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Qualitative results in thermoelasticity of type III for dipolar bodies

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Abstract

In our study we formulated the mixed initial boundary value problem corresponding to the thermoelasticity of type III for bodies with dipolar structure. In main section we approached four qualitative results regarding the solutions for this problem. In two of these (in the first two theorems) we obtained two results of uniqueness, proved in different ways. Also, we proven two results which show that the solutions of the considered problem depend continuously with respect to the supply terms. We use different procedures in the two theorems on continuous dependence, but we essentially rely on the auxiliary results from Section 3 and Gronwall-type inequalities. It is important to emphasize that all results are obtained by imposing on the basic equations and basic conditions, average constraints that are common in the mechanics of continuous solids.

1. Introduction

In our study we approach the thermoelasticity of type III for bodies having a dipolar structure, starting from the theories proposed by Green and Naghdi in [1-3].
Specific to this type of thermoelesticity is the consideration of a new independent variable, denoted by θ, which is called the thermal displacement and

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which is introduced by means of the variation of the absolute temperature T, using the relation:

$$\dot{\theta} = T. \tag{1}$$

After the appearance of this thermoelasticity theory of type III, the researchers have shown great interest in this theory. Many studies with this topic have been published, among which we mention the works [4-7]. There are studies that address the problem of the uniqueness of the solution of the mixed problem as well as the continuous dependence of the solutions, as we did in the present study. But in other studies the bodies are generally isotropic, which allows to apply the logarithmic convexity method or the Lagrange identity method, both to obtain the uniqueness as well the continuous dependence results, see for instance, [8]. However, in the case of dipolar bodies approached in the context of thermoelasticity of type III, there are no results either in terms of uniqueness or in terms of continuous dependence.
It should be noted that this type of structure is a part of such called non-classical theories, namely the theories of materials having microstructure. One of the pioneers of these theories is Eringen (see, for instance, [9], [10]). The dipolar structure has attracted the attention of many researchers, the importance of this type of structure can be deduced in the base of the large number of published studies dedicated to this topic, of which we can mention [11-13]. We consider our study as a modest continuation of these concerns.
The plane of our study is as follows. In Section 2 we summarize the main equations, the initial conditions and the boundary data of the mixed problem. In Section 3 we prove some results, more precisely four inequalities of integral type, useful for the basic results. Section 4 is devoted to the main results of our paper last part of our study we prove the , namely two results on uniqueness for the solution of the mixed problem and two theorems regarding the continuous dependence of solutions with regards to the supply terms

2. Basic equations and conditions

Our study is dedicated to a dipolar body in the context of the thermoelasticity of type III, as proposed by Green and Naghdi. This body is considered by non-homogeneous and anisotropic which occupies a specific regular region D, as a apart of the three-dimensional Euclidean physical space E^3. The border of the domain D is denoted by ∂D and is assumed be piecewise a surface, enough smooth. Let us denote by $\bar{D}$ the closure of D and, of course, we have $\bar{D} = D \cup \partial D$. An orthonormal system of references is introduced and then tensors and vectors have components with Latin subscripts over 1,2,3. The

Einstein convention for summation is implied. A superposed dot for a function f means the partial derivative of f with respect the time variable and the writing f,i is for partial derivative of f with respect to the spatial variable x_i. To characterize the behaviour of a dipolar thermoelastic we will use in our study the following functions:

$$v_k(t,x),\ \phi_{kl}(t,x),\ T(t,x),\ (t,x)\in D\times[0,t_0),$$

that is, the displacement vector, the dipolar displacement vector and the absolute temperature.
With the help of the variables $v_k(t,x),\ \phi_{kl}(t,x)$, we can define the deformation tensors, having the components $e_{kl},\ \kappa_{kl}$ and χ_{ijk}, as well the thermal displacement gradient of components α_i, as follows:

$$e_{kl}=\frac{1}{2}\left(v_{l,k}+v_{k,l}\right),\ \kappa_{kl}=v_{k,l}-\phi_{kl},\ \chi_{jkl}=\phi_{kl,j},\ \alpha_k=\theta_{,k}. \tag{2}$$

We consider the linear theory of the thermoelasticity of type III for dipolar bodies. As such it is natural to take the internal specific energy is a form with quadratic terms regarding its specific variables. Consider that in the initial state of the body, the internal energy is e. According to the energy conservation principle, we expand the function e in series and take into account only the terms of first and second order.
Considering that the initial state of the body is free of charges, we can write the internal energy is of the following form ((see [12])):

$$\begin{aligned}\rho e=&\frac{1}{2}A_{klmn}e_{kl}e_{mn}+D_{klmn}e_{kl}\kappa_{mn}+F_{klmnr}e_{kl}\chi_{mnr}+\\&+\frac{1}{2}B_{klmn}\kappa_{kl}\kappa_{mn}+G_{klmnr}\kappa_{kl}\chi_{mnr}+\frac{1}{2}C_{kljmnr}\chi_{klj}\chi_{mnr}+\\&+\left(P_{klm}e_{kl}+Q_{klm}\kappa_{kl}+R_{klnm}\chi_{kln}\right)\alpha_m+\frac{1}{2}K_{mn}\alpha_m\alpha_n+\frac{1}{2}cT^2.\end{aligned} \tag{3}$$

As a consequence, the Helmholtz free energy H receives the form:

$$\begin{aligned}\rho H=&\frac{1}{2}A_{klmn}e_{kl}e_{mn}+D_{klmn}e_{kl}\kappa_{mn}+F_{klmnr}e_{kl}\chi_{mnr}+\\&+\frac{1}{2}B_{klmn}\kappa_{kl}\kappa_{mn}+G_{klmnr}\kappa_{kl}\chi_{mnr}+\frac{1}{2}C_{kljmnr}\chi_{klj}\chi_{mnr}+\\&+\left(P_{klm}e_{kl}+Q_{klm}\kappa_{kl}+R_{klnm}\chi_{kln}\right)\alpha_m+\frac{1}{2}K_{mn}\alpha_m\alpha_n-\\&-a_{kl}e_{kl}T-b_{kl}\kappa_{kl}T-c_{klj}\chi_{klj}T+d_k\alpha_kT-\frac{1}{2}cT^2.\end{aligned} \tag{4}$$

We will use this form of the Helmholtz free energy in the inequality of entropy production in order to obtain the basic relations that characterize the the thermoelasticity of type III for dipolar bodies, namely (see [15], [19]):

- the motion equations:

$$
\begin{aligned}
&(\tau_{kl}+\eta_{kl})_{,l}+\rho f_k=\rho\ddot{v}_k,\\
&\mu_{jkl,j}+\eta_{jk}+\rho g_{jk}=I_{kr}\ddot{\phi}_{lr};
\end{aligned}
\tag{5}
$$

- the equation of energy:

$$
\rho T_0\dot{S}=q_{m,m}+\rho r. \tag{6}
$$

By using the same entropy production inequality, we can also deduce the constitutive relations. These relations are for the expression of the stress tensors of stress as functions depending on the strain tensors. So, if we write the elements of the tensors of stress as τ_{kl}, η_{kl} and μ_{jkl}, then the constitutive equations give the connection between these tensors and the strain tensors e_{kl}, κ_{kl}, χ_{jkl}.
Our approach is based on specific technique that are used by Green, Rivlin in paper [12], so that starting from the free energy of Helmholtz defined in (4) we get the following constitutive relations:

$$
\begin{aligned}
&\tau_{kl}=\frac{\partial W}{\partial e_{kl}}=A_{klmn}e_{mn}+D_{mnkl}\kappa_{mn}+F_{mnrkl}\chi_{mnr}+P_{klm}\alpha_m-a_{kl}\theta,\\
&\eta_{kl}=\frac{\partial W}{\partial \kappa_{kl}}=D_{klmn}e_{mn}+B_{klmn}\kappa_{mn}+G_{klmnr}\chi_{mnr}+Q_{klm}\alpha_m-b_{kl}\theta,\\
&\mu_{jkl}=\frac{\partial W}{\partial \chi_{jkl}}=F_{jklmn}e_{mn}+G_{mnjkl}\kappa_{mn}+C_{jklmnr}\chi_{mnr}+R_{jklm}\alpha_m-c_{jkl}\theta,\\
&S=-\frac{\partial W}{\partial T}=a_{kl}e_{kl}+b_{kl}\kappa_{kl}+c_{jkl}\chi_{jkl}-d_m\alpha_m+cT,
\end{aligned}
\tag{7}
$$

that take place in cylinder $D\times[0,t_0)$. By S we denoted the specific entropy in unit mass.
For the components q_i of the entropy flux vector we have the following constitutive relation:

$$
q_m=P_{klm}e_{kl}+Q_{klm}\kappa_{kl}+R_{klnm}\chi_{kln}+K_{nm}\alpha_n+d_mT+\mathcal{K}_{nm}\dot{\alpha}_n, \tag{8}
$$

where $\mathcal{K}_{nm}$ is the heat conductivity tensor which is symmetric and satifies the following dissipation inequality:

$$
\mathcal{K}_{nm}\dot{\alpha}_n\dot{\alpha}_m\geq 0. \tag{9}
$$

The semnifications of the notations which we have used in previous relations is the following: ρ-the density of mass, supposed be constant; I_{kl}-the microinertia, which is a symmetric tensor; f_k-the external body forces; g_{jk}-the external dipolar charges; r-the externalrate of supply of heat; $A_{klmn}, B_{klmn}, ..., a_{kl}$-the constitutive coefficients that characterize the material properties of the material from the elasticity point of view and which satisfy the relations of symmetry that follows:

$$\begin{aligned} &A_{klmn} = A_{lkmn} = A_{mnkl},\ C_{jklmnr} = C_{mnrjkl},\\ &B_{klmn} = B_{mnkl},\ F_{jklmn} = F_{jklnm},\ D_{klmn} = D_{klnm},\\ &K_{mn} = K_{nm},\ P_{klm} = P_{lkm},\ a_{mn} = a_{nm},\ \mathcal{K}_{mn} = \mathcal{K}_{nm}. \end{aligned} \tag{10}$$

To simplify writing, in the following we will consider that the domaun D is occupied by a centrosymmetric body.
Consequently, we have:

$$P_{klm} = Q_{klm} = R_{klnm} = d_m = 0. \tag{11}$$

Considering the constitutive relations (7) and (8) and the kinematic equations (2), the basic equations (5) and (6) are transformed in system of partial differential equations of the following form:

$$\begin{aligned} \rho\ddot{v}_k = &\left[\left(C_{klmn} + G_{klmn}\right) v_{n,m} + \left(G_{mnij} + B_{klmn}\right)\left(v_{n,m} - \phi_{mn}\right) + \right.\\ &\left. + \left(F_{mnrij} + D_{klmnr}\right)\phi_{nr,m} - \left(a_{kl} + b_{kl}\right)\theta\right]_{,l} + \rho f_k,\\ I_{kr}\ddot{\phi}_{lr} = &\left[F_{jklmn}v_{n,m} + D_{mnjkl}\left(v_{n,m} - \phi_{mn}\right) + A_{kljmnr}\phi_{nr,m} - c_{jkl}\theta\right]_{,j} + \\ &+ G_{jkmn}v_{m,n} + B_{jkmn}(v_{n,m} - \phi_{mn}) + D_{jkmnr}\phi_{nr,m} - b_{jk}\theta + \rho g_{kl},\\ c\dot{T} = &-\left[a_{kl}\dot{v}_{l,k} + b_{kl}\left(\dot{v}_{l,k} - \dot{\phi}_{kl}\right) + c_{jkl}\dot{\phi}_{kl,j}\right] + \left(K_{mn}T, n + \mathcal{K}_{mn}\dot{T}, n\right)_{,m}. \end{aligned} \tag{12}$$

Regarding the system of differential equations (12), we will construct a mixed problem with boundary and initial data. For this purpose, we will add the next boundary relations:

$$\begin{aligned} &v_k(t,x) = \tilde{v}_k(t,x),\ (t,x) \in S_1 \times [0,\infty),\ t_l(t,x) = \tilde{t}_l(t,x),\ (t,x) \in S_1^c \times [0,\infty),\\ &\phi_{kl}(t,x) = \tilde{\phi}_{kl}(t,x), (t,x) \in S_2 \times [0,\infty), m_{kl}(t,x) = \tilde{m}_{kl}(t,x), (t,x) \in S_2^c \times [0,\infty),\\ &T(t,x) = \tilde{T}(t,x),\ (t,x) \in S_3 \times [0,\infty),\ q(t,x) = \tilde{q}(t,x),\ (t,x) \in S_3^c \times [0,\infty). \end{aligned} \tag{13}$$

Here $\tilde{v}_k(t,x)$, $\tilde{t}_k(t,x)$, $\tilde{\phi}_{kl}(t,x)$, $\tilde{m}_{kl}(t,x)$, $\tilde{T}(t,x)$ and $\tilde{q}(t,x)$ are given and regular functions on their domains of definition.
Also, we considered the surface tractions of components t_i, the surface couple of components μ_{jk}, and the heat flux q defined by

$$t_l = \left(\tau_{kl} + \eta_{kl}\right) n_k,\ m_{kl} = \mu_{jkl}n_j,\ q = q_k n_k,$$

where we denoted by $n = (n_k)$ the normal that is oriented outward of the boundary ∂D.

In (13) we used the surfaces S_1, S_2, S_3 and their complements S_1^c, S_2^c, S_3^c which are subsets of the surfaces ∂D which meet the conditions:

$$S_1 \cup S_1^c = S_2 \cup S_2^c = S_3 \cup S_3^c = \partial D,$$
$$S_1 \cap S_1^c = S_2 \cap S_2^c = S_3 \cap S_3^c = \emptyset.$$

We also consider the initial conditions:

$$\begin{aligned} &v_k(0,x) = v_k^0(x),\ \dot{v}_k(0,x) = v_k^1(x),\ \phi_{kl}(0,x) = \phi_{kl}^0(x), \\ &\dot{\phi}_{kl}(0,x) = \phi_{kl}^1(x),\ T(0,x) = T^0(x),\ \dot{T}(0,x) = T^1(x),\ x \in \bar{D}. \end{aligned} \tag{14}$$

Here the functions $v_k^0(x)$, $v_k^1(x)$, $\phi_{kl}^0(x)$, $\phi_{kl}^1(x)$, $T^0(x)$ and $T^1(x)$ are continuous and prescribed in their domains of definition and are in accordance with relations (13) on the corresponding subsurfaces of ∂D.
In the following we will use the notation $\mathcal{P}$ for the mixed problem consists of the system of equations (12), the boundary conditions (13) and the initial conditions (14).

3. Auxiliary results

First, we will establish some estimations regarding the solutions $\mathbf{u} = (v_k, \phi_{kl}, T)$ of the mixed problem $\mathcal{P}$.

Proposition 1.. *Suppose that the array $\mathbf{u} = (v_k, \phi_{kl}, T)$ satisfies the mixed problem $\mathcal{P}$. Then, we have the following identity:*

$$\begin{aligned} &\rho f_k \dot{v}_k + I_{kl} g_{ks} \dot{\phi}_{ls} + \rho r T + \left[(\tau_{kl} + \eta_{kl})\, \dot{v}_k + \mu_{jkl} \dot{\phi}_{jk} + q_l T \right]_{,l} = \\ &\qquad \frac{\partial}{\partial s} \left(\frac{1}{2} \rho \dot{v}_k \dot{v}_k + I_{jk} \dot{\phi}_{jm} \dot{\phi}_{km} + \rho e \right) + \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n. \end{aligned} \tag{15}$$

Proof. This identity is obtained by direct calculations, taking into account the equations of motion (5), the energy equation (6) and the expression of the internal energy e which is defined in (3). ■
In the following we will have some integrals for which the integration domain is of the form $D(t)$. This means that the evaluation of the quantity under integral is made at time t.
As a measure of the deformation, we will can use the following function:

$$\begin{aligned} M(t) = &\int_{D(t)} \left(\frac{1}{2} \rho \dot{v}_k \dot{v}_k + I_{jk} \dot{\phi}_{jm} \dot{\phi}_{km} + \rho e \right) dV + \\ &\qquad + \int_0^t \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n dV d\tau. \end{aligned} \tag{16}$$

In the following proposition we will prove a conservation law relative to the measure defined (16).

Proposition 2.. *The following conservation law takes place:*

$$M(t)-M(0)=\int_0^t\int_{D(\tau)}\left(\rho f_k\dot{v}_k+I_{kl}g_{ks}\dot{\phi}_{ls}+\rho rT\right)dVd\tau+$$
$$+\int_0^t\int_{\partial D(\tau)}\left[\left(\tau_{kl}+\eta_{kl}\right)\dot{v}_kn_l+\mu_{jkl}\dot{\phi}_{jk}n_l+q_lTn_l\right]dAd\tau. \tag{17}$$

Proof. This important conservation law can be immediately obtained by integrating the identity (15) on the cylinder $[0,t]\times D(t)$ and using the divergence theorem and the initial conditions (14). ■
We can prove the next results only if we impose some conditions. So, we suppose that c, which is the specific heat, is strictly positive, the mass density ρ is strictly positive, the heat conductivity tensor $\mathcal{K}_{mn}$ is positive definite and the internal energy e is positive, that is:

$$\begin{gathered}c>0,\ \rho>0;\\ A_{klmn}x_{kl}x_{mn}+2D_{klmn}x_{kl}y_{mn}+2F_{klmnr}x_{kl}z_{mnr}+B_{klmn}y_{kl}y_{mn}+\\ +2G_{klmnr}y_{kl}z_{mnr}+C_{kljmnr}z_{klj}z_{mnr}+2\left(P_{klm}x_{kl}+Q_{klm}y_{kl}+R_{klnm}z_{kln}\right)u_m+\\ +K_{mn}u_mu_n+cw^2\geq 0,\ \forall x_{mn}=x_{nm},\ y_{mn},\ z_{mnr},\ u_m,\ w;\\ \mathcal{K}_{mn}\xi_m\xi_n\geq k_0\xi_m\xi_m,\ \forall\xi_m.\end{gathered} \tag{18}$$

Here we denoted by k_0 an appropriate positive constant which is related to the minimum eigenvalue of the tensor $\mathcal{K}_{mn}$.
The next two inequalities are useful in proving the main results.

Proposition 3.. *Assume that $\alpha_m(t,x)$ is a function of class C^1 regarding the time variable t and satisfies the condition:*

$$\alpha_m(0,x)=0,\ \forall x\in\bar{D}. \tag{19}$$

Then we can find a constant $b_1>0$ such that it is fulfilled the next inequality:

$$\int_0^t\int_{D(\tau)}\mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_ndVd\tau\geq b_1\int_{D(t)}\mathcal{K}_{mn}\alpha_m\alpha_ndV,\ \forall t\geq 0. \tag{20}$$

Proof. First, because the function $\alpha_m(t,x)$ is of class C^1, regarding the variable t, based on (19), we can find $t_1>0$ so that:

$$\alpha_m(t,x)=0,\ \forall(t,x)\in[0,t_1]\times\bar{D}. \tag{21}$$

We will prove the inequality (20) by reducing it to the absurd.
With other words, we suppose that the inequality is false on the interval (t_1, t_2), $t_2 > t_1$, i.e.

$$\int_0^t \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n dV d\tau < b_1 \int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV, \ \forall t \in (t_1, t_2)\,, \tag{22}$$

such that we can deduce that:

$$\int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV > 0, \ \forall t \in (t_1, t_2)\,. \tag{23}$$

Also, taking into account (21) and the Chauchy-Schwarz inequality, we deduce:

$$\int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV = \int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV + 2 \int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \alpha_n dV d\tau \leq$$
$$\leq 2 \left(\int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n dV d\tau \right)^{1/2} \left(\int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \alpha_m \alpha_n dV d\tau \right)^{1/2},$$

and from this we obtain a Gronwall-type inequality:

$$\int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV < 4b_1 \int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \alpha_m \alpha_n dV d\tau, \ \forall t \in (t_1, t_2)\,. \tag{24}$$

If we derive in (24), we obtain

$$\frac{d}{dt} \left(\int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \alpha_m \alpha_n dV d\tau \right) \leq 2b_1 \int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV d,$$

from where we arrive to the conclusion that:

$$\int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \alpha_m \alpha_n dV d\tau = 0,$$

such that, based on $(18)_3$, we contradicted (23).
Given the start of the demonstration, we deduce that the inequality (20) is true. ■

Proposition 4.. *For any solution* $\mathbf{u} = (v_k, \phi_{kl}, T)$ *of the initial boundary value problem* $\mathcal{P}$, *then takes place the next inequality:*

$$\begin{aligned} &\int_{D(t)} [A_{klmn} e_{kl} e_{mn} + 2D_{klmn} e_{kl} \kappa_{mn} + 2F_{klmnr} e_{kl} \chi_{mnr} + \\ &\quad + B_{klmn} \kappa_{kl} \kappa_{mn} + 2G_{klmnr} \kappa_{kl} \chi_{mnr} + C_{kljmnr} \chi_{klj} \chi_{mnr} + \\ &\quad\quad + 2 \left(P_{klm} e_{kl} + Q_{klm} \kappa_{kl} + R_{klnm} \chi_{kln} \right) \alpha_m + K_{mn} \alpha_m \alpha_n] \, dV + \\ &\quad\quad\quad + 2 \int_{t_1}^t \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n dV d\tau \geq 0. \end{aligned} \tag{25}$$

Proof. Let us consider the aritmethic-geometric mean inequalities in the following form:

$$ab \leq \frac{1}{2}\left(\frac{a^2}{p^2} + p^2 b^2\right),$$

for suitable chosen parameters p.
We will this inequality for each product from the expression

$$\left(P_{klm} e_{kl} + Q_{klm} \kappa_{kl} + R_{klnm} \chi_{kln}\right) \alpha_m$$

that appears in (25).
By conveniently choosing the parameters p, with this inequality and the Cauchy-Schwarz inequality, we can raise this product with terms in which only the elasticity tensors appear. In this way, we find the positive constants C_1 and C_2 for which we have:

$$\begin{aligned}
&\int_{D(t)} [A_{klmn} e_{kl} e_{mn} + 2D_{klmn} e_{kl} \kappa_{mn} + 2F_{klmnr} e_{kl} \chi_{mnr} + \\
&\quad + B_{klmn} \kappa_{kl} \kappa_{mn} + 2G_{klmnr} \kappa_{kl} \chi_{mnr} + C_{kljmnr} \chi_{klj} \chi_{mnr} + \\
&\quad\quad + 2\left(P_{klm} e_{kl} + Q_{klm} \kappa_{kl} + R_{klnm} \chi_{kln}\right) \alpha_m + K_{mn} \alpha_m \alpha_n]\, dV + \\
&\quad\quad\quad + 2\int_{t_1}^{t} \int_{D(\tau)} \mathcal{K}_{mn} \dot{\alpha}_m \dot{\alpha}_n dV d\tau \geq \qquad (26) \\
&\geq C_1 \int_{D(t)} [A_{klmn} e_{kl} e_{mn} + 2D_{klmn} e_{kl} \kappa_{mn} + \\
&\quad + 2F_{klmnr} e_{kl} \chi_{mnr} + B_{klmn} \kappa_{kl} \kappa_{mn} + 2G_{klmnr} \kappa_{kl} \chi_{mnr} + \\
&\quad\quad + C_{kljmnr} \chi_{klj} \chi_{mnr}]\, dV + C_2 \int_{D(t)} \mathcal{K}_{mn} \alpha_m \alpha_n dV.
\end{aligned}$$

If we take into account the hypotheses (18), from (26) the desired inequlity (25) is obtained and the proof of proposition is completed. ■

4. Main results

Our following main results are based, for the most part, on the auxiliary results from previous section. In the first two theorems we prove in two different ways the uniqueness of the solution of the mixed initial boundary value problem $\mathcal{P}$.

Theorem 1.. *We assume that the hypotheses (18) are satisfied. Then the mixed problem $\mathcal{P}$ admits only one solution.*

Proof. Since the problem $\mathcal{P}$ is linear, the difference of its two supposed solutions is also a solution, but which correspond to null initial data and homogeneous boundary conditions. Then, based on the proposition 2, we deduce:

$$\int_{D(t)} \left(\frac{1}{2}\rho \dot{v}_k \dot{v}_k + I_{jk}\dot{\phi}_{jm}\dot{\phi}_{km} + \rho e \right) dV + \int_0^t \int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau = 0, \quad (27)$$

so that, considering the hypotheses (18), we obtain:

$$\dot{v}_m(t,x) = 0,\ \dot{\phi}_{mn}(t,x) = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D}, \quad (28)$$

and from this we deduce

$$v_m(t,x) = 0,\ \phi_{mn}(t,x) = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D}, \quad (29)$$

because the initial data are null.
On the other hand, if we take into account Eqs. (16), (27) and (28), we are led to the identity:

$$\int_{D(t)} \frac{1}{2} \left(cT^2 + K_{mn}\alpha_m\alpha_n \right) dV + \int_0^t \int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D},$$

from where, based on the hypothese (18), we deduce

$$T(t,x) = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D}. \quad (30)$$

Considering (29) and (30), the proof of Theorem 1 is complete. ■

Theorem 2.. *We assume that the hypotheses (18) are satisfied. Then the mixed problem $\mathcal{P}$ admits only one solution.*

Proof. The result can be obtained as a consequence of the Proposition 3 and the Proposition 4. The difference of two solutions of the problem $\mathcal{P}$ is also a solution, but corresponding to null initial data and homogeneous boundary conditions. As a consequence, we have

$$M(t) = 0,\ \forall t \in [0,\infty),$$

the function $M(t)$ being defined in (16).
Based on (16), (18)-(20) and (25), we get

$$\left(\dot{v}_k, \dot{\phi}_{kl}, T \right)(t,x) = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D},$$

so that, considering the null initial data, we deduce

$$\left(v_k, \phi_{kl}, T \right)(t,x) = 0,\ \forall (t,x) \in [0,\infty) \times \bar{D},$$

and this ends the proof of Theorem 2. ■
Our next main results are regarding the continuous dependence of a solution of the mixed problem $\mathcal{P}$ regarding the supply terms.

Theorem 3.. *We assume that the hypotheses (18) are satisfied and consider the homogeneous form of the boundary conditions (13). Then, for any solution* (v_k, ϕ_{kl}, T) *of the mixed problem* $\mathcal{P}$*, we have the following estimate:*

$$[M(t)]^{1/2} \le [M(0)]^{1/2} + \left[\frac{1}{2}\int_0^t \int_{D(\tau)} \rho\left(f_k f_k + g_{kl}g_{kl} + \frac{1}{c}r^2\right) dV d\tau\right]^{1/2}, \forall t \ge 0. \quad (31)$$

Proof. Taking into account the null boundary data and using the Cauchy-Schwarz inequality in (17), we deduce

$$M(t) \le M(0) +$$
$$+\int_0^t \left(\int_{D(\tau)} \rho\left(f_k f_k + g_{kl}g_{kl} + \frac{1}{c}r^2\right) dV\right)^{\frac{1}{2}} \left(\int_{D(\tau)} (\rho \dot{v}_k \dot{v}_k + I_{kl}\phi_{km}\phi_{lm} + cT^2) dV\right)^{\frac{1}{2}} d\tau \quad (32)$$

If we take into account (16) and (19), from (32) we deduce the following Gronwall inequality

$$M(t) \le M(0) + \int_0^t \left(2M(\tau)\int_{D(\tau)} \rho\left(f_k f_k + g_{kl}g_{kl} + \frac{1}{c}r^2\right) dV\right)^{1/2} d\tau. \quad (33)$$

Now, we can use Gronwall's lemma so that we arrive at the estimate (31), which concludes the proof of the theorem. ■
Before we approach our last result, we must specify that the result is valid only in the situation where $S_3 = \partial D$, as such $S_3^c = \emptyset$, that is, the whole boundary surface is thermally insulated.
Let us denote by $\mathcal{P}_0$ the particular form of $\mathcal{P}$ which respect this situation. According to Eqs. (5) and (6), the supply therms are (f_k, g_{kl}, r).
To simplify writing, we introduce the following notations:

$$\begin{aligned} F(t) &= \left(\int_0^t \int_{D(\tau)} \rho\,(f_k f_k + g_{kl}g_{kl})\, dV d\tau\right)^{\frac{1}{2}}, \\ R(t) &= \int_0^t \int_0^s \int_{D(\tau)} \rho^2 r^2 dV d\tau ds, \\ G(t) &= 2R(t) + 2\left(\int_0^t F(\tau) d\tau\right)^2, \\ w &= \sqrt{\max_{\bar{D}} |c|}. \end{aligned} \quad (34)$$

Theorem 4.. *We assume that the hypotheses (18) are satisfied and consider the homogeneous form of the initial conditions (13). If* (v_k, ϕ_{kl}, T) *is a specific* *solution of the particular problem $\mathcal{P}_0$ for which the following conditions are met:*

$$\int_0^\infty e^{-\frac{t}{w^2}} G(t)dt < \infty,$$
$$\lim_{t\to\infty} e^{-\frac{t}{2w^2}} \int_0^t M(\tau)d\tau = 0, \tag{35}$$

then we obtain the next estimation:

$$0 \le \int_0^t M(\tau)d\tau \le \frac{1}{2w^2}\int_t^\infty e^{-\frac{\tau-t}{2w^2}} G(\tau)d\tau, \forall t \ge 0. \tag{36}$$

Proof. First, taking into account the hypotheses (18), fom (16) and (26) we find:

$$\int_0^t M(\tau)d\tau \ge \frac{1}{2}\int_{D(t)} \left(\rho \dot{v}_k \dot{v}_k + I_{jk}\dot{\phi}_{jm}\dot{\phi}_{km}\right) dV +$$
$$+\frac{1}{2}\int_0^t\int_0^s\int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau ds \ge 0,\ \forall t \ge 0. \tag{37}$$

Also,

$$\int_0^t M(\tau)d\tau = 0,\ \forall t\ge 0 \Rightarrow \left(v_k, \phi_{kl}, T\right)(t,x) = 0,\ \forall (t,x)\in[0,\infty)\times \bar{D}. \tag{38}$$

Considering (16) and (26) we can obtain the estimate:

$$M(t) \ge \frac{1}{2}\int_0^t\int_{D(\tau)} T^2 ddV\tau. \tag{39}$$

But we considered the case of null initial and boundary data, so that from (19) we can write

$$\int_0^t M(\tau)d\tau) = \frac{1}{2}\int_0^t\int_{D(\tau)} -cT^2 dV d\tau +$$
$$+\int_0^t\int_0^s\int_{D(\tau)} \left(\rho f_k \dot{v}_k + I_{kl} g_{ks}\dot{\phi}_{ls} + \rho r T\right) dV d\tau ds,\ \forall t \ge 0. \tag{40}$$

Here, we take the aritmethic-geometric mean inequalities in the same form as that proposed in Proposition 4, the Cauchy-Schwarz inequality and the the estimate (39) so that we obtain:

$$\int_0^t M(\tau)d\tau \le w^2 M(t) + \frac{1}{4}\int_0^t \int_0^s \int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau ds + R(t) + \\ + \int_0^t F(s)\left(\int_0^s \int_{D(\tau)} \left(\rho\dot{v}_k\dot{v}_k + I_{jk}\dot{\phi}_{jm}\dot{\phi}_{km}\right) dV d\tau\right)^{1/2} ds,$$

that can be reformulated as:

$$\int_0^t M(\tau)d\tau \le w^2 M(t) + \frac{1}{4}\int_0^t \int_0^s \int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau ds + R(t) + \\ + \left(\int_0^t \int_{D(\tau)} \left(\rho\dot{v}_k\dot{v}_k + I_{jk}\dot{\phi}_{jm}\dot{\phi}_{km}\right) dV d\tau\right)^{1/2} \int_0^t F(\tau)d\tau. \quad (41)$$

From (41), by using the aritmethic-geometric mean inequalities, we deduce

$$\int_0^t M(\tau)d\tau \le w^2 M(t) + R(t) + \\ + \frac{1}{4}\int_0^t \int_{D(\tau)} \left(\rho\dot{v}_k\dot{v}_k + I_{jk}\dot{\phi}_{jm}\dot{\phi}_{km}\right) dV d\tau + \\ + \frac{1}{4}\int_0^t \int_0^s \int_{D(\tau)} \mathcal{K}_{mn}\dot{\alpha}_m\dot{\alpha}_n dV d\tau ds. \quad (42)$$

If we analyze the estimates (37) and (42) we come to the conclusion that:

$$\int_0^t M(\tau)d\tau \le w^2 M(t) + G(t),\ \forall t \ge 0,$$

or, in another form, we have

$$\frac{d}{dt}\left(e^{-\frac{t}{2w^2}} \int_0^t M(\tau)d\tau + \frac{1}{2w^2}\int_0^t e^{-\frac{\tau}{2w^2}} G(\tau)d\tau\right) \ge 0,\ \forall t \ge 0. \quad (43)$$

Now, we take into account $(35)_2$ as such from (42) it follows that for all $t \ge 0$ we deduce

$$e^{-\frac{t}{2w^2}} \int_0^t M(\tau)d\tau + \frac{1}{2w^2}\int_0^t e^{-\frac{\tau}{2w^2}} G(\tau)d\tau \le \\ \le \frac{1}{2w^2}\int_0^\infty e^{-\frac{\tau}{2w^2}} G(\tau)d\tau,$$

and from here we arrive at the estimate (36), and the theorem is demonstrated.
■

5. Conclusions

We have considered the initial boundary value problem specific to the thermoelasticity of type III for dipolar materials. We then proven four ancillary results, namely four integral inequalities, in the four propositions of the study. In section "Main results" we approached four qualitative results regarding the solutions of the above proposed problem. In two of these (in the first two theorems) we obtained two results of uniqueness, proved in different ways. Also, we proven two results which show that the solutions of the considered problem depend continuously with respect to the supply terms. We use different procedures in the two theorems on continuous dependence, but we essentially rely on the auxiliary results from Section 3 and Gronwall-type inequalities. It is important to emphasize that all results are obtained by imposing on the basic equations and basic conditions, average constraints that are common in the mechanics of continuous solids.

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