The converse is clear. \square

The following example proves that the condition “ $\delta(I_i) = P$ for all $i \in \{1, 2, \dots, n\}$ ” is necessary in Proposition 2.18.

Example 2.30. Let $R = R_1 \times R_2$ where $R_1 = R_2 = \mathbb{Z}$ and $\delta_1 = \delta_2 = \delta$ where $\delta(I) = \sqrt{I} + 2\mathbb{Z}$ for every ideal I of \mathbb{Z} . Consider the ideals $K = 4\mathbb{Z} \times \mathbb{Z}$ and $L = \mathbb{Z} \times 9\mathbb{Z}$ of R . Thus $\delta_\times(K) = \delta_1(4\mathbb{Z}) \times \delta_2(\mathbb{Z}) = 2\mathbb{Z} \times \mathbb{Z}$ and $\delta_\times(L) = \delta_1(\mathbb{Z}) \times \delta_2(9\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Moreover, K and L are 1-absorbing δ_\times -primary ideal. But $K \cap L = 4\mathbb{Z} \times 9\mathbb{Z}$ is not a 1-absorbing δ_\times -primary ideal by Theorem 2.29.

In view of Theorem 2.29, we have the following result.

Theorem 2.31. Let R_1, R_2, \dots, R_n be commutative rings with nonzero identity and $R = R_1 \times R_2 \times \dots \times R_n$ where $n \geq 2$. Let δ_i be an expansion function of $\mathcal{J}(\mathcal{R}_i)$ for each $i = 1, 2, \dots, n$. Then the following statements are equivalent.

- (1) I is a 1-absorbing δ_\times -primary ideal of R .
- (2) $I = I_1 \times I_2 \times \dots \times I_n$ and either for some $k \in \{1, 2, \dots, n\}$ such that I_k is a 1-absorbing δ_k -primary ideal of R_k and $I_j = R_j$ for each $j \in \{1, 2, \dots, n\} \setminus \{k\}$ or I_{α_i} 's are proper ideals of R_{α_i} for $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subseteq \{1, 2, \dots, n\}$ and $|\{\alpha_1, \alpha_2, \dots, \alpha_k\}| \geq 2$ with $\delta_{\alpha_i}(I_{\alpha_i}) = R_{\alpha_i}$, and $I_j = R_j$ for all $j \in \{1, 2, \dots, n\} \setminus \{\alpha_1, \alpha_2, \dots, \alpha_k\}$.

Proof. It can be obtained by using mathematical induction on n . \square

Let A be a ring and E an A -module. Then $A \rtimes E$, the *trivial (ring) extension of A by E* , is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+E)$.) The basic properties of trivial ring extensions are summarized in the books [10], [9]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [1, 6, 7, 11]). In addition, for an ideal I of

A and a submodule F of E , $I \times F$ is an ideal of $A \times E$ if and only if $IE \subseteq F$. Moreover, for an expansion function δ of A , it is clear that δ_{\times} defined as $\delta_{\times}(I \times F) = \delta(I) \times E$ is an expansion function of $A \times E$. Also as usual, if $c \in A$ then $(F : c) = \{e \in E \mid ce \in F\}$.

Theorem 2.32. *Let A be a ring, E an A -module and δ be an expansion function of $\mathcal{J}(A)$. Let I be an ideal of A and F a submodule of E such that $IE \subseteq F$. Then the following statement hold:*

- (1) *If $I \times F$ is a 1-absorbing δ_{\times} -primary ideal of $A \times E$, then I is a 1-absorbing δ -primary ideal of A .*
- (2) *Assume that $(F : c) = F$ for every $c \in A \setminus I$. Then $I \times F$ is a 1-absorbing δ_{\times} -primary ideal of $A \times E$ if and only if I is a 1-absorbing δ -primary ideal of A .*

Proof. (1) Assume that $I \times F$ is a 1-absorbing δ_{\times} -primary ideal of $A \times E$ and let a, b, c be nonunit elements of A such that $abc \in I$. Thus $(a, 0)(b, 0)(c, 0) = (abc, 0) \in I \times F$ which implies that $(a, 0)(b, 0) \in I \times F$ or $(c, 0) \in \delta_{\times}(I \times F) = \delta(I) \times E$. Therefore $ab \in I$ or $c \in \delta(I)$ and so (1) holds.

(2) By (1), it suffices to prove the "if" assertion. Let $(a, s), (b, t), (c, r)$ be nonunit elements of $A \times E$ such that $(a, s)(b, t)(c, r) = (abc, bcs + act + abr) \in I \times F$. Clearly, $abc \in I$ and so $ab \in I$ or $c \in \delta(I)$ since I is a 1-absorbing δ -primary ideal of A . If $c \in \delta(I)$, then $(c, r) \in \delta(I) \times E = \delta_{\times}(I \times F)$. Hence, we may assume that $c \notin \delta(I)$. Then $ab \in I$. As $bcs + act + abr \in F$ and $abr \in F$, we get that $bcs + act \in F$. This implies $bs + at \in (F : c) = F$ and so $(a, s)(b, t) = (ab, at + bs) \in I \times F$. Therefore $I \times F$ is a 1-absorbing δ_{\times} -primary ideal of $A \times E$. \square

Corollary 2.33. *Let A be a ring, E an A -module and δ be an expansion function of $\mathcal{J}(A)$. Let I be a proper ideal of A . Then $I \times E$ is a 1-absorbing δ_{\times} -primary ideal of $A \times E$ if and only if I is a 1-absorbing δ -primary ideal of A .*

Acknowledgement. The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

References

- [1] Anderson, D. D. and Winders, M., Idealization of a module. J. Comm. Alg., 1(1) (2009), 3-56.
- [2] Badawi, A., On 2-absorbing ideals of commutative rings. Bulletin of the Australian Mathematical Society, 75(3) (2007), 417-429.

- [3] Badawi, A. and Fahid, B., On weakly 2-absorbin δ -primary ideals of commutative ring. *Georgian. Math. J.*, 27(4) (2017), 503-516.
- [4] Badawi, A. and Celikel, E. Y., On 1-absorbing primary ideals of a commutative rings. *J. Algebra Appl.*, (2020), 2050111.
- [5] Badawi, A., Sonmez, D., Yesilot, G., On weakly δ -semiprimary ideals of commutative rings. *Algebra Colloq.*, 25 (2018), 387-398.
- [6] Dobbs, D. E., Elkhalfi, A., Mahdou, N., Trivial extensions satisfying certain valuation-like properties, *Comm. Algebra*, 47(5) (2019), 2060-2077.
- [7] Dumitrescu, T., Mahdou, N., Zahir, Y., Radical factorization for trivial extensions and amalgamated duplication rings. *Journal of Algebra and Its Applications*, 20(02) (2021), 2150025.
- [8] Fahid, B. and Dongsheng, Z., 2-absorbing δ -primary ideals in commutative rings. *Kyungpook Mathematical Journal*, 57 (2) (2017), 193-198.
- [9] Glaz, S., *Commutative Coherent Rings*. Lecture Notes in Math. 1371, Springer-Verlag, Berlin, 1989.
- [10] Huckaba, J. A., *Commutative Rings with Zero Divisors*. Dekker, New York, 1988.
- [11] Kabbaj, S. and Mahdou, N., Trivial extensions defined by coherent-like conditions. *Comm. Algebra*, 32 (10) (2004), 3937-3953.
- [12] Kaplansky, I., *Commutative Rings*. rev. ed., Univ. Chicago Press, Chicago, 1974.
- [13] Ulucak, G., Tekir, Ü., Koc, S., On n -absorbing δ -primary ideals. *Turkish Journal of Mathematics*, 42 (4) (2018), 1833-1844.
- [14] Yassine, A., Nikmehr, M. J., Nikandish, R., On 1-absorbing prime ideals of commutative rings. *J. Algebra Appl.*, (2020), 2150175.
- [15] Zhao, D., δ -primary ideals of commutative rings. *Kyungpook Math. J.*, 41 (1) (2001), 17-22

Abdelhaq El Khalfi,
Laboratory of Modelling and Mathematical Structures
Department of Mathematic, Faculty of Science and Technology of Fez, Box
2202, University S.M. Ben Abdellah Fez, Morocco.
Email: abdelhaq.elkhalfi@usmba.ac.ma

Najib Mahdou,
Laboratory of Modelling and Mathematical Structures
Department of Mathematics, Faculty of Science and Technology of Fez, Box
2202, University S.M. Ben Abdellah Fez, Morocco.
Email: mahdou@hotmail.com

Ünsal Tekir,
Department of Mathematics,
Marmara University,
Istanbul, Turkey.
Email: utekir@marmara.edu.tr

Suat Koç,
Department of Mathematics,
Marmara University,
Istanbul, Turkey.
Email: suat.koc@marmara.edu.tr