On weakly S-prime ideals of commutative rings

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Abstract

Let R be a commutative ring with identity and S be a multiplicative subset of R. In this paper, we introduce the concept of weakly S-prime ideals which is a generalization of weakly prime ideals. Let P be an ideal of R disjoint with S. We say that P is a weakly S-prime ideal of R if there exists an $s \in S$ such that, for all $a, b \in R$, if $0 \neq ab \in P$, then $sa \in P$ or $sb \in P$. We show that weakly S-prime ideals have many analog properties to these of weakly prime ideals. We also use this new class of ideals to characterize S-Noetherian rings and S-principal ideal rings.

1 Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. An ideal I of a ring R is said to be proper if $I \neq R$. Let R be a ring and I be an ideal of R. Then, $\sqrt{0}$ denotes the nilradical of R and $\sqrt{I}$ denotes the radical of I.

Recall that a proper ideal $\mathfrak{q}$ of a ring R is said to be prime if, whenever $a, b \in R$ with $ab \in \mathfrak{q}$, then $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. The prime ideals play a very important role in the commutative rings theory so this notion has been generalized and studied in several directions. The importance of some of these generalizations is same as the prime ideals. For instance, in 2003, D.D. Anderson and E. Smith [1] defined the weakly prime, that is a proper ideal P of R such that for all $a, b \in R$ with $0 \neq ab \in P$, $a \in P$ or $b \in P$. Every proper prime ideal of R

is weakly prime. However, the converse is not true. For instance, (0) is always weakly prime of R, and it is prime if and only if R is an integral domain. In a very recent paper[8], the authors introduced the notion of S-prime ideal which is also a generalization of prime ideals. As in [8], let $S \subseteq R$ a multiplicative subset of R and P an ideal of R disjoint with S, then P is said to be S-prime P if there exists an $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in P$, $sa \in P$ or $sb \in P$. Note that every (weakly) prime ideal of R which is disjoint with S is an S-prime ideal. However, the converse is not true in general [8, Example 1].

The main purpose of this paper is to introduce and to investigate the notion of S-prime ideals which is also a generalization of (weakly) prime ideals. Let R be a commutative ring and $S \subseteq R$ be a multiplicative set of R. An ideal P of R satisfying $P \cap S = \emptyset$ is said to be weakly S-prime if there exists an element $s \in S$ such that, whenever $a, b \in R$, $0 \neq ab \in P$ implies $sa \in P$ or $sb \in P$. In this case, we say that P is associated to s. Among many results in this paper, it is shown that if P is a weakly S-prime ideal of R which is not S-prime, then $P^2 = 0$. Theorem 7 proves that an ideal P is weakly S-prime of R (associated to $s \in S$) if and only if for each $x \notin (P : s)$ we have either $(P : x) \subseteq (P : s)$ or $(P : x) = (0 : x)$. In the case when S is consisting only of regular element, it is proved that P is a weakly S-prime ideal of R (associated to $s \in S$) if and only if $(P : s)$ is a weakly S-prime ideal of R (associated to $s \in S$) if and only if $S^{-1}P$ is a weakly prime ideal of $S^{-1}R$ and $S^{-1}P \cap R = (P : s)$.
Let R be a commutative ring and S a multiplicative subset of R. Recall from[1] that an ideal I of R is called S-finite (resp., S-principal) if $sI \subseteq J \subseteq I$ for some finitely generated (resp., principal) ideal J of R and some $s \in S$. Also, R is called S-Noetherian (resp., S-PIR) if each ideal of R is S-finite (resp., S-principal). Theorem 18 shows that R is S-Noetherian (resp., S-PIR) if and only if every weakly S-prime ideal of R is S-finite (resp., S-principal). Let $R := R_1 \times R_2$, $S = S_1 \times S_2$, where S_i is a multiplicative subset of R_i and P_1 and P_2 be an nonzero ideals of R_1 and R_2, respectively. Theorem 24 proves that weakly S-prime of R coincide with S-prime of R, and then characterizes them. Hence, we deduce that if, over a ring $R = R_1 \times R_2$, every proper ideal is weakly S-prime then R_1 and R_2 are fields. We turn off this paper, with a study of some kind of weakly S-prime ideals of the idealization of *Nagata* and the duplication of a ring along an ideal.

2 Weakly S-prime ideals of commutative rings

Definition 1. *Let R be a commutative ring and $S \subseteq R$ be a multiplicative set of R. An ideal P of R satisfying $P \cap S = \emptyset$ is said to be weakly S-prime if there exists an element $s \in S$ such that, whenever $a, b \in R$, $0 \neq ab \in P$*

implies $sa \in P$ or $sb \in P$. In this case, we say that P is associated to s.

Obviously, an S-prime ideal of R is weakly S-prime. However, the next example shows that the converse implication is not true in general.

Example 2. *Set $R := \mathbb{Z}/12\mathbb{Z}$ and $S = \{\overline{3}, \overline{9}\}$. The ideal $(\overline{0})$ is weakly S-prime. Now, $\overline{2}.\overline{6} = \overline{0}$. However, $\overline{2}.\overline{3} = \overline{2}.\overline{9} = \overline{6}.\overline{3} = \overline{6}.\overline{9} = \overline{6} \neq \overline{0}$. Hence, $(\overline{0})$ is not S-prime.*

A weakly-prime ideal P of R is weakly S-prime for each multiplicative set S of R such that $P \cap S = \emptyset$. Conversely, if $S \subseteq U(R)$ is a multiplicative set of R then weakly S-prime ideals are just the weakly-prime ideals of R. In particular, if $(R, \mathfrak{m})$ is a local ring and P is a proper ideal of R, then P is a weakly prime ideal of R if and only if P is a weakly $(R \backslash \mathfrak{m})$-prime ideal of R. However, in general, weakly prime ideals and weakly S-prime ideals are different concepts as shown by the following example.

Example 3. *Set $R := \mathbb{Z}[X]$ and $S = \{2^n | n \in \mathbb{N}\}$. By [8, Example 1], $P = 4X\mathbb{Z}[X]$ is an S-prime ideal of R, and so weakly S-prime. Moreover, $0 \neq 4X \in P$ but neither $4 \in P$ nor $X \in P$. Thus, P is not weakly prime.*

Proposition 4. *Let R be a ring, S be a multiplicative subset of R, and P be a weakly S-prime ideal of R. If P is not S-prime, then $P^2 = 0$.*

Proof. There exists $s \in S$ such that, whenever $a, b \in R$, $0 \neq ab \in P$ implies $sa \in P$ or $sb \in P$. Suppose that $P^2 \neq 0$. We claim that P is S-prime.
Let $a, b \in R$ such that $ab \in P$. If $0 \neq ab \in P$, then $sa \in P$ or $sb \in P$. Now, suppose that $ab = 0$. If $aP \neq 0$, there is $p \in P$ such that $0 \neq ap$, and so $0 \neq ap = a(p + b) \in P$. Hence, $sa \in P$ or $s(p + b) \in P$. Hence, $sa \in P$ or $sb \in P$. Similarly, if $bP \neq 0$, we obtain the same result. Finally, assume that $aP = 0$ and $bP = 0$. Since, $P^2 \neq 0$, there exists $p, q \in P$ such that $pq \neq 0$. Thus, $0 \neq pq = (a + p)(b + q) \in P$. Then, $s(a + p) \in P$ or $s(b + q) \in P$. Therefore, $sa \in P$ or $sb \in P$. Consequently, we conclude that P is an S-prime ideal of R, as desired. $\square$

Corollary 5 ([1], Theorem 1). *Let P be a weakly prime ideal of ring R. If P is not prime, then $P^2 = 0$.*

Proof. It suffices to take $S = \{1\}$ in Proposition 4. $\square$

Seeing [8, Proposition 2(4)], it is clear that if P is an S-prime ideal of R for some multiplicative subset S of R, then there exists an $s \in S$ such that $s\sqrt{0_R} \subseteq P$. Hence, using Proposition 4, we can deduce easily the following corollary.

Corollary 6. *Let R be a ring, S be a multiplicative subset of R, and P be a weakly S-prime ideal of R. Then, $P \subseteq \sqrt{0_R}$ or $s\sqrt{0_R} \subseteq P$ for some $s \in S$. In particular, if R is reduced then $P = (0)$ or P is S-prime.*

Theorem 7. *Let R be a ring, $S \subseteq R$ be a multiplicative subset and P an ideal of R disjoint with S. Then the following assertions are equivalent:*

1. *P is weakly S-prime of R.*

2. *There exists $s \in S$ such that for each $x \notin (P : s)$ we have either $(P : x) \subseteq (P : s)$ or $(P : x) = (0 : x)$.*

3. *There is an $s \in S$, such that for all I and J two ideals of R, if $0 \neq IJ \subseteq P$, then $sI \subseteq P$ or $sJ \subseteq P$.*

Proof. $(1) \Rightarrow (2)$ There exists $s \in P$ such that, whenever $x, y \in R$, $xy \in P$ implies $sx \in P$ or $sy \in P$.
Let $x \in R \setminus (P : s)$ (that is $sx \notin P$) and suppose that $(P : x) \neq (0 : x)$. Since $(0 : x) \subseteq (P : x)$, there exists $a \in (P : x)$ with $ax \neq 0$. Thus, $0 \neq ax \in P$. Hence, since $sx \notin P$, we have $as \in P$. Now, let $y \in (P : x)$. We have $xy \in P$. If $xy \neq 0$ then $sy \in P$, and so $y \in (P : s)$. Now, suppose that $xy = 0$. We have $0 \neq xa = x(a + y) \in P$. Then, $s(a + y) \in P$. Therefore, $sy \in P$. So, $y \in (P : s)$. Consequently, $(P : x) \subseteq (P : s)$.
$(2) \Rightarrow (1)$ Let $x, y \in R$ such that $0 \neq xy \in P$ and suppose that $sx \notin P$. Thus, $x \notin (P : s)$. Since $y \in (P : x)$ and $xy \neq 0$, we get immediately, $(P : x) \subseteq (P : s)$. Hence, $y \in (P : s)$, and so $sy \in P$. Accordingly, P is weakly S-prime.
$(2) \Rightarrow (3)$ Let I and J be two ideals of R such that $IJ \subseteq P$ and, for the element s of (2), we have $sI \not\subseteq P$ and $sJ \not\subseteq P$. We claim that $IJ = 0$. Let $x \in I \setminus (P : s)$. We have $xJ \subseteq P$. Then, $J \subseteq (P : x)$. Since $J \not\subseteq (P : s)$, we get $J \subseteq (P : x) = (0 : x)$. Hence, $xJ = 0$. Next, suppose that $x \in I \cap (P : s)$. Let $y \in J$. If $y \notin (P : s)$ then, as previously, we obtain $yI = 0$, and so $yx = 0$. If $y \in (P : s)$, consider $j \in J$ such that $sj \notin P$. Hence, $j \notin (P : s)$ and $y + j \notin (P : s)$. Hence, $xj = 0$ and $x(y + j) = 0$. Then $xy = 0$. So, $xJ = 0$. Consequently, $IJ = 0$.
$(3) \Rightarrow (1)$ Let $a, b \in R$ with $0 \neq ab \in P$. Then, $0 \neq (a)(b) \subseteq P$. Hence, by hypothesis we have $s(a) \subseteq P$ or $s(b) \subseteq P$. Thus, $sa \in P$ or $sb \in P$, and so we conclude that P is weakly S-prime of R. $\square$

Corollary 8 ([1], Theorem 3). *For proper ideal P of R, the following are equivalent:*

1. *P is weakly prime.*

2. For each $x \notin P$ we have either $(P : x) = P$ or $(P : x) = (0 : x)$.

3. For ideals I and J of R with $0 \neq IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

Proof. Take $S = \{1\}$ in Theoerm7. $\qquad \square$

Proposition 9. *Let R be a ring, S be a multiplicative subset of R, and P an ideal of R. If P is a weakly S-prime ideal that is not S-prime, then $sP\sqrt{0} = 0$ for some $s \in S$.*

Proof. By Theorem 7, there exists $s \in S$ such that for each $x \notin (P : s)$ we have either $(P : x) \subseteq (P : s)$ or $(P : x) = (0 : x)$.
Let $a \in \sqrt{0}$. If $a \in (P : s)$ then $sa \in P$. Thus, by Proposition 4, $saP = 0$. So, suppose that $a \notin (P : s)$. Then, $(P : a) \subseteq (P : s)$ or $(P : a) = (0 : a)$. Since $P \subseteq (P : a)$, the case $(P : a) = (0 : a)$ implies that $aP = 0$, and then $saP = 0$. Hence, suppose that $(P : a) \subseteq (P : s)$. Let $n \geq 1$ be a minimal integer such that $a^n = 0$. Then, $a^{n-1} \in (P : a) \subseteq (P : s)$. Therefore, $sa^{n-1} \in P$. Clearly, $n - 1 \geq 1$ since $P \cap S = \emptyset$. If $sa^{n-1} \neq 0$ then $sa \in P$, a contradiction. Hence, $sa^{n-1} = 0$. Let j be minimal such that $sa^j = 0$. Since $sa \neq 0$, we get that $j > 1$.
Suppose that there exists $p \in P$ such that $sap \neq 0$. We have $0 \neq sa(a^{j-1}+p) = sap \in P$. Then, $s(a^{j-1} + p) \in P$. Hence, $0 \neq sa^{j-1} \in P$. Hence, $sa \in P$, a contradiction. Thus, $saP = 0$. Consequently, $sP\sqrt{0} = 0$. $\qquad \square$

Corollary 10 ([1], Theorem 4)**.** *Let P be a weakly prime ideal of R. If P is not prime, then $P \subseteq \sqrt{0}$ and $P\sqrt{0} = 0$.*

Proof. Take $S = \{1\}$. $\qquad \square$

Corollary 11. *Let R be a ring and S a multiplicative subset of R. If P and Q are weakly S-prime ideals that are not S-prime, then $sPQ = 0$ for some $s \in S$.*

Proof. By Proposition4, we have $P, Q \subseteq \sqrt{0}$. Thus, by Proposition9 there is $s \in S$ such $sPQ \subseteq sP\sqrt{0} = 0$. $\qquad \square$

Proposition 12. *Let R be a commutative ring and S a multiplicative subset of R. If P is a weakly S-prime ideal, then $S^{-1}P$ is a weakly prime ideal of $S^{-1}R$.*

Proof. As $S \cap P = \emptyset$, we have that $S^{-1}P \neq S^{-1}R$. Let $0 \neq \frac{a}{s_1}\frac{b}{s_2} \in S^{-1}P$ where $a, b \in R$ and $s_1, s_2 \in S$. Then, $\frac{a}{s_1}\frac{b}{s_2} = \frac{p}{s_3}$ for some $p \in P$ and $s_3 \in S$. So, there is $t \in S$ such $0 \neq ts_3ab = ts_1s_2p \in P$. Since P is weakly S-prime, there exists $s \in S$ such that $sts_3 \in P$ or $0 \neq sab \in P$. Thus $0 \neq sab \in P$,

since $sts_3 \notin P$. Hence, $0 \neq s^2 a \in P$ or $sb \in P$, and so $sa \in P$ or $sb \in P$. This implies that $\frac{a}{s_1} = \frac{sa}{ss_1} \in S^{-1}P$ or $\frac{b}{s_2} = \frac{sb}{ss_2} \in S^{-1}P$, and so we have the desired result.

$\square$

Let R be a ring, $S \subseteq R$ be a multiplicative subset of R, and P an ideal of R with $P \cap S = \emptyset$. It is clear that if $(P : s)$ is a weakly prime ideal of R for some $s \in S$, then P is a weakly S-prime ideal. However, the converse implication is not true in general.

Example 13. *Set $R := \mathbb{Z}/12\mathbb{Z}$ and $S = \{\overline{3}, \overline{9}\}$. The ideal $(\overline{0})$ of R is weakly S-prime, but $(\overline{0} : \overline{3}) = (\overline{0} : \overline{9}) = \{\overline{0}, \overline{4}\}$ is not weakly prime.*

Proposition 14. *Let R be a ring, S be a multiplicative subset of R consisting of regular elements, and P be an ideal of R disjoint with S. Then the following assertions are equivalent:*

1. *P is weakly S-prime of R.*

2. *$(P : s)$ is a weakly prime ideal of R for some $s \in S$.*

3. *$S^{-1}P$ is a weakly prime ideal of $S^{-1}R$ and there is $s \in S$ such that $(P : t) \subseteq (P : s)$ for all $t \in S$.*

4. *$S^{-1}P$ is a weakly prime ideal of $S^{-1}R$ and $S^{-1}P \cap R = (P : s)$ for some $s \in S$.*

Proof. Since P is weakly S-prime, then there exist $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in P$, we have $sa \in P$ or $sb \in P$.
$(1) \Rightarrow (2)$ Let $0 \neq ab \in (P : s)$. Then, $0 \neq sab \in P$. Hence, $0 \neq s^2 a \in P$ or $sb \in P$. Thus $sa \in P$ or $sb \in P$, since $S \cap P = \emptyset$. Hence, we conclude that $a \in (P : s)$ or $b \in (P : s)$, and so $(P : s)$ is a weakly prime ideal of R.
$(1) \Rightarrow (3)$ By Proposition12, $S^{-1}P$ is a weakly prime ideal of $S^{-1}R$. Let $s \in S$ be the element associated to P. Let $t \in S$ and $0 \neq a \in (P : t)$. So, $0 \neq ta \in P$. Hence, $st \in P$ or $sa \in P$. Since $P \cap S = \emptyset$, $st \notin P$ which implies that $sa \in P$. Consequently, $a \in (P : s)$.
$(1) \Rightarrow (4)$ Again, $S^{-1}P$ is a weakly prime ideal of $S^{-1}R$ (by Proposition12). Let $0 \neq a \in (P : s)$. Then, $sa \in P$ and $a = \frac{as}{s} \in S^{-1}P$. Hence, $a \in S^{-1}P \cap R$, and so $(P : s) \subseteq S^{-1}P \cap R$. Now, let $0 \neq a \in S^{-1}P \cap R$. Then, $a \in R$ and $a = \frac{p}{t}$ with $p \in P$ and $t \in S$. So, $0 \neq ta = p \in P$. Hence, $st \in P$ or $sa \in P$. Thus, $sa \in P$ since $S \cap P = \emptyset$. Consequently $a \in (P : s)$, and then $S^{-1}P \cap R \subseteq (P : s)$.
$(2) \Rightarrow (1)$ Clear.

$(3) \Rightarrow (1)$ Let a, $b \in R$ such that $0 \neq ab \in P$. Since $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}P$, we have $\frac{a}{1} \in S^{-1}P$ or $\frac{b}{1} \in S^{-1}P$. If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{p}{t}$ for some $p \in P$ and $t \in S$. Hence, $ta = p$, and so $a \in (P : t) \subseteq (P : s)$. Similarly, if $\frac{b}{1} \in S^{-1}P$ then $sb \in P$, which means that P is a weakly S-prime ideal of R.

$(4) \Rightarrow (1)$ Let a, $b \in R$ such that $0 \neq ab \in P$. Since $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}P$, we have $\frac{a}{1} \in S^{-1}P$ or $\frac{b}{1} \in S^{-1}P$. If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{p}{t}$ for some $p \in P$ and $t \in S$. Hence, $ta = p \in P$ and so $a = \frac{ta}{t} \in S^{-1}P \cap R$. Then, by hypothesis $a \in (P : s)$, that is $sa \in P$. Similarly, if $\frac{b}{1} \in S^{-1}P$ then $sb \in P$. Thus, P is a weakly S-prime ideal of R. $\square$

Proposition 15. *Let R be a commutative ring and S a multiplicative subset of R. Then the following assertions are equivalent:*

1. *(0) is the only weakly S-prime ideal of R.*

2. *(0) is the only S-prime ideal of R.*

3. *R is a domain and $S^{-1}R$ is a field.*

Proof. $(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (3)$ By [9, Proposition 2.12], there is a prime ideal P with $P \cap S = \emptyset$. Hence, P is an S-prime ideal of R. Then, $P = (0)$, and so R is a domain. Let $a \in R \setminus \{0\}$ and $s \in S$. We claim that $\frac{a}{s}$ is invertible in $S^{-1}R$. If $a \in S$ then we have the desired result. Hence, we may assume $a \notin S$. Suppose that $< a > \cap S = \emptyset$. Then, by [9, Proposition 2.12], there is a prime ideal P of R such that $< a > \subseteq P$. Hence, $< a > \subseteq P = \{0\}$, a contradiction. Thus, $< a > \cap S \neq \emptyset$. Let $s' \in < a > \cap S$. Set $s' = at$. We have, $\frac{st}{s'} \in S^{-1}R$ and $\frac{a}{s} \cdot \frac{st}{s'} = \frac{ast}{ss'} = \frac{ss'}{ss'} = \frac{1}{1}$. Then, $\frac{a}{s}$ is invertible in $S^{-1}R$ as desired. So, $S^{-1}R$ is a field.

$(3) \Rightarrow (1)$ Let P be a nonzero weakly S-prime ideal of R. Let $p \in P \setminus \{0\}$. Since $S^{-1}R$ is a field, there exists $a \in R \setminus \{0\}$ and $s \in S$ such that $\frac{p}{1} \cdot \frac{a}{s} = \frac{1}{1}$. Since R is a domain, we deduce that $pa = s \in P \cap S = \emptyset$, a contradiction. Consequently, (0) is the only weakly S-prime ideal of R. $\square$

Proposition 16. *Let R be a ring, $S \subseteq R$ a multiplicative set and P an ideal of R disjoint with S. If Q is an ideal of R such that $Q \cap S \neq \emptyset$ and P is a weakly S-prime, then $P \cap Q$ and PQ are weakly S-prime ideals of R.*

Proof. It is clear that $(P \cap Q) \cap S = \emptyset$ and $PQ \cap S = \emptyset$. First, we will prove that $P \cap Q$ is a weakly S-prime of R. Pick $s' \in Q \cap S$ and let a, $b \in R$ such that $0 \neq ab \in P \cap Q \subseteq P$. Then, there is $s \in S$ such that $sa \in P$ or $sb \in P$. Thus, $ss'a \in P \cap Q$ or $ss'b \in P \cap Q$, and $ss' \in S$. Consequently, $P \cap Q$ is a weakly S-prime ideal of R. The proof is similar for PQ. $\square$

Proposition 17. *Let R be a commutative ring and S a multiplicative subset of R. Then the following assertions are equivalent:*

1. *Every weakly S-prime ideal of R is prime.*

2. *R is a domain and every S-prime ideal of R is prime.*

Proof. (1) $\Rightarrow$ (2) Since (0) is weakly S-prime ideal of R, and every weakly S-prime ideal of R is prime, R is a domain.
(2) $\Rightarrow$ (1) As R is domain, then every weakly S-prime ideal of R is S-prime. Consequently, every weakly S-prime ideal of R is prime. $\qquad\square$

Let R be a commutative ring and S a multiplicative subset of R. Recall from[1] that an ideal I of R is called S-finite (resp., S-principal) if $sI \subseteq J \subseteq I$ for some finitely generated (resp., principal) ideal J of R and some $s \in S$. Also, R is called S-Noetherian (resp., S-PIR) if each ideal of R is S-finite (resp., S-principal).

Theorem 18. *Let R be a ring and S be a multiplicative subset of R. Then the following assertions are equivalent:*

1. *R is S-Noetherian (resp., S-PIR).*

2. *Every weakly S-prime ideal of R is S-finite (resp., S-principal).*

3. *Every S-prime ideal of R is S-finite (resp., S-principal).*

4. *Every prime ideal of R is S-finite (resp., S-principal).*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.
(3) $\Rightarrow$ (4) Follows from [8, Theorem 3] (the proof of "resp., S-principal" statement is analogue to that of (2) $\Rightarrow$ (3) in [8, Theorem 3]).
(4) $\Rightarrow$ (1) Follows from [1, Corollary 5] (resp., [1, Proposition 5]). $\qquad\square$

Remark 19. *Let $S_1 \subseteq S_2$ be multiplicative subsets of R and P an ideal of R disjoint with S_2. Clearly, if P is a weakly S_1-prime of R, then P is weakly S_2-prime. However, the converse is not true general. To see that, consider the ideal $P = (4X)$ of the rings $\mathbb{Z}[X]$ and set $S_1 = \{1\}$ and $S_2 = \{2^n | n \in \mathbb{N}\}$. By Example 3, P is a weakly S_2-prime ideal of $\mathbb{Z}[X]$ but not weakly S_1-prime.*

Proposition 20. *Let $S_1 \subseteq S_2$ be multiplicative subsets of R such that for any $s \in S_2$, there is an element $t \in S_2$ satisfying $st \in S_1$. If P is a weakly S_2-prime ideal of R, then P is a weakly S_1-prime ideal of R.*

Proof. Let $a, b \in R$ such that $0 \neq ab \in P$. So, there is $s \in S_2$ such that $sa \in P$ or $sb \in P$. By the assumption, $s' = st \in S_1$ for some $t \in S_2$, and then $s'a \in P$ or $s'b \in P$. Consequently, P is a weakly S_1-prime ideal of R. $\square$

Let S be a multiplicative subset of R, $S^* = \{r \in R \mid \frac{r}{1}$ is unit in $S^{-1}R\}$ denotes the saturation of S. Note that, S^* is a multiplicative subset containing S. A multiplicative subset S of R is called saturated if $S = S^*$. It is clear that S^* is always a saturated multiplicative subset of R [10].

Proposition 21. *Let R be a ring, S be a multiplicative subset of R, and P an ideal of R disjoint with S. Then, P is a weakly S-prime ideal of R if and only if P is a weakly S^*-prime ideal.*

Proof. It is clear that $S^* \cap P = \emptyset$. We will show that for any $r \in S^*$, there is $r' \in S^*$ such that $rr' \in S$. Let $r \in S^*$, then $\frac{r}{1}\frac{a}{s} = 1$ for some $s \in S$ and $a \in R$. This implies that $tar = ts \in S$, for some $t \in S$. Now, take $r' = ta$. Then, we have $r' \in S^*$ with $rr' \in S$, and so the desired condition is satisfied. Therefore, by putting $S = S_1$ and $S_2 = S^*$, we conclude immediately the result from the proposition20. $\square$

Proposition 22. *Let $f : R \to R'$ be a ring homomorphism and S a multiplicative subset of R. Then, the followings hold:*

1. *If f is an epimorphism and P is a weakly S-prime ideal of R containing $\mathrm{Ker}(f)$, then $f(P)$ is a weakly $f(S)$-prime ideal of R'.*

2. *If f is a monomorphism and P' is a weakly $f(S)$-prime ideal of R', then $f^{-1}(P')$ is a weakly S-prime ideal of R.*

Proof. (1) Let $r \in f(S) \cap f(P)$. Then, $r = f(p) = f(s)$ for some $p \in P$ and $s \in S$. So, $s - p \in \mathrm{Ker}(f) \subseteq P$, which implies that $s \in P$, a contradiction. Hence, $f(S) \cap f(P) = \emptyset$. Now, let $0 \neq a'b' \in f(P)$. Then, there is $a, b \in R$ such that $f(a) = a'$, $f(b) = b'$ and $0 \neq f(ab) = a'b' \in f(P)$. Since $\mathrm{Ker}(f) \subseteq P$, we get $0 \neq ab \in P$, and so $sa \in P$ or $sb \in P$ for some $s \in S$. It means that $f(s)a' \in f(P)$ or $f(s)b' \in f(P)$. Thus, $f(P)$ is weakly $f(S)$-prime ideal of R'. (2) Let P' be an $f(S)$-prime ideal of R'. So there exists $s \in S$ such that, for all $a', b' \in R'$, $0 \neq a'b' \in P'$ implies $f(s)a' \in P'$ or $f(s)b' \in P'$. The condition $f^{-1}(P') \cap S = \emptyset$ is trivial. Let $a, b \in R$ such that $0 \neq ab \in f^{-1}(P')$. Since $\mathrm{Ker}(f) = \{0\}$, we get $0 \neq f(ab) = f(a)f(b) \in P'$. Then, $f(s)f(a) = f(sa) \in P'$ or $f(s)f(b) = f(sb) \in P'$. Thus, $sa \in f^{-1}(P')$ or $sb \in f^{-1}(P')$, and so we conclude that $f^{-1}(P')$ is a weakly S-prime of R. $\square$

Let R be a ring, S be a multiplicative subset of R, and I an ideal of R disjoint with S. Let $s \in S$, we denote by $\overline{s}$ the class of s in R/I. Let $\overline{S} = \{\overline{s} \mid s \in S\}$. It is easy to check that $\overline{S}$ is a multiplicative subset of R/I.

Corollary 23. *Let R be a ring and S be a multiplicative subset of R.*

1. *If $I \subseteq J$ be two ideals of R. If J is a weakly S-prime of R, then J/I is a weakly $\overline{S}$-prime of R/I.*

2. *If R is sub-ring of R' and J' is a weakly S-prime of R', then $J' \cap R$ is a weakly S-prime of R.*

Proof. (1) Follows by applying Proposition22(1) to the canonical surjection $\pi : R \to R/I$.
(2) It suffices to apply Proposition22(2) to the naturel injection $\iota : R \hookrightarrow R'$, since $\iota^{-1}(J') = J' \cap R$. $\qquad\square$

Proposition 24. *Let $R := R_1 \times R_2$, $S = S_1 \times S_2$, where S_i is a multiplicative subset of R_i and P_1 and P_2 be an nonzero ideals of R_1 and R_2, respectively. Then the following are equivalent:*

1. *$P := P_1 \times P_2$ is a weakly S-prime of R.*

2. *P_1 is an S_1-prime ideal of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is an S_2-prime ideal of R_2 and $S_1 \cap P_1 \neq \emptyset$.*

3. *$P := P_1 \times P_2$ is an S-prime of R.*

Proof. (1) $\Rightarrow$ (2) Let $0_R \neq (p, q) \in P$ with $p \in R_1$ and $q \in R_2$. Then, $0 \neq (p, q) = (p, 1)(1, q) \in P$. Since, P is weakly S-prime ideal of R, then there is $s = (s_1, s_2) \in S$ such that $s(p, 1) = (s_1 p, s_2) \in P$ or $s(1, q) = (s_1 1, s_2 q) \in P$. Thus, $S_1 \cap P_1 \neq \emptyset$ or $S_2 \cap P_2 \neq \emptyset$. Assume, $S_2 \cap P_2 \neq \emptyset$. As, $P \cap S = \emptyset$, we have $S_1 \cap P_1 = \emptyset$. Now, we show that P_1 is an S_1-prime ideal of R_1. Let $pp' \in P_1$ for some $p, p' \in R_1$. Since $S_2 \cap P_2 \neq \emptyset$, then there is $0_{R_2} \neq t \in S_2 \cap P_2$, and so we have $0_R \neq (p, t)(p', 1) \in P$. Hence, $s(p, t) = (s_1 p, s_2 t) \in P$ or $s(p', 1) = (s_1 p', s_2) \in P$. So, we get $s_1 p \in P_1$ or $s_1 p' \in P_1$, as desired.
(2) $\Rightarrow$ (3) Follows directly from [7, Lemma 2.13].
(3) $\Rightarrow$ (1) Clear. $\qquad\square$

Proposition 25. *Let $R := R_1 \times R_2$ and $S = S_1 \times S_2$, where S_i is a multiplicative subset of R_i. Then the following are equivalent.*

1. *Every proper ideal of R is weakly S-prime.*

2. *R_1 and R_2 are fields.*

Proof. $(1) \Rightarrow (2)$ Let I_1 be a proper ideal of R_1. Then, $I_1 \times R_2$ is a weakly S-prime ideal of R, and so I_1 is an S_1-prime ideal of R_1 (by Proposition 24). Thus, every proper ideal of R_1 is S_1-prime. So, by [7, Corollary 2.27], R_1 is a field. Similarly, R_2 is a field.

$(2) \Rightarrow (1)$ Since R_1 and R_2 are fields. Then, we have exactly three proper ideals $\{0\} \times \{0\}$, $\{0\} \times R_2$, and $R_1 \times \{0\}$ which are S-prime (by Proposition24). $\square$

Proposition 26. *Let R be a commutative ring and $S \subseteq R$ a multiplicative set. Then the following are equivalent.*

1. *Every proper ideal of R is weakly S-prime.*

2. *$S \subseteq U(R)$ and "$(R, \mathfrak{m})$ is a quasilocal ring (possibly a field) with $\mathfrak{m}^2 = 0$" or "$R = F_1 \times F_2$ where F_1 and F_2 are fields".*

Proof. Suppose that (1) holds and let $s \in S$. Since every maximal ideal of R is weakly S-prime by hypothesis, then there is no maximal ideal of R which contains s. This implies that s is a unit element of R, and so $S \subseteq U(R)$.

Hence, if (1) or (2) holds, $S \subseteq U(R)$. In this situation, the concepts of weakly S-prime ideals and weakly prime ideals coincide, and the result follows directly from [1, Theorem 8]. $\square$

Let R be a ring and M be an R-module. The additive group $R \times M$ with multiplication $(a, m)(b, m') = (ab, am' + bm)$ is a commutative ring with identity, noted $R \propto M$ and called the Nagata idealization. If P is an ideal of R then, $P \propto M$ is an ideal of $R \propto M$. Also, if S is a multiplicative subset of R then $S \propto 0$ and $S \propto M$ are multiplicative subset of $R \propto M$.

Proposition 27. *Let R be a ring, M be an R-module, S be a multiplicative subset of R and P be an ideal of R disjoint with S. Then the following are equivalent:*

1. *$P \propto M$ is a weakly $(S \propto M)$-prime ideal of $R \propto M$.*

2. *$P \propto M$ is a weakly $(S \propto 0)$-prime ideal of $R \propto M$.*

3. *P is a weakly S-prime of R associated to $s \in S$, and if $a, b \in R$ with $ab = 0$, but $sa \notin P$ and $sb \notin P$ then $a \in \operatorname{ann}_R(M)$ and $b \in \operatorname{ann}_R(M)$.*

Proof. $(2) \Rightarrow (1)$ Follows from Remark19 since $S \propto 0 \subseteq S \propto M$.

$(1) \Rightarrow (3)$ Let $a, b \in R$ such that $0 \neq ab \in P$. Then, $(0,0) \neq (a,0)(b,0) \in P \propto M$. As $P \propto M$ is a weakly $(S \propto M)$-prime ideal of $R \propto M$, there is $(s,n) \in S \propto M$ such that $(s,n)(a,0) = (sa, an) \in P \propto M$ or $(s,n)(b,0) = (sb, bn) \in P \propto M$. Thus, $sa \in P$ or $sb \in P$, and so P is a weakly S-prime

of R. Now, suppose that $ab = 0$, but $sa \notin P$ and $sb \notin P$. Assume that $a \notin \mathrm{ann}_R(M)$. Then, there is $m \in M$ such that $am \neq 0$, and so we have $(0,0) \neq (a,0)(b,m) \in P \propto M$. Hence, $(s,n)(a,0) = (sa, an) \in P \propto M$ or $(s,n)(b,m) = (sb, sm + bn) \in P \propto M$, a contradiction.

$(3) \Rightarrow (2)$ Let $(0,0) \neq (a,m)(b,n) \in P \propto M$, where $(a,m), (b,n) \in R \propto M$. If $ab \neq 0$, then $sa \in P$ or $sb \in P$, and hence $(s,0)(a,m) \in P \propto M$ or $(s,0)(b,n) \in P \propto M$. Assume that $ab = 0$, but $sa \notin P$ and $sb \notin P$. Then a, $b \in \mathrm{ann}_R(M)$. Consequently, we get $(a,m)(b,n) = (0,0)$, a contradiction. $\square$

Let R be a ring and I be an ideal of R. The amalgamated duplication of R along I, denoted by $R \bowtie I$, is the subring of $R \times R$ given by $A \bowtie I = \{(a, a + i \mid a \in I, i \in I\}$. This construction was introduced and its basic properties were studied by D'Anna and Fontana in [4, 5] and then it was investigated by D'Anna in [3] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve.

Recently, the authors in [6] studied some kind of weakly prime ideals in amalgamated algebras (a generalization of a the duplication construction). In the next result of this paper, we initiate the study of weakly S-prime ideals over amalgamated duplication of a ring along an ideal.

Proposition 28. *Let R be a ring, I be an R-module, S be a multiplicative subset of R, and P be an ideal of R disjoint with S. Then the following are equivalent:*

1. *$P \bowtie I$ is a weakly $(S \bowtie I)$-prime ideal of $R \bowtie I$.*

2. *$P \bowtie I$ is a weakly $(S \bowtie 0)$-prime ideal of $R \bowtie I$.*

3.
   - *P is a weakly S-prime of R associated to $s \in S$, and*
   - *if there exists a, $b \in R$ with $ab = 0$, but $sa \notin P$ and $sb \notin P$ then $a \in \mathrm{ann}_R(I)$, $b \in \mathrm{ann}_R(I)$, and $I^2 = (0)$.*

Proof. $(2) \Rightarrow (1)$ Follows from Remark19 since $S \bowtie 0 \subseteq S \bowtie I$.

$(1) \Rightarrow (3)$ Suppose that $P \bowtie I$ is a weakly $(S \bowtie I)$-prime ideal of $R \bowtie I$ associated to $(s, s + i)$. Let $a, b \in R$ such that $0 \neq ab \in P$. Then, $(0,0) \neq (a,a)(b,b) \in P \bowtie I$. Thus, $(s, s+i)(a,a) \in P \bowtie I$ or $(s, s+i)(b,b) \in P \bowtie I$. So, $sa \in P$ or $sb \in P$, and so P is a weakly S-prime of R associated to s. Now, suppose that there exists $a, b \in R$ such that $ab = 0$ but $sa \notin P$ and $sb \notin P$. Assume that $a \notin \mathrm{ann}_R(I)$. Then, there is $j \in I$ such that $aj \neq 0$, and so we have $(0,0) \neq (a,a)(b, b+j) \in P \bowtie I$. Hence, $(s, s+i)(a,a) = (sa, sa + ia) \in P \bowtie I$ or $(s, s+i)(b, b+j) = (sb, sb + ib + sj + ij) \in P \bowtie I$, a contradiction.

Hence, $a \in \mathrm{ann}_R(I)$. Similarly, $b \in \mathrm{ann}_R(I)$. Let $j, k \in I$. We have $(a, a + j)(b, b+k) = (0, jk) \in P \bowtie I$. If $jk \neq 0$ then $(a, a+j)(b, b+k) \neq (0,0)$. Thus, $(s, s+i)(a, a) = (sa, sa) \in P \bowtie I$ or $(s, s+i)(b, b+j) = (sb, sb+ij) \in P \bowtie I$, a contradiction. Thus, $I^2 = (0)$.

$(3) \Rightarrow (2)$ Let $(0,0) \neq (a, a+j)(b, b+k) \in P \bowtie I$, where $(a, a+j)$, $(b, b+k) \in R \bowtie I$. If $ab \neq 0$, then $sa \in P$ or $sb \in P$, and hence $(s, s)(a, a+j) \in P \bowtie I$ or $(s, s)(b, b+k) \in P \bowtie I$. Assume that $ab = 0$, but $sa \notin P$ and $sb \notin P$. Then a, $b \in \mathrm{ann}_R(I)$. Consequently, we get $(a, a+j)(b, b+k) = (0, 0)$, a contradiction. $\qquad\square$

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