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Mixed Mode Crack Propagation in Iliac Bone

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Abstract

Bone is a complex material that can be regarded as an anisotropic elastic composite material. The problem of crack propagation in human bone is analyzed by using a generalization of the maximum tensile stress criterion (MTS). The results concern the critical stress for crack propagation and the direction of the crack path in Iliac bone.

1 Introduction

In this paper we study the mathematical problem for mixed mode I + II of classical fracture of the Iliac bone, regarded as a composite material. By using the representation formulae by complex potentials (see [1]-[3]), we obtain the displacement and stress fields. Generalizing the maximum stress criterion (see [3]-[5]) for anisotropic materials, we find the critical stress to produce crack propagation, as well as the direction of the crack in Iliac bone (see Figure 1). Several authors studied the properties of fracture and resistance of the composite materials used in biomechanics and medicine (see [7]). Bone is a composite tissue consisting of mineral, matrix consist of collagen and non-collagenous proteins, cells, and water. Its elastic modulus varies with the type of loading: tension-compression, bending-shear or with orientation: transverse versus axial. Human bone is often considered to be orthotropic composite material with organized microstructure; the structural axes of orthotropic symmetry being defined by the bone microstructure, (see [8]-[11]).

Key Words: crack in mixed mode of fracture, anisotropic materials, generalized MTS criteria, Iliac bone
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Figure 1: Iliac bone

2 Representation of the fields by complex potentials

The state of the human bone is a *plane state* relative to the plane x_1x_2. The involved nominal stresses t_{21} and t_{22} must satisfy the following boundary conditions on the two faces of the crack, represented as a cut of length equal with $2a$:

$$t_{21}(x_1,0^+) = t_{21}(x_1,0^-) = -p\sin\beta\cos\beta, \text{ for } |x_1| < a, \tag{1}$$

$$t_{22}(x_1,0^+) = t_{22}(x_1,0^-) = -p\sin^2\beta, \text{ for } |x_1| < a, \tag{2}$$

where p is the given value of normal force acting on the crack faces and having direction of Ox_2 axis, β represents the angle between crack line and Ox_2 axis. The displacement, nominal stresses and complex potentials are vanishing at large distances from the crack.

Using the representation of nominal stresses with complex potentials (see Annex) in the boundary conditions (1)-(2) we obtain two Riemann-Hilbert problems. After long manipulations, we get the following representation of the complex potentials $\Psi_j(z_j), j = 1, 2$:

$$\Psi_1(z_1) = -\frac{p}{2\pi\Delta\sqrt{z_1^2 - a^2}} \int_{-a}^{a} \frac{(a_2\mu_2 \sin^2\beta + \sin\beta\cos\beta)\sqrt{a^2 - t^2}}{t - z_1} dt,$$

$$\Psi_2(z_2) = \frac{p}{2\pi\Delta\sqrt{z_2^2 - a^2}} \int_{-a}^{a} \frac{(a_1\mu_1 \sin^2\beta + \sin\beta\cos\beta)\sqrt{a^2 - t^2}}{t - z_2} dt,$$

$$\Psi_j(z_j) = \Phi_j'(z_j), j = 1, 2. \tag{3}$$

3 Asymptotical values

We shall analyze the *asymptotical behavior* of the fields in the neighborhood of the crack tips. This analysis is important since in this way the relationship between the stresses and the input energy rates, in crack extension may be established.

The fields distribution around the (right) tip can be obtained by letting

$$x_1 = a + r\cos\varphi,\ x_2 = r\sin\varphi$$

In a small neighborhood of the crack tip $x_1 \approx a, x_2 \approx 0$ we have

$$z_1 \approx z_2 \approx a.$$

The Plemelj functions may be approximated by

$$\sqrt{z_j^2 - a^2} = \sqrt{2ar}\chi_j(\varphi), \chi_j(\varphi) = \sqrt{\cos\varphi + \mu_j \sin\varphi},\ j = 1, 2.$$

The asymptotic values of the complex potentials:

$$\Psi_1(z_1) = \frac{a_2\mu_2 K_I + K_{II}}{2\Delta\sqrt{2\pi r}} \frac{1}{\chi_1(\varphi)},\ \Psi_2(z_2) = -\frac{a_1\mu_1 K_I + K_{II}}{2\Delta\sqrt{2\pi r}} \frac{1}{\chi_2(\varphi)}. \tag{4}$$

The asymptotic expressions of the fields corresponding to the mixed mode:

$$t_{11} = \frac{K_I}{\sqrt{2\pi r}} Re \frac{1}{\Delta} [a_1\mu_1^2 \frac{a_2\mu_2 K_I + K_{II}}{\chi_1(\varphi)} - a_1\mu_2^2 \frac{a_1\mu_1 K_I + K_{II}}{\chi_2(\varphi)}],$$

$$t_{21} = -\frac{K_I}{\sqrt{2\pi r}} Re \frac{1}{\Delta} [a_1\mu_1 \frac{a_2\mu_2 K_I + K_{II}}{\chi_1(\varphi)} - a_2\mu_2 \frac{a_1\mu_1 K_I + K_{II}}{\chi_2(\varphi)}],$$

$$t_{12} = -\frac{K_I}{\sqrt{2\pi r}} Re \frac{1}{\Delta} [\mu_1 \frac{a_2\mu_2 K_I + K_{II}}{\chi_1(\varphi)} - \mu_2 \frac{a_1\mu_1 K_I + K_{II}}{\chi_2(\varphi)}],$$

$$t_{22} = \frac{K_I}{\sqrt{2\pi r}} Re \frac{1}{\Delta} [\frac{a_2\mu_2 K_I + K_{II}}{\chi_1(\varphi)} - \frac{a_1\mu_1 K_I + K_{II}}{\chi_2(\varphi)}]; \quad (5)$$

$$u_1 = 2\sqrt{\frac{r}{2\pi}} Re \frac{1}{\Delta} [b_1(a_2\mu_2 K_I + K_{II})\chi_1(\varphi) - b_2(a_1\mu_1 K_I + K_{II})\chi_2(\varphi)],$$

$$u_2 = 2\sqrt{\frac{r}{2\pi}} Re \frac{1}{\Delta} [c_1(a_2\mu_2 K_I + K_{II})\chi_1(\varphi) - c_2(a_1\mu_1 K_I + K_{II})\chi_2(\varphi)]. \quad (6)$$

4 Crack propagation criteria. Erdogan and Sih's maximum tangential stress criterion (MTS)

Erdogan and Sih's maximum tangential stress criterion states the following hypothesis for the extension of cracks in a brittle material under slowly applied plane loads

- The crack extension starts at its tip in radial direction;
- The crack extension starts in the plane perpendicular to the direction of greatest tension.

These hypotheses imply that the crack will start to initialize in a perpendicular direction on φ_c along which the tangential stress $t^*_{\varphi\varphi}$ is maximum,

$$t^*_{\varphi\varphi}(\varphi_c, \beta) = \max_{\varphi \in [-\pi, \pi]} t^*_{\varphi\varphi}(\varphi, \beta). \quad (7)$$

The physical tangential stress $t^*_{\varphi\varphi}$ by the components of the stress t^* in the new system of coordinates $Ox_1^* x_2^*$ has the following form (see [1]):

$$t^*_{\varphi\varphi}(\varphi, \beta) = t^*_{11} \sin^2\varphi - (t^*_{12} + t^*_{21}) \sin\varphi \cos\varphi + t^*_{22} \cos^2\varphi, \quad (8)$$

Due to the fact that the new system of coordinates $Ox_1^* x_2^*$ is obtained from the initial one Ox_1x_2, with a 90^o clockwise rotation we have:

$$\begin{aligned}
t^*_{11} &= t_{11} \sin^2\beta + (t_{12} + t_{21}) \sin\beta \cos\beta + t_{22} \cos^2\beta, \\
t^*_{12} &= (t_{22} - t_{11}) \sin\beta \cos\beta + t_{12} \sin^2\beta - t_{21} \cos^2\beta, \\
t^*_{12} &= (t_{22} - t_{11}) \sin\beta \cos\beta - t_{12} \cos^2\beta + t_{21} \sin^2\beta, \\
t^*_{11} &= t_{11} \cos^2\beta - (t_{12} + t_{21}) \sin\beta \cos\beta + t_{22} \sin^2\beta.
\end{aligned} \quad (9)$$

5 Numerical results. Conclusions

We consider an cracked Iliac bone with an inclined crack, regarded as an orthotropic composite material characterized by the following technical constants:

$$\begin{aligned} E_1 = 11.6GPa, \quad E_2 = 12.2GPa, E_3 = 19.9GPa, \quad G_{12} = 4GPa, \\ G_{13} = 5GPa, G_{23} = 5.4GPa, \quad \nu_{12} = 0.42, \quad \nu_{13} = \nu_{23} = 0.23, \\ \nu_{21} = 0.44, \quad \nu_{31} = 0.39, \quad \nu_{32} = 0.38. \end{aligned} \tag{10}$$

where E_1, E_2, E_3 are Young's moduli in the corresponding symmetry directions of the material, ν_{12}, ..., ν_{23} are the Poisson's ratios and G_{12}, G_{13}, G_{23} are the shear moduli.

Using Eqs. (5)-(7) we plotted the physical tangential stress $t^*_{\varphi\varphi}(\varphi, \beta)$ versus inclination angle β, (see Figure 2). When the inclination angle β increases from $\beta = 0^o$ to $\beta = 90^o$ the crack propagation angle $\varphi_c(\beta)$ decreases to 0^o. We note that in the particular case of $\beta = 90^o$ corresponding to the first mode of fracture the crack will propagate along its line, result well known in Fracture Mechanics.

The presented 2D quasistatic mathematical model provides a means to find crack propagation angle of a mixed mode crack in a human bones regarded as anisotropic materials.

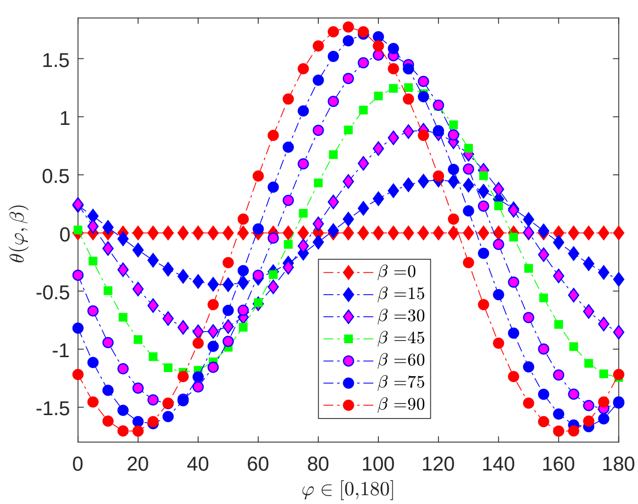

Figure 2: Plots of maximum tangential stress $t_{\varphi\varphi}$ vs. inclination angle β

6 Annex: The plane state

A plane state can exist in the body relative to the plane Ox_1x_2, and it is characterized by the two nonvanishing components of the displacement field

$$u_1 = u_1(x_1, x_2), \; u_2 = u_2(x_1, x_2). \tag{11}$$

The only nonvanishing components of the nominal stress $t_{mn}, m, n = 1, 2, 3$ are

$$\begin{array}{ll} t_{11} = \omega_{1111}u_{1,1} + \omega_{1122}u_{2,2}, & t_{21} = \omega_{2112}u_{1,2} + \omega_{2121}u_{2,1} \\ t_{12} = \omega_{1212}u_{1,2} + \omega_{1221}u_{2,1}, & t_{22} = \omega_{2211}u_{1,1} + \omega_{2222}u_{2,2} \\ t_{33} = \omega_{3311}u_{1,1} + \omega_{3322}u_{2,2}. & \end{array} \tag{12}$$

The only equilibrium equations that must be satisfied are

$$t_{11,1} + t_{21,2} = 0, \; t_{12,1} + t_{22,2} = 0, \tag{13}$$

or, equivalently,

$$\begin{gathered} \omega_{1111}u_{1,11} + \omega_{1122}u_{2,21} + \omega_{2112}u_{1,22} + \omega_{2121}u_{2,12} = 0, \\ \omega_{1212}u_{1,21} + \omega_{1221}u_{2,11} + \omega_{2211}u_{1,12} + \omega_{2222}u_{2,22} = 0. \end{gathered} \tag{14}$$

The instantaneous elasticities used in Eqs. (3) and (4) are given by:

$$\begin{array}{lll} \omega_{1111} = C_{11} & \omega_{2222} = C_{22} & \omega_{1122} = \omega_{2211} + C_{12}, \\ \omega_{1212} = \omega_{2121} + C_{66}, & \omega_{1221} = C_{66}, & \omega_{2112} = C_{66} \\ \omega_{3311} = C_{13}, & \omega_{3322} = C_{23}. & \end{array} \tag{15}$$

We have the following expressions of the non-vanishing independents components of the stiffness matrix $[C]$ of an orthotropic material, as function of its engineering constants (see [1]):

$$\begin{aligned} & C_{11} = \frac{1 - \nu_{23}\nu_{32}}{E_2E_3\Delta}, \; C_{12} = \frac{\nu_{21+}\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12+}\nu_{32}\nu_{13}}{E_1E_3\Delta}, \\ & C_{13} = \frac{\nu_{31+}\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13+}\nu_{12}\nu_{23}}{E_1E_2\Delta}, \\ & C_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3\Delta}, \; C_{23} = \frac{\nu_{32+}\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23+}\nu_{21}\nu_{13}}{E_1E_2\Delta}, \\ & C_{33} = \frac{1 - \nu_{12}\nu_{21}}{E_1E_2\Delta}, \end{aligned} \tag{16}$$

$$C_{44} = G_{23},\ C_{55} = G_{13},\ C_{66} = G_{12},$$

where

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{23}\nu_{31}}{E_1E_2E_3}. \tag{17}$$

We have the following *representation of the fields by two arbitrary analytic complex potentials* $\Phi_j = \Phi_j(z_j), j = 1, 2$ and their derivatives:

$$\begin{aligned}
&t_{22} = 2Re\left\{\Phi_1'(z_1) + \Phi_2'(z_2)\right\}, t_{21} = -2Re\left\{a_1\mu_1\Phi_1'(z_1) + a_2\mu_2\Phi_2'(z_2)\right\},\\
&t_{12} = -2Re\left\{a_1\mu_1\Phi_1'(z_1) + a_2\mu_2\Phi_2'(z_2)\right\}, t_{11} = 2Re\left\{a_1\mu_1^2\Phi_1'(z_1) + a_2\mu_2^2\Phi_2'(z_2)\right\},\\
&u_1 = 2Re\left\{b_1\Phi_1(z_1) + b_2\Phi_2(z_2)\right\}, u_2 = 2Re\left\{c_1\Phi_1(z_1) + c_2\Phi_2(z_2)\right\},
\end{aligned} \tag{18}$$

where:

$$a_j = \frac{\omega_{2112}\omega_{1122}\mu_j^2 - \omega_{1111}\omega_{1212}}{B_j\mu_j^2}, b_j = -\frac{\omega_{1122} + \omega_{1212}}{B_j}, c_j = \frac{\omega_{2112}\mu_j^2 + \omega_{1111}}{B_j\mu_j},$$

with $z_j = x_1 + \mu_j x_2$ and $\mu_j, j = 1, 2$ are the roots of characteristic equation, supposed unequal in all what it follows.

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