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Dynamics and Ulam Stability for Fractional q-Difference Inclusions via Picard Operators Theory

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Abstract

In this manuscript, by using weakly Picard operators we investigate the Ulam type stability of fractional q-difference An illustrative example is given in the last section.

1 Introduction

Not only fractional differential inclusions (FDIs) but also fractional differential equations (FDEs) have applications in mathematics, and other applied sciences, see e.g. [18, 6, 7, 35, 38, 40, 21, 22, 37, 9, 17, 4, 5]. Fractional q-difference equations received much attention from many authors; see e.g. [12]. Other interesting results about this subject can be found in [24].

Functional differential inclusions and coupled systems of differential inclusions are a generalization of the concept of ordinary differential equation of the form $\frac{d}{dt}x(t) \in F(t, x(t))$, where F is a multivalued map containing one element (single-valued map). Differential inclusions arise in many situations as differential variational inequalities, projected dynamical systems, linear and nonlinear complementarity dynamical systems, discontinuous ordinary differential equations, and fuzzy set arithmetic; see e.g.[14, 19, 36].

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Ulam stability for functional differential equations and inclusions has been widely considered; see e.g.[26, 27]. Picard operators [28, 29] seemed to be a powerful method in the processing of Ulam stability theory [10, 27, 16], and ordinary differential inclusions and equations; see e.g.[1, 29, 30, 31].

In this paper we first discuss the stability of the fractional q-difference inclusion below in the sense of Ulam-Rassias

$$({}^c\mathfrak{D}_q^\alpha \mathfrak{h})(\mathfrak{t}) \in F(\mathfrak{t}, \mathfrak{h}(\mathfrak{t})); \ \mathfrak{t} \in \mathfrak{J} := [0, T], \tag{1.1}$$

along the initial condition

$$\mathfrak{h}(0) = \mathfrak{h}_0 \in \mathbb{R}, \tag{1.2}$$

with $T > 0, \alpha \in (0, 1], q \in (0, 1)$, and $F : \mathfrak{J} \times \mathbb{R} \to \mathcal{N}(\mathbb{R})$ is a given multi-valued map, $\mathcal{N}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and ${}^c\mathfrak{D}_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

After getting a solution of (1.1), we shall investigate the coupled fractional q-difference inclusions

$$\begin{cases} ({}^c\mathfrak{D}_{q_1}^{\alpha_1} \mathfrak{g}_1)(\mathfrak{t}) \in F_1(\mathfrak{t}, \mathfrak{g}_1(\mathfrak{t}), \mathfrak{g}_2(\mathfrak{t})), \\ ({}^c\mathfrak{D}_{q_2}^{\alpha_2} \mathfrak{g}_2)(\mathfrak{t}) \in F_1(\mathfrak{t}, \mathfrak{g}_1(\mathfrak{t}), \mathfrak{g}_2(\mathfrak{t})) \end{cases} ; \ \mathfrak{t} \in \mathfrak{J}, \tag{1.3}$$

with the initial conditions

$$\begin{cases} \mathfrak{g}_1(0) = \mathfrak{i}_1 \\ \mathfrak{g}_2(0) = \mathfrak{i}_2, \end{cases} \tag{1.4}$$

where $T > 0, \ q_i \in (0, 1), \ \alpha_i \in (0, 1], \ \mathfrak{i}_i \in \mathbb{R}, \ F_i : \mathfrak{J} \times \mathbb{R} \to \mathcal{N}(\mathbb{R}); \ i = 1, 2.$

This paper initiates the application of Picard operators for the study of Ulam stability for problems (1.1)-(1.2) and (1.3)-(1.4).

2 Preliminaries

We deal with the following collection

$$C(\mathfrak{J}) := \{\mathfrak{g} : \mathfrak{J} \to \mathbb{R} | \quad \mathfrak{g} \text{ is continuous } \}.$$

Then, $C(\mathfrak{J})$ forms a Banach space by regarding the supremum (uniform) norm $\|\mathfrak{g}\|_C := \sup_{\tau \in \mathfrak{J}} |\mathfrak{g}(\tau)|$.

$$L^1(\mathfrak{J}) := \{\mathfrak{g} : \mathfrak{J} \to \mathbb{R} | \mathfrak{g} \text{ is measurable and Lebesgue integrable function.}\}$$

Then, $L^1(\mathfrak{J})$ forms a Banach space by regarding $\|\mathfrak{g}\|_{L^1} = \|\mathfrak{g}\|_1 = \int_{\mathfrak{J}} |\mathfrak{g}(\tau)| d\tau$.

Over a metric space $(\mathfrak{M}, \delta)$, the symbol $\mathfrak{P}(E)$ denotes the family of all nonempty subsets of $E \subset \mathfrak{M}$. Then, we set

$$\mathfrak{P}_{\pi}(E) = \{F \in \mathfrak{P}(E) : F \text{ fulfills the property } \pi\},$$

where, π can be, for instance, bounded, closed, compact, convex (in short, bd, cl, cp, cv). For clarification, consider, for example
$\mathfrak{P}_{bd,cl}(E) = \{F \in \mathfrak{P}(E) : F \text{ is bounded and closed}\}$.

A multivalued function $G : \mathfrak{J} \to \mathfrak{P}_{cl}(E)$ is called measurable whenever the mapping

$$\tau \to dist(\mathfrak{u}, G(\mathfrak{t})) = \inf\{\|\mathfrak{u} - \nu\| : \nu \in G(\tau)\}$$

is measurable for each $\mathfrak{u} \in E$.

A mapping $H_d : \mathfrak{P}(E) \times \mathfrak{P}(E) \to [0, \infty) \cup \{\infty\}$ described by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

is called Hausdorff metric, where $d(a, B) = \inf_{b \in B} d(a, b)$, $d(A, b) = \inf_{a \in A} d(a, b)$ and $A, B \subset E$. Then, the coupled $(\mathfrak{P}_{bd,cl}(E), H_d)$ is named as Hausdorff metric space.

Definition 2.1. *[14] The set*

$$S_G = \{\mathfrak{g} \in L^1(\mathfrak{J}) : \mathfrak{g}(\tau) \in G(\tau)\ ,\ a.e.\ \tau \in \mathfrak{J}\},$$

is the selection set of G. Moreover, the set selector $S_{F \circ \mathfrak{g}}$, for each $\mathfrak{g} \in C(\mathfrak{J})$ from $F \circ \mathfrak{g}$ is formulated by $S_{F \circ \mathfrak{g}} := \{\mathfrak{u} \in L^1(\mathfrak{J}) : \mathfrak{u}(\tau) \in F(\tau, \mathfrak{g}(\tau)),\ a.e.\ \tau \in \mathfrak{J}\}$

A selfmapping $\mathcal{O}$ on a metric space $(\mathfrak{M}, \delta)$ is called

($P.o.$) Picard operator ($P.o.$) if $\mathcal{F}ix_{\mathcal{O}} = \{\mathfrak{z}^*\}$ for $\mathfrak{z}^* \in \mathfrak{M}$ and $(\mathcal{O}^n(\mathfrak{z}_0))_{n \in \mathbb{N}} \to \mathfrak{z}^*$ for any $\mathfrak{z}_0 \in \mathfrak{M}$.

($w.P.o.$) weakly Picard operator (w.P.o) if $(\mathcal{O}^n(\mathfrak{z}))_{n \in \mathbb{N}} \to \mathfrak{z}^* \in \mathfrak{M}$, in a way that $\mathfrak{z}^* \in \mathcal{F}ix_{\mathcal{O}}$, (limit may depend on $\mathfrak{z}$).

($k.w.P.o.$) k-weakly Picard operator (c.w.P.o) if it is (w.P.o) and $d(\mathfrak{z}, \mathcal{O}^{\infty}(\mathfrak{z})) \leq k\ d(\mathfrak{z}, \mathcal{O}(\mathfrak{z}))$; $\mathfrak{z} \in X$.

where $\mathcal{F}ix_{\mathcal{O}} = \{\mathfrak{z} : \mathfrak{z} = \mathcal{O}\mathfrak{z}\}$. Further, for a (w.P.o) $\mathcal{O}$, we set $\mathcal{O}^{\infty} = \mathcal{O}^{\infty}(\mathfrak{z}) = \lim_{n \to \infty} \mathcal{O}^n(\mathfrak{z})$. Notice that $\mathcal{O}^{\infty}(\mathfrak{M}) = \mathcal{F}ix_{\mathcal{O}}$.
A multivalued mapping $\mathcal{Q} : \mathfrak{M} \to \mathfrak{P}(\mathfrak{M})$ on $(\mathfrak{M}, \delta)$ is called weakly Picard operator (m.w.P.o.)[23, 32], if for each $\mathfrak{g} \in \mathfrak{M}$ and $\mathfrak{y} \in \mathcal{Q}(x)$, there is $(\mathfrak{g}_n)_{n \in \mathbb{N}}$ where

(i) $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = \mathfrak{y}$;

(ii) $\mathfrak{g}_{n+1} \in \mathcal{Q}(\mathfrak{g}_n)$, $n \in \mathbb{N}$;

(iii) $(\mathfrak{g}_n)_{n\in\mathbb{N}} \to \mathfrak{g}^*$ so that $\mathfrak{g}^* \in \mathcal{Fix}_{\mathcal{Q}}$.

Set $\mathfrak{A} := \{\varphi : [0,\infty) \to [0,\infty) |\ \varphi$ increasing, and $\lim_{n\to\infty} \varphi^n(t) \to 0$ for every $t \in [0,\infty\}$, where φ^n is the n-th iterate of φ. Here φ is called comparison function [34]. If $\varphi \in \mathfrak{A}$ then φ is continuous at 0 and $\varphi(t) < t$ for all $t > 0$. Furthermore, we set

$$\mathfrak{S} := \{\varphi : [0,\infty) \to [0,\infty) |\ \varphi \text{ strictly increasing} \& \sum_{n=1}^{\infty} \varphi^n(t) < \infty \text{ for all } t \in [0,\infty)\},$$

Here, $\varphi \in \mathfrak{S}$ is called strictly comparison function and $\mathfrak{S} \subset \mathfrak{A}$.

Definition 2.2. *For $\varphi \in \mathfrak{A}$, operator $\mathcal{Q} : \mathfrak{M} \to \mathfrak{P}_{cl}(\mathfrak{M})$ is called φ-multivalued weakly Picard (briefly $\varphi-$m.w.P. operator) if it is a m.w.P. and there is a selection $\mathcal{O}^\infty : \Lambda_{\mathcal{Q}} \to \mathcal{Fix}_{\mathcal{Q}}$ of $\mathcal{Q}^\infty$ so that*

$$d(\theta, \mathcal{O}^\infty(\theta,\nu)) \le \varphi(d(\theta,\nu)); \ for\ all\ (\theta,\nu) \in \Lambda_{\mathcal{Q}}.$$

In particular, if $\varphi(\mathfrak{z}) = k\mathfrak{z}$, for all $\mathfrak{z} \in \mathbb{R}_+$, for some $k > 0$ then $\mathcal{Q}$ is named as k-multivalued weakly Picard operator (k-m.w.P.o.).

Definition 2.3. *An operator $\mathcal{Q} : \mathfrak{M} \to \mathfrak{P}_{cl}(\mathfrak{M})$ is named*

a) multivalued k-Lipschitz if there is $k \ge 0$ with

$$H_\delta(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\nu)) \le \gamma\delta(\mathfrak{q}, v); \ for\ each\ \ \mathfrak{q},\ \nu \in \mathfrak{M}, \tag{2.1}$$

b) a multivalued $k-$contraction if (2.1) holds for $k \in [0,1)$,

c) a multivalued $\varphi-$contraction if there is a $\varphi \in \mathfrak{S}$ with

$$H_\delta(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\nu)) \le \varphi(\delta(\mathfrak{q},\nu)); \ for\ each\ \ \mathfrak{q},\ \nu \in \mathfrak{M}.$$

Definition 2.4. *[1]. The inclusion $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ is named generalized Ulam type (g.U.t) stable if there is $\varphi \in \mathfrak{S}$ such that for each $\varepsilon > 0$ and solution $\mathfrak{g} \in C(\mathfrak{J})$ of*

$$H_\delta(\mathfrak{g}(\tau), (\mathcal{Q}\mathfrak{g})(\tau)) \le \varepsilon; \ \tau \in \mathfrak{J},$$

there is a solution $\mathfrak{u} \in C(\mathfrak{J})$ of $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ (inclusion) so that

$$|\mathfrak{g}(\tau) - \mathfrak{u}(\tau)| \le \theta_{\mathcal{Q}}(\varepsilon); \ \tau \in \mathfrak{J}.$$

In case of $\varphi(t) = kt;\ k > 0$, it is called Ulam type stable.

Definition 2.5. *[1, 2, 3]. The fixed point inclusion $\mathfrak{g} \in \mathfrak{Q}(\mathfrak{g})$ is named generalized Ulam-Rassias type stable with respect to ϕ if there is a real number $c_{N,\phi} > 0$ such that for each solution $\mathfrak{g} \in C_\gamma$ of*

$$H_d(\mathfrak{g}(\tau), (\mathfrak{Q}\mathfrak{g})(\tau)) \leq \phi(\tau);\ t \in I,$$

there is a solution $\mathfrak{u} \in C(\mathfrak{J})$ of $\mathfrak{g} \in \mathfrak{Q}(\mathfrak{g})$ (inclusion) such that

$$|\mathfrak{g}(\tau) - \mathfrak{u}(\tau)| \leq c_{N,\phi}\phi(\tau);\ t \in I.$$

In case of $\phi(t) = kt;\ k > 0$, it is called Ulam-Rassias type stable.

Lemma 2.6. *[39] Let $\mathfrak{Q}$ be a multivalued mapping from a complete metric space $(\mathfrak{M}, \delta)$ to $\mathfrak{P}_{cl}(\mathfrak{M})$, and $\lambda \in \mathfrak{S}$. If $\mathfrak{Q}$ is a λ-contraction, then $\mathcal{Fix}_{\mathfrak{Q}} \neq \emptyset$. Moreover, $\{\mathfrak{Q}$ is (w.P.o).*

Theorem 2.7. *[20] Let $\mathfrak{Q}$ be a multivalued mapping from a complete metric space $(\mathfrak{M}, \delta)$ to $\mathfrak{P}_{cl}(\mathfrak{M})$, and $\lambda \in \mathfrak{S}$. If $\mathfrak{Q}$ forms a multivalued $\lambda-$contraction, then*

(1) *h is a m.w.P. operator;*

(2) *If additionally $\lambda(\kappa\tau) \leq \kappa\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$ (where $\kappa > 1$), then h is a φ-m.w.P. operator, with $\varphi(\tau) := \tau + \sum_{n=1}^{\infty} \lambda n(\tau)$, for each $\tau \in \mathbb{R}^+$;*

(3) *Let $S : \mathfrak{M} \to \mathfrak{P}_{cl}(\mathfrak{M})$ be a $\lambda-$contraction and $\eta > 0$ be such that $H_\delta(S(\tau), \mathfrak{Q}(\tau)) \leq \eta$ for each $\tau \in E$. Suppose that $\lambda(\kappa\tau) \leq \kappa\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$ (where $\kappa > 1$). Then,*

$$H_\delta(\mathcal{Fix}_S, \mathcal{Fix}_F) \leq \varphi(\eta).$$

For $\varepsilon \in \mathbb{R}$, we set

$$[\varepsilon]_q = \frac{1 - q^\varepsilon}{1 - q}.$$

Definition 2.8. *[15] The q-derivative of order $n \in \mathbb{N}$ of $\mathfrak{g} : \mathfrak{J} \to \mathbb{R}$ is described as $(D_q^0\mathfrak{g})(\tau) = \mathfrak{g}(\tau)$,*

$$(D_q\mathfrak{g})(\tau) := (D_q^1\mathfrak{g})(\tau) = \frac{\mathfrak{g}(\tau) - \mathfrak{g}(q\tau)}{(1 - q)\tau};\ \tau \neq 0, \quad (D_q\mathfrak{g})(0) = \lim_{t \to 0}(D_q\mathfrak{g})(\tau),$$

and

$$(D_q^n\mathfrak{g})(\tau) = (D_q D_q^{n-1}\mathfrak{g})(\tau);\ \tau \in \mathfrak{J},\ n \in \{1, 2, \ldots\}.$$

Set $I_s := \{sq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.9. *[15] The q-integral of* $\mathfrak{g} : I_s \to \mathbb{R}$ *is described as*

$$(I_q\mathfrak{g})(s) = \int_0^s \mathfrak{g}(\tau)d_q\tau = \sum_{n=0}^{\infty} t(1-q)q^n\mathfrak{g}(sq^n).$$

$(D_qI_q\mathfrak{g})(s) = \mathfrak{g}(s)$, while if $\mathfrak{g}$ is continuous at 0, then

$$(I_qD_q\mathfrak{g})(s) = \mathfrak{g}(s) - \mathfrak{g}(0).$$

Definition 2.10. *[8] The Riemann-Liouville fractional q-integral of order* $\alpha \in \mathbb{R}_+ := [0, \infty)$ *of a function* $\mathfrak{g} : \mathfrak{J} \to \mathbb{R}$ *is defined by* $(I_q^0\mathfrak{g})(s) = \mathfrak{g}(s)$, *and*

$$(I_q^\alpha\mathfrak{g})(s) = \int_0^t \frac{(s-q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)}\mathfrak{g}(\tau)d_q\tau;\ t \in \mathfrak{J}.$$

Definition 2.11. *[25] The Caputo fractional q-derivative of order* $\alpha \in \mathbb{R}_+$ *of a function* $\mathfrak{g} : \mathfrak{J} \to \mathbb{R}$ *is defined by* $({}^CD_q^0\mathfrak{g})(s) = \mathfrak{g}(s)$, *and*

$$({}^CD_q^\alpha\mathfrak{g})(s) = (I_q^{[\alpha]-\alpha}D_q^{[\alpha]}\mathfrak{g})(s);\ s \in \mathfrak{J}.$$

Lemma 2.12. *[25] Let* $\alpha \in \mathbb{R}_+$. *Then*

$$(I_q^\alpha\ {}^CD_q^\alpha\mathfrak{g})(s) = \mathfrak{g}(s) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)}(D_q^k\mathfrak{g})(0).$$

In particular, if $\alpha \in (0, 1)$, *then*

$$(I_q^\alpha\ {}^CD_q^\alpha\mathfrak{g})(s) = \mathfrak{g}(s) - \mathfrak{g}(0).$$

Lemma 2.13. *Assume that* $S_{F\circ\mathfrak{g}} \subset C(\mathfrak{J})$ *for each* $\mathfrak{g} \in C(\mathfrak{J})$. *Then (1.1)-(1.2) is equivalent to* $\mathfrak{g} \in \mathfrak{Q}(\mathfrak{g})$, *where* $\mathfrak{Q} : C(\mathfrak{J}) \to \mathfrak{P}(C(\mathfrak{J}))$ *is the multi-function described as*

$$(\mathfrak{Q}\mathfrak{g})(\tau) = \{\mathfrak{g}_0 + (I_q^\alpha g)(\tau) : g \in S_{F\circ\mathfrak{g}}\}.$$

In this manuscript, we launch the study of the Ulam stability for Caputo fractional q-difference inclusions and related coupled systems via Picard operators theory, and it is structured as follows: Section 2 the first main result, existence and stability of (1.1) and (1.2), is expressed. Additionally, in Section 3; we obtain similar results for the coupled system (1.3)-(1.4). Lastly, in Section 4 an example is expressed to indicate the applicability of the derived theorem of the paper.

3 Caputo Fractional q-Difference Inclusions

Definition 3.1. $\mathfrak{g}$ *is a solution of (1.1)-(1.2) if it achieves the condition (1.2), and the equation* $\mathfrak{g}(\tau) = \mathfrak{g}_0 + (I_q^\alpha g)(\tau)$ *on* $\mathfrak{J}$, *where* $g \in S_{F\circ\mathfrak{g}}$.

(H_1) The multifunction $\tau \longmapsto F(\tau, \mathfrak{g})$ is jointly measurable for each $\mathfrak{g} \in \mathbb{R}$

(H_2) The multifunction $\mathfrak{g} \longmapsto F(\tau, \mathfrak{g})$ is l.s.c. for a.a. $\tau \in \mathfrak{J}$;

(H_3) There exists a function $\varrho \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\lambda \in \mathfrak{S}$ so that

$$H_\delta(F(\tau, \mathfrak{g}), F(\tau, \overline{\mathfrak{g}})) \leq \varrho(\tau)\lambda(|\mathfrak{g} - \overline{\mathfrak{g}}|), \tag{3.1}$$

and

$$\frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1+\alpha)} \leq 1, \tag{3.2}$$

for almost all $\tau \in \mathfrak{J}$, and each $\mathfrak{g}, \overline{\mathfrak{g}} \in \mathbb{R}$;

(H_4) There exists $q \in L^1(\mathfrak{J}, \mathbb{R}_+)$ such that

$$F(\tau, \mathfrak{g}) \subset q(\tau)B_0,$$

where $B_0 = \{\mathfrak{g} \in C(\mathfrak{J}) : \|\mathfrak{g}\|_C < 1\}$, for almost all $\tau \in \mathfrak{J}$ and each $\mathfrak{g} \in \mathbb{R}$.

A self-mapping λ on $[0, \infty)$ is called quasi-homogenous function if If $\lambda(\mathfrak{z}\tau) \leq \mathfrak{d}\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$, where $\mathfrak{d} > 1$.

Theorem 3.2. *Assume* $(H_1) - (H_4)$ *hold. Then:*

(a) *Pb. (1.1)-(1.2) admits least one solution and* $\mathfrak{Q}$ *is a m.w.P.o.;*

(b) *Furthermore, if* λ *is quasi-homogenous, then Pb. (1.1)-(1.2) is g.U.t stable, and* N *is a* φ*-m.w.P.o., with*

$$\varphi(\tau) := t + \sum_{n=1}^{\infty} \lambda^n(\tau), \ \tau \in \mathbb{R}^+.$$

Proof. First, we assert $\mathfrak{Q}(\mathfrak{g}) \in \mathfrak{P}_{cp}(C(\mathfrak{J}))$ for each $\mathfrak{g} \in C(\mathfrak{J})$.
For each $\mathfrak{g} \in C(\mathfrak{J})$ there exists $f \in S_{F\circ\mathfrak{g}}$, (see [33]). Thus $\nu(\tau) = \mathfrak{g}_0 + (I_q^\alpha f)(\tau)$ verify $\nu \in \mathfrak{Q}(\mathfrak{g})$. From (H_1) and (H_4), via Theorem 8.6.3. in [11], the set $\mathfrak{Q}(\mathfrak{g})$ is compact, for each $\mathfrak{g} \in C(\mathfrak{J})$.

Next, we assert $H_\delta(\mathfrak{Q}(\mathfrak{g}), \mathfrak{Q}(\overline{\mathfrak{g}})) \leq \lambda(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C)$ *for each* $\mathfrak{g}, \overline{\mathfrak{g}} \in C(\mathfrak{J})$.
Let $\mathfrak{g}, \overline{\mathfrak{g}} \in C(\mathfrak{J})$ and $h \in \mathfrak{Q}(\mathfrak{g})$. Then, there exists $f \in S_{F\circ\mathfrak{g}}$, with

$$h(\tau) = \mathfrak{g}_0 + (I_q^\alpha f)(\tau).$$

We get, from (H_3)

$$H_\delta(F(\tau,\mathfrak{g}(\tau)),F(\tau,\overline{\mathfrak{g}}(\tau)))\leq\varrho(\tau)\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C).$$

Consequently, there is $w\in S_{F\circ\overline{\mathfrak{g}}}$, with

$$|f(\tau)-w(\tau)|\leq\varrho(\tau)\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C);\ \tau\in I.$$

Construct a map $\mathfrak{G}:\mathfrak{J}\to\mathfrak{P}(\mathbb{R})$ by

$$\mathfrak{G}(\tau)=\{\mathfrak{b}\in\mathbb{R}:|f(\tau)-\mathfrak{b}(\tau)|\leq\varrho(\tau)\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C)\}.$$

Due to the measurability of $\mathfrak{g}(\tau)=\mathfrak{G}(\tau)\cap F(\tau,\overline{\mathfrak{g}}(\tau))$ (Proposition III.4 in [13]), then there is $\overline{f}$ which is a measurable selection function for $\mathfrak{g}$. Thus, $\overline{f}\in S_{F\circ\overline{\mathfrak{g}}}$, and for each $\tau\in\mathfrak{J}$,

$$|f(\tau)-\overline{f}(\tau)|\leq\varrho(\tau)\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}|_C).$$

Let the function

$$\overline{h}(\tau)=\mathfrak{g}_0+(I_q^\alpha\overline{f})(\tau).$$

Then

$$\begin{aligned}|h(\tau)-\overline{h}(\tau)| &\leq I_q^\alpha|f(\tau)-\overline{f}(\tau)|\\ &\leq I_q^\alpha(\varrho(\tau)\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C))\\ &\leq \|\varrho\|_{L^\infty}\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C)\left(\int_0^\tau\frac{|\tau-qs|^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs\right)\\ &\leq \frac{T^\alpha\|\varrho\|_{L^\infty}}{\Gamma_q(1+\alpha)}\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C).\end{aligned}$$

Thus, from (4.2) yields

$$\|h-\overline{h}\|_C\leq\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C).$$

By verbatim with changing the roles of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$, it yields

$$H_\delta(\mathcal{Q}(\mathfrak{g}),\mathcal{Q}(\overline{\mathfrak{g}}))\leq\lambda(\|\mathfrak{g}-\overline{\mathfrak{g}}\|_C).$$

Hence, $\mathcal{Q}$ is a $\lambda-$contraction.
(a) Lemma 2.6 infer that $\mathcal{Q}$ possesses a fixed point on $\mathfrak{J}$, and from [Theorem 2.7, (i)] we conclude that $\mathcal{Q}$ is a m.w.P.o.
(b) The problem (1.1)-(1.2) is g.U-H stable. For clarification, for $\varepsilon>0$ and $\nu\in C(\mathfrak{J})$ there is $\mathfrak{g}\in C(\mathfrak{J})$ so that

$$\mathfrak{g}(\tau)-\mathfrak{g}_0\in(I_q^\alpha F)(\tau,\nu(\tau));\ \tau\in\mathfrak{J},$$

and

$$\|\mathfrak{g} - \nu\|_C \leq \varepsilon,$$

where

$$(I_q^\alpha F)(\tau, \nu(\tau)) = \{(I_q^\alpha w)(\tau);\ w \in S_{F\circ\nu}\};\ \tau \in \mathfrak{J}.$$

Then $H_\delta(y, \mathfrak{Q}(y)) \leq \varepsilon$. Moreover, the multivalued map $\mathfrak{Q}$ is $\lambda-$contraction, and from [Theorem 2.7, (i)-(ii)], $\mathfrak{Q}$ is a φ-m.w.P.o. Tus, $\mathfrak{g} \in \mathfrak{Q}(\mathfrak{g})$ is g.U-H stable. Hence, our problem (1.1)-(1.2) is g.U-H stable. Theorem 2.7,(iii) concludes the result.

4 Caputo Fractional q-Difference Inclusions

Ulam stability of the problem (1.1)-(1.2) shall be discussed in this section.

Definition 4.1. *If a continuous $\mathfrak{g}$ along the initial condition (1.2) achieve $\mathfrak{g}(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha g)(\mathfrak{t})$ on $\mathfrak{J}$, where $g \in S_{F\circ\mathfrak{g}}$, then we say that it is a solution of the problem (1.1)-(1.2)*

Now, we present requirements for both Ulam stability of problem (1.1)-(1.2).

The following are the basic requirements for our aim:

(H_1) The multifunction $\mathfrak{t} \longmapsto F(\mathfrak{t}, \mathfrak{g})$ is jointly measurable for each $\mathfrak{g} \in \mathbb{R}$

(H_2) The multifunction $\mathfrak{g} \longmapsto F(\mathfrak{t}, \mathfrak{g})$ is lower semi-continuous for almost all $\mathfrak{t} \in \mathfrak{J}$;

(H_3) There exists $\rho \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\varphi \in \mathfrak{S}$ so that for for almost all $\mathfrak{t} \in \mathfrak{J}$, and each $\mathfrak{g}, \overline{\mathfrak{g}} \in \mathbb{R}$, we have

$$H_d(F(\mathfrak{t}, \mathfrak{g}), F(\mathfrak{t}, \overline{\mathfrak{g}})) \leq \varrho(\mathfrak{t})\varphi(|\mathfrak{g} - \overline{\mathfrak{g}}|), \tag{4.1}$$

and

$$\frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1+\alpha)} \leq 1; \tag{4.2}$$

(H_4) we have $F(\mathfrak{t}, \mathfrak{g}) \subset q(\mathfrak{t})B(0,1)$, for almost all $\mathfrak{t} \in \mathfrak{J}$ and each $\mathfrak{g} \in \mathbb{R}$, where $q : \mathfrak{J} \to \mathbb{R}$ is integrable and $B(0,1) = \{\mathfrak{g} \in C(\mathfrak{J}) : \|\mathfrak{g}\|_C < 1\}$.

Theorem 4.2. *Suppose that $(H_1) - (H_4)$ are achieved by the multifunction $F : \mathfrak{J} \times \mathbb{R} \to \mathfrak{P}_{cp}(\mathbb{R})$ Then,*

(a) *Problem (1.1)-(1.2) possesses a solution and $\mathfrak{Q}$ is a m.w.P.o;*

(b) *In addition, if λ is quasi-homogenous, then the problem defined by (1.1)-(1.2) is g.U.t. stable, and $\mathfrak{Q}$ is a λ-m.w.P.o, where λ is described as* $\lambda(\mathfrak{t}) := t + \sum_{n=1}^{\infty} \varphi^n(\mathfrak{t})$, *for each* $\mathfrak{t} \in [0, \infty)$.

Remark 4.3. *Note that $S_{F\circ\mathfrak{g}}$ is nonempty for all $\mathfrak{g} \in C(\mathfrak{J})$ on account of (H_1) F has a measurable selection (see [13], Theorem III.6)*

Proof. Let $\mathfrak{Q}$ a mapping as described in Lemma 2.13. We assert that it achieves the hypothesis of Theorem 2.7. We assert first that $\mathfrak{Q}(x) \in \mathfrak{P}_{cp}(C(\mathfrak{J}))$ for each $\mathfrak{g} \in C(\mathfrak{J})$.
On account of Theorem 2 in [33], for each $\mathfrak{g} \in C(\mathfrak{J})$ there is $f \in S_{F\circ\mathfrak{g}}$, for all $\mathfrak{t} \in \mathfrak{J}$. Then, $\nu(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha f)(\mathfrak{t})$ has the property $\nu \in \mathfrak{Q}(\mathfrak{g})$. In addition, taking (H_1) and (H_4), together with Theorem 8.6.3. in [11], we find that for each $\mathfrak{g} \in C(\mathfrak{J})$, the set $\mathfrak{Q}(\mathfrak{g})$ is compact.

Next, we assert that $H_d(\mathfrak{Q}(\mathfrak{g}), \mathfrak{Q}(\overline{\mathfrak{g}})) \le \varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C)$ for each $\mathfrak{g}, \overline{\mathfrak{g}} \in C(\mathfrak{J})$. Let $\mathfrak{g}, \overline{\mathfrak{g}} \in C(\mathfrak{J})$ and $h \in \mathfrak{Q}(\mathfrak{g})$. So, there is $f \in S_{F\circ\mathfrak{g}}$, so that

$$h(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha f)(\mathfrak{t}),$$

for each $\mathfrak{t} \in \mathfrak{J}$. Due to (H_3), we have

$$H_d(F(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})), F(\mathfrak{t}, \overline{\mathfrak{g}}(\mathfrak{t}))) \le \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C).$$

Consequently, there is $w \in S_{F\circ\overline{\mathfrak{g}}}$, with

$$|f(\mathfrak{t}) - w(\mathfrak{t})| \le \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C); \ \mathfrak{t} \in \mathfrak{J}.$$

We set $\mathfrak{G} : \mathfrak{J} \to \mathfrak{P}(\mathbb{R})$ as follows

$$\mathfrak{G}(\mathfrak{t}) = \{w \in \mathbb{R} : |f(\mathfrak{t}) - w(\mathfrak{t})| \le \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C)\}.$$

Note that $\mathfrak{g}(\mathfrak{t}) = \mathfrak{G}(\mathfrak{t}) \cap F(\mathfrak{t}, \overline{\mathfrak{g}}(\mathfrak{t}))$ is a measurable multivalued operator due to Proposition III.4 in [13]. Consequently, there is a measurable selection function $\overline{f}$ for $\mathfrak{g}$. Thus, $\overline{f} \in S_{F\circ\overline{\mathfrak{g}}}$, and

$$|f(\mathfrak{t}) - \overline{f}(\mathfrak{t})| \le \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}|_C),$$

for all $\mathfrak{t} \in \mathfrak{J}$. Define

$$\overline{h}(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha \overline{f})(\mathfrak{t}),$$

for each $\mathfrak{t} \in \mathfrak{J}$. Consequently, we find

$$\begin{aligned} |h(\mathfrak{t}) - \overline{h}(\mathfrak{t})| &\leq \mathfrak{I}_q^{\alpha}|f(\mathfrak{t}) - \overline{f}(\mathfrak{t})| \\ &\leq \mathfrak{I}_q^{\alpha}(\varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C)) \\ &\leq \|\varrho\|_{L^\infty}\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C)\left(\int_0^{\mathfrak{t}} \frac{|\mathfrak{t} - qs|^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s\right) \\ &\leq \frac{T^{\alpha}\|\varrho\|_{L^\infty}}{\Gamma_q(1+\alpha)}\varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C), \end{aligned}$$

for each $t \in \mathfrak{J}$. On account of (4.2), we find

$$\|h - \overline{h}\|_C \leq \varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C).$$

Regarding the analogy, changing the roles of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$, yields

$$H_d(\mathcal{Q}(\mathfrak{g}), \mathcal{Q}(\overline{\mathfrak{g}})) \leq \varphi(\|\mathfrak{g} - \overline{\mathfrak{g}}\|_C).$$

As a result, $\mathcal{Q}$ is a $\varphi-$contraction.

By taking Lemma 2.6 into account, we deduce that fixed point of $\mathcal{Q}$ possesses a solution of the inclusion (1.1)-(1.2) on $\mathfrak{J}$. Further, [Theorem 2.7, (i)] yields that $\mathcal{Q}$ is a m.w.P.o.

Now, we assert that the problem (1.1)-(1.2) is g.U.t stable.
For this purpose, take $\epsilon > 0$ and $\nu \in C(\mathfrak{J})$ for which there is $\mathfrak{g} \in C(\mathfrak{J})$ so that

$$\mathfrak{g}(\mathfrak{t}) - \mathfrak{g}_0 \in (\mathfrak{I}_q^{\alpha}F)(\mathfrak{t}, \vartheta(\mathfrak{t}));\ \mathfrak{t} \in \mathfrak{J},$$

and

$$\|\mathfrak{g} - \nu\|_C \leq \epsilon,$$

with

$$(\mathfrak{I}_q^{\alpha}F)(\mathfrak{t}, \nu(\mathfrak{t})) = \{(\mathfrak{I}_q^{\alpha}w)(\mathfrak{t});\ w \in S_{F\circ\nu}\};\ \mathfrak{t} \in \mathfrak{J}.$$

Then $H_d(y, \mathcal{Q}(y)) \leq \epsilon$. In addition, we conclude that $\mathcal{Q}$ is a multivalued $\varphi-$contraction. Regarding [Theorem 2.7, (i)-(ii)], we deduce that $\mathcal{Q}$ is a Ψ-m.w.P.o. Thus, the fixed point problem $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ is g.U.t. stable. In conclusion, the problem (1.1)-(1.2) is g.U.t. stable. The rest follows from [Theorem 2.7,(iii)].

5 Coupled System of Caputo Fractional q-Difference Inclusions

This section is devoted for the existence, uniqueness and Ulam stability of (1.3)-(1.4). Here, $\mathcal{C} := C(\mathfrak{J}) \times C(\mathfrak{J})$ denotes the Banach space with the norm

$$\|(\mathfrak{g}, \nu)\|_{\mathcal{C}} = \|\mathfrak{g}\|_C + \|\nu\|_C.$$

Lemma 5.1. *Let $G : \mathcal{C} \to \mathcal{C}$ described as*

$$(G(\mathfrak{g}_1, \mathfrak{g}_2))(\mathfrak{t}) = ((G_1\mathfrak{g}_1)(\mathfrak{t}), G_2\mathfrak{g} - 2)(\mathfrak{t})); \;\; \mathfrak{t} \in \mathfrak{J}, \tag{5.1}$$

where $G_i : C(\mathfrak{J}) \to C(\mathfrak{J})$; $i = 1, 2$, are defined by

$$(G_i(\mathfrak{g}_1, \mathfrak{g}_2))(\mathfrak{t}) = \mathfrak{i}_i + (\mathfrak{J}_{q_i}^{\alpha_i} g_i)(\mathfrak{t}); \;\; \mathfrak{t} \in \mathfrak{J}, \tag{5.2}$$

where $g_i \in S_{F_i \circ \mathfrak{g}_i}$; $i = 1, 2$. Then, fixed points of G form the solutions of the system (1.3)-(1.4).

Definition 5.2. *By a coupled solutions of problem (1.3)-(1.4) we mean a continuous coupled functions $(\mathfrak{g}_1, \mathfrak{g}_2)$ those satisfy the initial condition (1.4), and the equations $\mathfrak{g}_i(\mathfrak{t}) = \mathfrak{i}_i + (\mathfrak{J}_{q_i}^{\alpha_i} \nu_i)(\mathfrak{t})$ on $\mathfrak{J}$, where $\nu_i \in S_{F_i \circ \mathfrak{g}_i}$; $i = 1, 2$.*

Keeping Lemma 5.1 on mind, we shall investigate the existence and Ulam stability of (1.3)-(1.4), as in Theorem 4.2.

Theorem 5.3. *Assume that the multifunctions $F_i : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathcal{N}_{cp}(\mathbb{R})$ satisfy the following hypotheses*

(H_{01}) *The multifunctions $\mathfrak{t} \longmapsto F_i(\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_2)$ are jointly measurable for each $\mathfrak{g}_i \in \mathbb{R}$; $i = 1, 2$,*

(H_{02}) *The multifunctions $\mathfrak{g}_i \longmapsto F(\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_2)$ are lower semi-continuous for almost all $\mathfrak{t} \in \mathfrak{J}$;*

(H_{03}) *There exist $p_i \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\varphi_i :\in \mathfrak{S}$ such that*

$$H_d(F_i(\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_2), F_i(\mathfrak{t}, \overline{\mathfrak{g}_1}, \overline{\mathfrak{g}_2}) \leq p_i(\mathfrak{t})\varphi_i(|\mathfrak{g}_i - \overline{\mathfrak{g}_i}|), \tag{5.3}$$

and

$$\frac{T^{\alpha_i}\|p_i\|_{L^\infty}}{\Gamma_q(1+\alpha_i)} \leq 1; \tag{5.4}$$

or for almost all $\mathfrak{t} \in \mathfrak{J}$, and each $\mathfrak{g}_1, \mathfrak{g}_2, \overline{\mathfrak{g}_1}, \overline{\mathfrak{g}_2} \in \mathbb{R}$.

(H_{04}) *There exist integrable functions $q_i : \mathfrak{J} \to \mathbb{R}$ such that for almost all $\mathfrak{t} \in \mathfrak{J}$ and each $\mathfrak{g}_i \in \mathbb{R}$; $i = 1, 2$, we have*

$$F_i(\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_2) \subset q_i(\mathfrak{t})B(0, 1),$$

where $B(0, 1) = \{\nu \in C(\mathfrak{J}) : \|\nu\|_C < 1\}$.

Then, we have

(a) *Problem (1.3)-(1.4) possess a solution and G is a m.w.P.o;*

(b) *In addition, if each* φ_i *is quasi-homogenous, (* $i = 1, 2,$ *) then the problem (1.3)-(1.4) is g. U. t. stable, and* G *is a* Ψ*-m.w.P.o, with* $\Psi = (\Psi_1, \Psi_2)$ *and the the functions* Ψ_i; $i = 1, 2$ *defined by* $\Psi_i(\mathfrak{t}) := \mathfrak{t} + \sum_{n=1}^{\infty} \varphi_i^n(\mathfrak{t})$, *for each* $\mathfrak{t} \in [0, \infty)$.

6 An Example

We aim to deal with the Cauchy problem of Caputo fractional $\frac{1}{4}-$difference inclusion

$$\begin{cases} ({}^C\mathfrak{D}_{\frac{1}{4}}^{\frac{1}{2}}\mathfrak{g})(\mathfrak{t}) \in F(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})); \ \mathfrak{t} \in [0, 1], \\ \mathfrak{g}(\mathfrak{t})|_{\mathfrak{t}=0} = 1, \end{cases} \tag{6.1}$$

for

$$F(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})) = \{\nu \in C([0, 1], \mathbb{R}) : |f_1(\mathfrak{t}, \mathfrak{g}(\mathfrak{t}))| \le |\nu| \le |f_2(\mathfrak{t}, \mathfrak{g}(\mathfrak{t}))|\}; \ \mathfrak{t} \in [0, 1],$$

where $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, such that

$$f_1(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})) = \frac{\mathfrak{t}^2\mathfrak{g}(\mathfrak{t})}{(1 + |\mathfrak{g}(\mathfrak{t})|)e^{10+\mathfrak{t}}}, \quad f_2(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})) = \frac{\mathfrak{t}^2\mathfrak{g}(\mathfrak{t})}{e^{10+\mathfrak{t}}}.$$

Set $\alpha = \frac{1}{2}$ and suppose that F is both convex and closed multivalued function. Notice that the solutions of the problem (6.1) are the solutions

$$\mathfrak{g} \in A(\mathfrak{g}) \text{ (the fixed point inclusion)}$$

where the multifunction operator $A : C([0, 1], \mathbb{R}) \to \mathfrak{P}(C([0, 1], \mathbb{R}))$ is described as

$$(A\mathfrak{g})(\mathfrak{t}) = \left\{1 + (\mathfrak{I}_{\frac{1}{4}}^{\frac{1}{2}} f)(\mathfrak{t}); \ f \in S_{F\circ\mathfrak{g}}\right\}; \ \mathfrak{t} \in [0, 1].$$

For each $\mathfrak{t} \in [0, 1]$ and all $z_1, z_2 \in C([0, 1], \mathbb{R})$, we have

$$\|f_2(\mathfrak{t}, z_2) - f_1(\mathfrak{t}, z_1)\|_C \le \mathfrak{t}^2 e^{-10-\mathfrak{t}}\|z_2 - z_1\|_C.$$

Consequently, we conclude that all hypotheses $(H_1) - (H_3)$ are achieved with $\varrho(\mathfrak{t}) = \mathfrak{t}^2e^{-10-\mathfrak{t}}$.

As a next step, we indicate that condition (4.2) is fulfilled for $T = 1$. For clarification, we note that $\|\varrho\|_{L^\infty} = e^{-9}$, $\Gamma_{\frac{1}{4}}(1 + \frac{1}{2}) > \frac{1}{2}$. After an elementary calculation, one can get that

$$\Delta := \frac{T^{\frac{3}{4}}\|\varrho\|_{L^\infty}}{\Gamma_{\frac{1}{4}}(1 + \frac{1}{2})} < 2e^{-9} < 1.$$

Furthermore, (H_4) is fulfilled with $q(\mathfrak{t}) = \frac{\mathfrak{t}^2 e^{-10-\mathfrak{t}}}{\|F\|_{\mathfrak{P}}}$; $\mathfrak{t} \in [0,1]$, where

$$\|F\|_{\mathfrak{P}} = \sup\{\|f\|_C : f \in S_{F\circ\mathfrak{g}}\}; \quad \text{for all} \quad \mathfrak{g} \in C([0,1],\mathbb{R}).$$

As a result, Theorem 4.2 implies that:

(a) The problem (6.1) possesses a solution and A is a m.w.P.o.

(b) The function $\varphi(\mathfrak{t}) = \Delta\mathfrak{t}$ forms quasi-homogenous. Hence, the problem (6.1) is g.U.t. stable, and A is a Ψ-m.w.P.o, with the function Ψ defined by $\Psi(\mathfrak{t}) := t + (1-\Delta\mathfrak{t})^{-1}$, for each $\mathfrak{t} \in [0, \Delta^{-1})$.

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