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A Mean Ergodic Theorem for Affine Nonexpansive Mappings in Nonpositive Curvature Metric Spaces

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Abstract

In this paper, we consider the orbits of an affine nonexpansive mapping in Hadamard (nonpositive curvature metric) spaces and prove an ergodic theorem for the inductive mean, which extends the von Neumann linear ergodic theorem. The main result shows that the sequence given by the inductive means of iterations of an affine nonexpansive mapping with a nonempty fixed point set converges strongly to a fixed point of the mapping. A Tauberian theorem is also proved in order to ensure convergence of the iterations.

1 Introduction

The first mean ergodic theorem for linear nonexpansive mappings was studied by von Neumann [17] in Hilbert spaces. Birkhoff [4] extended this theorem to Banach spaces which asserts that for a linear nonexpansive mapping T on a uniformly convex Banach space $\mathscr{B}$, $\frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges strongly to a fixed point of T. The main aim of this paper is to extend this theorem to nonpositive curved geodesic metric spaces. Therefore, we first briefly present definitions and other preliminaries of nonpositive curvature metric spaces.

Let (X, d) be a metric space, a geodesic segment (or geodesic) between two points $x, y \in X$, is the image of an isometry mapping $\gamma : [0, d(x, y)] \longrightarrow$

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X, with $\gamma(0) = x, \gamma(d(x,y)) = y$ and $d\big(\gamma(t), \gamma(t')\big) = |t - t'|$ for all $t, t' \in [0, d(x,y)]$. A metric space (X,d) is said to be a geodesic metric space if every two points of X are jointed by a geodesic and it is said to be uniquely geodesic if between any two points there is exactly one geodesic that for two arbitrary points x, y is denoted by $[x, y]$.
A uniquely geodesic metric space (X,d) is a $CAT(0)$ space (or nonpositive curvature metric space) if the function d^2 is strongly convex, i.e., for every three points $x_0, x_1, y \in X$ and all $0 \leqslant t \leqslant 1$,

$$d^2(y, x_t) \leqslant (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1), \tag{1.1}$$

where $x_t = (1-t)x_0 \oplus tx_1$ is the unique point in the segment $[x_0, x_1]$ such that $d(x_0, x_t) = td(x_0, x_1)$ and $d(x_1, x_t) = (1-t)d(x_0, x_1)$. A $CAT(0)$ space is uniquely geodesic. A complete $CAT(0)$ space is called a Hadamard space. From now we denote Hadamard spaces by $\mathscr{H}$. We collect some other metric properties of Hadamard spaces in the following lemma.

Lemma 1.1. *(see [13, Proposition 8.1.2],[6, p. 983]) Let (X,d) be a $CAT(0)$ space. Then for all $x, y, z \in X$ and $t, s \in [0,1]$; we have:*

i. $d\big((1-t)x \oplus ty, z\big) \leqslant (1-t)d(x,z) + td(y,z)$.

ii. $d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) = |t-s|d(x,y)$.

iii. $d\big((1-t)z \oplus tx, (1-t)z \oplus ty\big) \leqslant td(x,y)$.

In order to simplify computations, for a, b, c, d in a $CAT(0)$ space (X,d), we denote $\frac{1}{2}\big(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\big)$ by $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ or $\langle ab, cd \rangle$. It is easy to see that:

Lemma 1.2. *Let (X,d) be a $CAT(0)$ space and $a, b, c, d, e \in X$. Then*

i. $\langle ab, ab \rangle = d^2(a,b)$.

ii. $\langle ab, cd \rangle = -\langle ab, dc \rangle = -\langle ba, cd \rangle$.

iii. $\langle ab, cd \rangle = \langle ae, cd \rangle + \langle eb, cd \rangle$.

Lemma 1.3. *(see [8, Lemma 2.3]). Let (X,d) be a $CAT(0)$ space and $x, y, z \in X$. Then for each $\lambda \in [0,1]$ we have:*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda^2 d^2(x,z) + (1-\lambda)^2 d^2(y,z) + 2\lambda(1-\lambda)\langle xz, yz \rangle.$$

In Hadamard spaces every nonempty closed convex subset S is Chebyshev i.e., $P_S x = \{s \in S : d(x,S) = d(x,s)\}$ is singleton, where

$d(x,S) := \inf_{s\in S} d(x,s)$[3]. Thus the metric projection on a nonempty closed convex subset S is the following map:

$$P : \mathscr{H} \underset{x\mapsto P_S x}{\longrightarrow} S,$$

where $P_S x$ is the nearest point of S to x for all $x \in \mathscr{H}$. A well-known fact implies that:

$$d^2(x, P_S x) + d^2(P_S x, y) \leqslant d^2(x,y),\ \forall y \in S \tag{1.2}$$

(see [3, Theorem 2.1.12]). For more facts about Hadamard spaces, we refer the reader to [3, 5].

Let (X,d) be a metric space. A mapping $T : X \longrightarrow X$ is called nonexpansive if $d(Tx,Ty) \leqslant d(x,y)$ for all $x,y \in X$. $F(T) = \{x \in X : Tx = x\}$, the set of all fixed points of the mapping T, is closed and convex [9]. T is said to be asymptotically regular at $x \in X$ if $d(T^n x, T^{n+1}x) \longrightarrow 0$, and in general a sequence $\{x_n\}$ in a metric space (X,d) is said to be asymptotically regular if $d(x_n, x_{n+1}) \longrightarrow 0$.

A mapping $T : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be affine if

$$T\big((1-t)x \oplus ty\big) = (1-t)Tx \oplus tTy, \quad t \in [0,1] \ \text{ and } \ x,y \in \mathscr{H}. \tag{1.3}$$

In fact, T maps the segment joining x,y into the segment joining Tx,Ty, i.e., a geodesic affine mapping T maps the points of geodesic $\gamma_{x,y}(t)$ joining x,y into the points of geodesic $\gamma_{Tx,Ty}(t)$ with preserving the distance from the endpoints of the segments.
It is easy to see that a continuous mapping $T : \mathscr{H} \longrightarrow \mathscr{H}$ is affine if it preserves midpoints of line segments, i.e.,

$$T\big(\frac{1}{2}x \oplus \frac{1}{2}y\big) = \frac{1}{2}Tx \oplus \frac{1}{2}Ty \quad x,y \in \mathscr{H}. \tag{1.4}$$

For extension of ergodic theorems in Hadamard spaces, we need the notion of mean in these spaces. Let $T : \mathscr{H} \to \mathscr{H}$ be an affine nonexpansive mapping. Consider the orbit $\{T^n p | n = 0,1,2,\dots\}$ for any $p \in \mathscr{H}$. There are at least two different ways to define a mean for an orbit in Hadamard spaces that both of them are extensions of the linear mean in linear spaces. We first state them for a general sequence as follows.

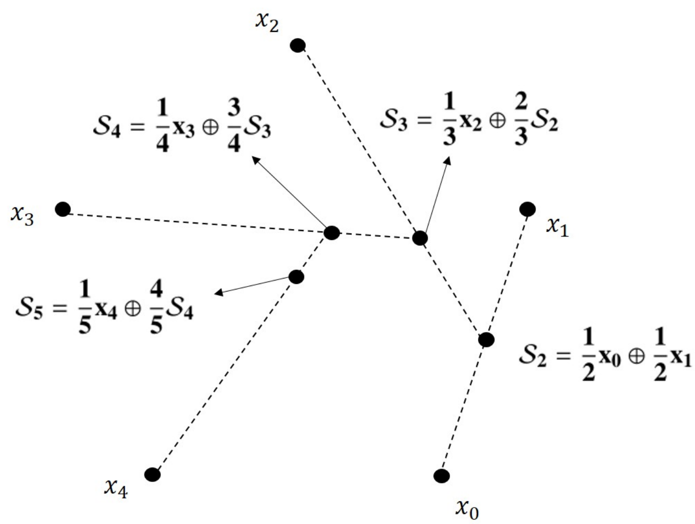
Definition 1.4. (Inductive mean) For each $k \geqslant 0$, we define the inductive mean as defined in [16] for n point $x_k, x_{k+1}, \cdots, x_{k+n-1}$ of the sequence $\{x_n\}$ in $\mathscr{H}$, by induction on n as follows.

$$\mathcal{S}_1^k := x_k \quad \text{and} \quad \mathcal{S}_n^k := \frac{1}{n}x_{k+n-1} \oplus \frac{n-1}{n}\mathcal{S}_{n-1}^k. \tag{1.5}$$

If $k = 0$, the mean of $x_0, \ldots, x_{n-1}$ denoted by $\mathcal{S}_n$ is defined by

$$\mathcal{S}_n := \frac{1}{n} x_{n-1} \oplus \frac{n-1}{n} \mathcal{S}_{n-1}. \tag{1.6}$$

The following figure shows the computation of the inductive mean for five points $x_0, x_1, \cdots, x_4$



and for every $n \geqslant 5$, we can continue this process. Also if T is affine, for the orbit $\{T^n x | n = 0, 1, 2, \cdots\}$ for some $x \in \mathscr{H}$, we denote the mean of n points $x_k = T^k x, \cdots, x_{k+n-1} = T^{k+n-1} x$ by $\mathcal{S}_n^k x$ which is defined as

$$\mathcal{S}_n^k x := \frac{1}{n} x_{k+n-1} \oplus \frac{n-1}{n} \mathcal{S}_{n-1}^k x. \tag{1.7}$$

If $k = 0$, the mean of n points $x_0, \cdots, x_{n-1}$ is inductively defined by

$$\mathcal{S}_n x := \frac{1}{n} x_{n-1} \oplus \frac{n-1}{n} \mathcal{S}_{n-1} x. \tag{1.8}$$

It is obvious that, since T is affine, for each $k \geqslant 1$, we have:

$$\mathcal{S}_n^k x = T^k \mathcal{S}_n x.$$

Remark 1.1. By induction and using Lemma 1.1, for each $z \in \mathscr{H}$ and each $k \geqslant 0$, for the mean $\mathcal{S}_n^k$ defined in (1.5), we derive:

$$d(\mathcal{S}_n^k, z) \leqslant \frac{1}{n} \sum_{i=0}^{n-1} d(x_{k+i}, z).$$

Definition 1.5. (Karcher mean) Given a finite number of points $x_0, \dots, x_{n-1}$ in a Hadamard space, we define the functions

$$\mathcal{F}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_i, x),$$

and for $k \geqslant 1$,

$$\mathcal{F}_n^k(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(x_{k+i}, x).$$

By [3, Proposition 2.2.17] these functions have unique minimizers. For $\mathcal{F}_n(x)$ the unique minimizer is denoted by $\sigma_n(x_0, \dots, x_{n-1})$ (or briefly σ_n) and is called the Karcher mean of $x_0, \dots, x_{n-1}$ [7]. Also for the function $\mathcal{F}_n^k(x)$ the unique minimizer is denoted by $\sigma_n^k(x_k, \dots, x_{k+n-1})$ (or briefly σ_n^k), which is the Karcher mean of $x_k, \dots, x_{k+n-1}$.

For the orbit $\{T^n p\}$, $\sigma_n(p)$ and $\sigma_n^k(p)$, are defined respectively as the unique minimizers of the functions

$$\mathcal{F}[p]_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(T^i p, x),$$

and

$$\mathcal{F}[p]_n^k(x) = \frac{1}{n}\sum_{i=0}^{n-1} d^2(T^{k+i} p, x).$$

Another approach for extension of von Neumann's mean ergodic theorem from Hilbert to Hadamard spaces has been presented by Liimatainen in [11]. He proved the following result for the Karcher mean. In this result affinity of the nonexpansive mapping is replaced with *distance convexity* (defined in blew), which is exactly equivalent to affinity in linear spaces for the linear mean. The distance convexity is defined in [11] as follows.

Definition 1.6. A mapping $T : \mathscr{H} \longrightarrow \mathscr{H}$ is called *distance convex* if for all $n \in \mathbb{N}$ and $q \in \mathscr{H}$ the mapping $d^2\big(\sigma_n(\cdot), q\big) : \mathscr{H} \longrightarrow \mathbb{R}^+$ is convex.

Liimatainen among other results in [11, Theorem 2.1] proved the following nice result.

Theorem 1.7. *Let $\mathscr{H}$ be a Hadamard space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be a nonexpansive distance convex mapping. Then for any $p \in \mathscr{H}$ whose orbit is bounded and any $q \in \mathscr{H}$ such that $Tq = q$ and $q \in \overline{\text{co}}\{p, Tp, T^2p, \cdots\}$, we have $\sigma_n(p) \underset{n}{\longrightarrow} q$.*

In this paper, we prove a mean ergodic theorem for the inductive mean, which has two advantages. First of all the inductive mean is much easier to compute than the Karcher mean. Therefore, this approach may be used easily for approximation of fixed points. The second one is to use a more natural assumption (affinity of the mapping) instead of *distance convexity*, which seems difficult to check.

2 A Mean Ergodic Theorem

In this section, we prove convergence of the inductive mean $\mathcal{S}_n x$ defined by (1.8) for the orbit of an affine nonexpansive mapping T at x. This result generalizes von Neumann's mean ergodic theorem to Hadamard spaces. We first state some lemmas.

Lemma 2.1. *(see [2, Lemma 2.3]). Suppose that*

$$s_{n+1} \leqslant (1-\alpha_n)s_n + \alpha_n t_n + u_n, \quad \forall n \in \mathbb{N}$$

where $\{s_n\}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$ and $\{t_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} t_n \leqslant 0$. Then $\lim_{n\to\infty} s_n = 0$.

The following lemma was proved in [14, Lemma 2] for Hilbert space. Kakavandi stated it in [1, Proposition 4.1] for an amenable semigroup of nonexpansive mappings in Hadamard space. We recall the proof of this lemma in the discrete case in Hadamard space.

Lemma 2.2. *Let $\mathscr{H}$ be a Hadamard space, $T : \mathscr{H} \longleftarrow \mathscr{H}$ be a nonexpansive mapping that $F(T)$ is nonempty and P be the metric projection from $\mathscr{H}$ onto $F(T)$. Then for any $x \in \mathscr{H}$, $\{PT^n x\}$ converges strongly to an element p of $F(T)$.*

Proof. It is well known that for a nonexpansive mapping T in a Hadamard space, $F(T)$ is closed and convex. By the definition of metric projection, we have:

$$\begin{aligned} d(PT^n x, T^n x) &\leqslant d(PT^{n-1}x, T^n x) \\ &= d(TPT^{n-1}x, T^n x) \\ &\leqslant d(PT^{n-1}x, T^{n-1}x). \end{aligned}$$

This implies that $\{d(PT^nx, T^nx)\}$ is nonincreasing. Let $m, n \in \mathbb{N}$ with $m \geqslant n$. Putting T^mx and PT^nx in (1.2), we obtain:

$$\begin{aligned} d^2(PT^mx, PT^nx) &\leqslant d^2(T^mx, PT^nx) - d^2(PT^mx, T^mx) \\ &\leqslant d^2(T^nx, PT^nx) - d^2(PT^mx, T^mx). \end{aligned}$$

Since $\{d(PT^nx, T^nx)\}$ is nonincreasing, so $\{PT^nx\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^nx\}$ converges strongly to an element p of $F(T)$. □

Lemma 2.3. *Let $\mathscr{H}$ be a Hadamard space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be an affine nonexpansive mapping with a nonempty fixed point set $F(T)$. Then for $\{\mathcal{S}_n^k x\}$ defined by* (1.7)*, and each $k \geqslant 1$, we have:*

i. The sequence $\{\mathcal{S}_n^k x\}$ is bounded.

ii. $d(\mathcal{S}_{n+1}^k x, \mathcal{S}_n^k x) \longrightarrow 0$ as $n \to +\infty$.

iii. $d(\mathcal{S}_n^k x, T\mathcal{S}_n^k x) \longrightarrow 0$ as $n \to +\infty$.

Proof. (i). Since $F(T) \neq \emptyset$, if $p \in F(T)$ by Remark 1.1 and nonexpansiveness of T, we see that:

$$\begin{aligned} d(\mathcal{S}_n^k x, p) &\leqslant \frac{1}{n}\sum_{i=0}^{n-1} d(T^{k+i}x, p) \\ &\leqslant d(x, p), \end{aligned}$$

thus $\{\mathcal{S}_n^k x\}$ is bounded.
(ii). By Lemma 1.1, Part (i) and this assumption that T is nonexpansive and affine, we have:

$$\begin{aligned} d(\mathcal{S}_{n+1}^k x, \mathcal{S}_n^k x) &= d\big(\frac{1}{n+1}T^{k+n}x \oplus \frac{n}{n+1}\mathcal{S}_n^k x, \mathcal{S}_n^k x\big) \\ &= \frac{1}{n+1} d(T^{k+n}x, \mathcal{S}_n^k x) \\ &\leqslant \frac{1}{n+1} d(T^{k+n}x, p) + \frac{1}{n+1} d(p, \mathcal{S}_n^k x) \to 0, \end{aligned}$$

therefore,

$$d(\mathcal{S}_{n+1}^k x, \mathcal{S}_n^k x) \longrightarrow 0.$$

(iii). To show that $d(\mathcal{S}_n^k x, T\mathcal{S}_n^k x) \to 0$, by Lemma 1.1, convexity of the distance, Part (i) of this lemma, also since T is affine, we have:

$$d(\mathcal{S}_{n+1}^k x, T\mathcal{S}_n^k x) = d\Big(\frac{1}{n+1}T^{k+n}x \oplus \frac{n}{n+1}\mathcal{S}_n^k x, \frac{1}{n}T^{k+n}x \oplus \frac{n-1}{n}T\mathcal{S}_{n-1}^k x\Big)$$

$$\begin{aligned}
&\leqslant d\Big(\frac{1}{n+1}T^{k+n}x \oplus \frac{n}{n+1}\mathcal{S}_n^k x, \frac{1}{n+1}T^{k+n}x \oplus \frac{n}{n+1}T\mathcal{S}_{n-1}^k x\Big)\\
&\quad + d\Big(\frac{1}{n+1}T^{k+n}x \oplus \frac{n}{n+1}T\mathcal{S}_{n-1}^k x, \frac{1}{n}T^{k+n}x \oplus \frac{n-1}{n}T\mathcal{S}_{n-1}^k x\Big)\\
\leqslant &\qquad \frac{n}{n+1}d(\mathcal{S}_n^k x, T\mathcal{S}_{n-1}^k x) + |\frac{1}{n+1} - \frac{1}{n}|d(T^{k+n}x, T\mathcal{S}_{n-1}^k x)\\
&= (1 - \frac{1}{n+1})d(\mathcal{S}_n^k x, T\mathcal{S}_{n-1}^k x)\\
&\quad + \frac{1}{n+1}\Big(\frac{1}{n}d(T^{k+n}x, T\mathcal{S}_{n-1}^k x)\Big),
\end{aligned}$$

Thereupon by Lemma 2.1, we have $d(\mathcal{S}_{n+1}^k x, T\mathcal{S}_n^k x) \to 0$. On the other hand by Part (ii) and since

$$d(\mathcal{S}_n^k x, T\mathcal{S}_n^k x) \leqslant d(\mathcal{S}_n^k x, \mathcal{S}_{n+1}^k x) + d(\mathcal{S}_{n+1}^k x, T\mathcal{S}_n^k x),$$

we obtain:

$$d(\mathcal{S}_n^k x, T\mathcal{S}_n^k x) \to 0. \tag{2.1}$$

□

Lemma 2.4. (*see [15, Lemma 2.1, Lemma 2.2]*). *Let C be a closed and convex subset of a Hadamard space $\mathscr{H}$, $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $u \in C$. For each $t \in (0, 1)$, set $z_t = tu \oplus (1 - t)Tz_t$. Then z_t converges to the unique fixed point of T as $t \to 0$, which is the nearest point of $F(T)$ to u.*

The following lemma is a consequence of [8, Lemma 2.5].

Lemma 2.5. *Let $\mathscr{H}$ be a Hadamard space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in C such that $d(x_n, Tx_n) \to 0$, then*

$$\limsup_n \langle uz, x_n z\rangle \leqslant 0,$$

where $u \in \mathscr{H}$ and z is the nearest point of $F(T)$ to u.

We rewrite this lemma by replacing u with $T^n x$ and prove it.

Lemma 2.6. *Let $\mathscr{H}$ be a Hadamard space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. If $\{x_n\}$ is a bounded sequence in C such that $d(x_n, Tx_n) \to 0$, then*

$$\limsup_n \langle T^n xp, x_n p\rangle \leqslant 0,$$

where $x \in \mathscr{H}$ and $p = \lim_n PT^n x$ by Lemma 2.2.

Proof. For each $t \in (0,1)$ and $n \in \mathbb{N}$, there exists a point z_t such that $z_t = tT^n x \oplus (1-t)Tz_t$. By Lemma 2.4, z_t converges strongly to $PT^n x$ as $t \to 0$ and remains bounded, since $F(T) \neq \emptyset$. In fact, for $p \in F(T)$, we have:

$$\begin{aligned} d(z_t, p) &= d(tT^n x \oplus (1-t)Tz_t, p) \\ &\leqslant td(T^n x, p) + (1-t)d(Tz_t, p) \\ &\leqslant td(x, p) + (1-t)d(z_t, p), \end{aligned}$$

hence,

$$d(z_t, p) \leqslant d(x, p).$$

By Lemmas 1.2 and 1.3, for each $t \in (0,1)$ and all $n \in \mathbb{N}$, we have:

$$\begin{aligned} d^2(z_t, x_n) &= d^2(tT^n x \oplus (1-t)Tz_t, x_n) \\ &\leqslant t^2 d^2(T^n x, x_n) + (1-t)^2 d^2(Tz_t, x_n) + 2t(1-t)\langle T^n x x_n, Tz_t x_n\rangle \\ &= t^2 d^2(T^n x, x_n) + (1-t)^2 d^2(Tz_t, x_n) + 2t(1-t)\langle T^n x Tz_t, Tz_t x_n\rangle \\ &\quad +2t(1-t)\langle Tz_t x_n, Tz_t x_n\rangle \\ &= t^2 d^2(T^n x, x_n) + \Big((1-t)^2 + 2t(1-t)\Big) d^2(Tz_t, x_n) \\ &\quad +2t(1-t)\langle T^n x Tz_t, Tz_t x_n\rangle \\ &\leqslant t^2 d^2(T^n x, x_n) + (1-t^2)\Big(d(Tz_t, Tx_n) + d(Tx_n, x_n)\Big)^2 \\ &\quad +2t(1-t)\langle T^n x Tz_t, Tz_t x_n\rangle \\ &= t^2 d^2(T^n x, x_n) + (1-t^2) d^2(Tz_t, Tx_n) + (1-t^2) d^2(Tx_n, x_n) \\ &\quad +2(1-t^2) d(Tz_t, Tx_n) d(Tx_n, x_n) + 2t(1-t)\langle T^n x Tz_t, Tz_t x_n\rangle \\ &\leqslant t^2 d^2(T^n x, x_n) + (1-t^2) d^2(z_t, x_n) + (1-t^2) d^2(Tx_n, x_n) \\ &\quad +2(1-t^2) d(z_t, x_n) d(Tx_n, x_n) + 2t(1-t)\langle T^n x Tz_t, Tz_t x_n\rangle, \end{aligned}$$

which by Lemma 1.2 implies

$$\begin{aligned} 2t(1-t)\langle T^n x Tz_t, x_n Tz_t\rangle &\leqslant t^2 d^2(T^n x, x_n) + (1-t^2) d^2(Tx_n, x_n) \\ &\quad + 2(1-t^2) d(z_t, x_n) d(Tx_n, x_n). \end{aligned}$$

Hence, by boundedness of $\{x_n\}$ and $d(x_n, Tx_n) \to 0$, for each $t \in (0,1)$ we obtain:

$$\limsup_n \langle T^n x Tz_t, x_n Tz_t\rangle \leqslant \frac{t}{2(1-t)} \limsup_n d^2(T^n x, x_n). \tag{2.2}$$

On the other hand, by Lemma 2.4, z_t converges to $PT^n x$, as $t \to 0$. So continuity of d and T as well as boundedness of $\{x_n\}$ imply

$$\langle T^n x Tz_t, x_n Tz_t\rangle \underset{t\to 0}{\longrightarrow} \langle T^n x PT^n x, x_n PT^n x\rangle, \quad \text{uniformly with respect to } n.$$

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\langle T^n x PT^n x, x_n PT^n x\rangle \leqslant \epsilon + \langle T^n x T z_t, x_n T z_t\rangle, \tag{2.3}$$

for all $0 \leqslant t \leqslant \delta$ and all $n \in \mathbb{N}$. Since by Lemma 2.2, $p = \lim_n PT^n x$, by (2.2), (2.3) and triangle inequality, we have:

$$\begin{aligned}
2\limsup_n \langle T^n xp, x_n p\rangle &= \limsup_n \big(d^2(T^n x, p) + d^2(x_n, p) - d^2(T^n x, x_n)\big)\\
&\leqslant \limsup_n \big(d^2(T^n x, PT^n x) + d^2(PT^n x, p)\\
&\quad +2d(T^n x, PT^n x)d(PT^n x, p)\\
&\quad +d^2(x_n, PT^n x) + d^2(PT^n x, p)\\
&\quad +2d(x_n, PT^n x)d(PT^n x, p) - d^2(T^n x, x_n)\big)\\
&\leqslant \limsup_n \big(d^2(T^n x, PT^n x) + d^2(x_n, PT^n x) - d^2(T^n x, x_n)\big)\\
&= \limsup_n \langle T^n x PT^n x, x_n PT^n x\rangle\\
&\leqslant \epsilon + \limsup_n \langle T^n x T z_t, x_n T z_t\rangle\\
&\leqslant \epsilon + \frac{t}{2(1-t)} \limsup_n d^2(T^n x, x_n).
\end{aligned}$$

Letting $t \to 0$, we get:

$$\limsup_n \langle T^n xp, x_n p\rangle \leqslant \frac{\epsilon}{2}.$$

Since ϵ is arbitrary, we deduce:

$$\limsup_n \langle T^n xp, x_n p\rangle \leqslant 0,$$

and this completes the proof. $\square$

Now we can prove the main result.

Theorem 2.7. *Let $\mathscr{H}$ be a Hadamard space and $T : \mathscr{H} \longrightarrow \mathscr{H}$ be an affine nonexpansive mapping with a nonempty fixed point set $F(T)$. Then $\{\mathcal{S}_n x\}$ defined by (1.8) converges to $p \in F(T)$, where $p = \lim_n PT^n x$ by Lemma 2.2.*

Proof. For $p = \lim_n PT^n x$, we get:

$$d^2(\mathcal{S}_{n+1}x, p) = d^2(\frac{1}{n+1}T^n x \oplus \frac{n}{n+1}\mathcal{S}_n x, p)$$

$$\begin{aligned}
&\leqslant \ (\frac{1}{n+1})^2 d^2(T^n x, p) + (\frac{n}{n+1})^2 d^2(\mathcal{S}_n x, p) \\
&\quad + 2\frac{1}{n+1}\frac{n}{n+1}\langle T^n xp, \mathcal{S}_n xp\rangle \\
&\leqslant \ \frac{n}{n+1} d^2(\mathcal{S}_n x, p) + \frac{1}{n+1}\Big(\frac{1}{n+1} d^2(T^n x, p) + \frac{2n}{n+1}\langle T^n xp, \mathcal{S}_n xp\rangle\Big).
\end{aligned}$$

By Lemmas 2.1, 2.3 and 2.6, we have that $\mathcal{S}_n x$ converges strongly to p. □

Remark 2.1. It is easily seen that the Mazur-Ulam theorem holds in Hadamard spaces, i.e., every surjective self-isometry in a Hadamard space is affine. For every $x, y \in \mathscr{H}$, $\frac{1}{2}T(x) \oplus \frac{1}{2}T(y)$ is the midpoint of the segment $[T(x), T(y)]$. Since T is surjective, there exists some $z \in \mathscr{H}$ such that $T(z) = \frac{1}{2}T(x) \oplus \frac{1}{2}T(y)$. We have:

$$d\big(T(x), T(z)\big) = d\big(T(y), T(z)\big) = \frac{1}{2} d(Tx, Ty).$$

Since T is distance preserving, we get

$$d(x, z) = d(y, z) = \frac{1}{2} d(x, y).$$

Therefore, z is the midpoint of x and y. Hence,

$$T(\frac{1}{2}x \oplus \frac{1}{2}y) = \frac{1}{2}T(x) \oplus \frac{1}{2}T(y).$$

Since an isometry mapping is also continuous, then T is affine. Consequently, any surjective self-isometry in a Hadamard space is a nonexpansive affine mapping and therefore it satisfies the assumptions of Theorem 2.7.

3 From Ergodic Convergence to Convergence

For a sequence $\{x_n\}$ in a linear space, $\frac{1}{n}\sum_{i=0}^{n-1} x_i$ is called the Cesaro mean and $\frac{1}{n}\sum_{i=0}^{n-1} x_{k+i}$ is called the Vallee-Poussin mean. Lorentz [12] introduced almost convergence of real-valued sequences in terms of uniform convergence of the Vallee-Poussin means with respect to k. The sequence $\{x_n\}$ is almost convergent to s if the Vallee-Poussin mean of $\{x_n\}$ is convergent to s uniformly in k. He showed the following relation between three kinds of convergence for a real sequence:
i.e., convergence of the sequence $\{x_n\}$ implies the almost convergence of the sequence and the almost convergence implies convergence of the Cesaro mean of the sequence. For the reverse directions, we need some sufficient conditions, which are called the Tauberian conditions. Kuo [10] extended these results from real sequences to vector sequences in Banach spaces. In the next theorem

we show that in Hadamard spaces, the almost convergence (with respect to the inductive mean) together with the asymptotic regularity implies convergence of the sequence. Although the proof is true for every geodesic metric space whose distance is geodesically convex, we state it in Hadamard spaces.

Theorem 3.1. *Let $\{x_n\}$ be a sequence in a Hadamard space $(\mathscr{H}, d)$. Then $\{x_n\}$ converges to y if and only if $\mathcal{S}_n^k$ defined by (1.5) converges to y uniformly in $k \geqslant 0$ (or the sequence $\{x_n\}$, is almost convergent to y) and $\{x_n\}$ is asymptotically regular.*

Proof. **Necessity.** If $\{x_n\}$ converges strongly to y, for an arbitrary $\epsilon > 0$, there is a positive integer i_0 such that:

$$d(x_n, y) < \frac{\epsilon}{4} \qquad \forall n \geqslant i_0. \tag{3.1}$$

Choose $m > i_0$ sufficiently large such that

$$\frac{d(x_0, y) + \cdots + d(x_{m-1}, y)}{m} = \frac{1}{m}\sum_{i=0}^{m-1} d(x_i, y) \leqslant \frac{\epsilon}{2}. \tag{3.2}$$

Because if $\sum_{i=0}^{i_0-1} d(x_i, y) := I$, since by (3.1)

$$\frac{1}{m}\sum_{i=0}^{m-1} d(x_i, y) \leqslant \frac{1}{m}\sum_{i=0}^{i_0-1} d(x_i, y) + \frac{m-i_0}{m}\frac{\epsilon}{4} \leqslant \frac{1}{m}I + \frac{\epsilon}{4},$$

by choosing $m > \max\{i_0, \frac{4I}{\epsilon}\}$, we get (3.2). For all $k \geqslant 0$ there are two cases, $k < i_0$ and $k \geqslant i_0$. If $k \geqslant i_0$, it is clear that

$$d(\mathcal{S}_N^k, y) \leqslant \frac{1}{N}\sum_{i=0}^{N-1} d(x_{k+i}, y) \leqslant \frac{\epsilon}{4} < \epsilon.$$

If $k < i_0$, since $i_0 < m$, by (3.1) and (3.2), for all $N \geqslant m$, we have:

$$d(\mathcal{S}_N^k, y) \quad \leqslant \quad \frac{1}{N}\sum_{i=0}^{N-1} d(x_{k+i}, y)$$

$$
\begin{aligned}
&\leqslant \frac{1}{N}\sum_{i=0}^{k+N-1} d(x_i, y) \\
&= \frac{m}{N}\sum_{i=0}^{m-1}\frac{d(x_i, y)}{m} + \frac{1}{N}\sum_{i=m}^{k+N-1} d(x_i, y) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.
\end{aligned}
$$

Thus for each $\epsilon > 0$ there is a positive integer m such that

$$d(\mathcal{S}_N^k, y) < \epsilon \qquad \forall N \geqslant m, \forall k \geqslant 0.$$

This shows that $\mathcal{S}_n^k$ converges to y uniformly in $k \geqslant 0$. Clearly convergence of $\{x_n\}$ implies the asymptotic regularity.

Sufficiency. Let $\mathcal{S}_n^k$ converge to y uniformly in $k \geqslant 0$ and $\{x_n\}$ be asymptotically regular. By Lemma 1.1, we have:

$$
\begin{aligned}
d(x_k, y) &\leqslant d(x_k, \mathcal{S}_n^k) + d(\mathcal{S}_n^k, y) \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n-1} d(x_k, x_{k+i}) + d(\mathcal{S}_n^k, y) \\
&= \frac{1}{n}\Big\{ d(x_k, x_{k+n-1}) + d(x_k, x_{k+n-2}) + \cdots + d(x_k, x_{k+1}) \Big\} + d(\mathcal{S}_n^k, y) \\
&\leqslant \frac{1}{n}\Big\{ \Big(\sum_{i=k}^{k+n-2} d(x_i, x_{i+1}) \Big) + \cdots + \Big(\sum_{i=k}^{k+1} d(x_i, x_{i+1}) \Big) + d(x_k, x_{k+1}) \Big\} \\
&\quad + d(\mathcal{S}_n^k, y) \\
&\leqslant \frac{1}{n}\sup_{i\geqslant k} d(x_i, x_{i+1})\big((n-1) + \cdots + 2 + 1\big) + d(\mathcal{S}_n^k, y) \\
&= \frac{1}{n}\frac{(n-1)n}{2}\sup_{i\geqslant k} d(x_i, x_{i+1}) + d(\mathcal{S}_n^k, y) \\
&= \frac{n-1}{2}\sup_{i\geqslant k} d(x_i, x_{i+1}) + d(\mathcal{S}_n^k, y).
\end{aligned}
$$

Taking limsup when $k \to \infty$, by asymptotic regularity of $\{x_n\}$, we get

$$\limsup_{k\to\infty} d(x_k, y) \leqslant \limsup_{k\to\infty} d(\mathcal{S}_n^k, y) \leqslant \sup_{k\geqslant 1} d(\mathcal{S}_n^k, y).$$

Now letting $n \to \infty$. Since $\mathcal{S}_n^k$ is uniformly convergent to y, the recent inequality completes the proof. $\square$

Corollary 3.2. *Let $\mathscr{H}$ be a Hadamard space, also T and p satisfy the assumptions of Theorem 2.7. Then $\{T^n x\}$ converges strongly to p if and only if $\{T^n x\}$ is asymptotically regular.*

Proof. For $\{\mathcal{S}_n^k x\}$ defined as (1.7), since T is affine and $p \in F(T)$, we have:

$$d(\mathcal{S}_n^k x, p) = d(T^k \mathcal{S}_n x, p) \leqslant d(\mathcal{S}_n x, p).$$

This shows that for the orbits of an affine nonexpansive mapping in a Hadamard space, the mean convergence is equivalent to the almost convergence. Therefore, Theorem 3.1 completes the proof. □

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