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Reducibility in Corsini hypergroups

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Abstract

In this paper, we study the reducibility property of special hypergroups, called Corsini hypergroups, named after the mathematician who introduced them. The concept of reducibility was introduced by Jantosciak, who noticed that it can happen that hyperproduct does not distinguish between a pair of elements. He defined a certain equivalences in order to identify elements which play the same role with respect to the hyperoperation. First we will determine specific conditions under which the Corsini hypergroups are reduced. Next, we will present some properties of these hypergroups necessary for studying the fuzzy reducibility property. The fuzzy reducibility will be considered with respect to the grade fuzzy set $\widetilde{\mu}$, used for defining the fuzzy grade of a hypergroup. Finally, we will study the reducibility and the fuzzy reducibility of the direct product of Corsini hypergroups.

1 Introduction

In a classical algebraic structure (group, ring, field, etc) the result of the synthesis, called operation, between two elements of the support set is an element of the same support set. Extending this property in a "hyper" way, one can consider the synthesis of two elements having as result a subset of the support set, so substituting the operation on a set with a hyperoperation. This genial idea came to F. Marty in 1934, when he proved that the quotient structure of a group by any arbitrary subgroup can be defined as a hypergroup. This is a set H endowed with a hyperoperation, i.e. a function $\circ : H \times H \to \mathcal{P}^*(H)$ defined from the Cartesian product $H \times H$ to the set of non-empty subsets of H, having two properties:

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1. the associativity: for any $x, y, z \in H, (x \circ y) \circ z = x \circ (y \circ z)$ and
2. the reproducibility: for any $x \in H, x \circ H = H = H \circ x$.

Here, $(x \circ y) \circ z$ must be read as $\bigcup_{u \in x \circ y} u \circ z$, and similarly $x \circ (y \circ z)$ must be read as $\bigcup_{u \in y \circ z} x \circ u$.

Since then, the theory of algebraic hypercompositional structures has been developed from different perspectives, becoming today not only a very well known branch of Modern Algebra, but also an important tool to solve problems in other areas, as graph heory, probability, geometry, number theory, coding theory, etc. For a collection of some applications obtained before 2003, we indicate the book [7]. One of the most studied applications of hypergroups is that one related with binary or n-ary relations ([3, 8, 9, 11, 12, 14, 16, 24]), that has been then extended to graphs and hypergraphs ([2, 17, 23, 25]). This idea was developed at the beginning by the Italian school, mainly by P. Corsini, who associated to an arbitrary hypergraph a particular commutative quasi-hypergoup and founded necessary and sufficient conditions so that it is a hypergroup [2]. Corsini called this new structure a "hypergraph-hypergroupoid". In 2019 Al Tahan and Davvaz [1] studied again this hypergroup, calling it "Corsini hypergroup", finding its properties related to cyclicity, regular relations, complete parts and direct products of hypergroups. It is also interesting to notice that a particular type of this Corsini hypergroup was studied by G. Massouros [23] for its applications in the Theory of Languages. More precisely, this is a B-hypergoup, where the hyperoperation is defined as $x \circ y = \{x, y\}$ for any two arbitrary elements. The B-hypergroup appears also in the study of fortified join hypergroups [21] or breakable semihypergroups [18].

In this paper we will study the reducibility property of Corsini hypergroups. This concept was first defined by Jantosciak [19] in 1990, when he noticed that in a hypergroup (and we can say that in any hypercompositional structure) some elements play "interchangeable roles" with respect to the hyperoperations. In particular, two arbitrary elements can belong to the same hyperproducts of elements, or their hyperproducts with all elements in the support set are the same. Mathematically speaking, two equivalence relations can be defined on a hypergroup in order to describe these properties. They have been introduced by Jantosciak [19], who called them inseparability and operational equivalence. Combining both of them, Jantosciak defined also a third equivalence, i.e. the essential indistinguishability. Moreover, he called a hypergroup to be reduced if the equivalence class of each element in the hypergroup is a singleton with respect to the last equivalence relation. This property was studied in deep by Cristea ([9, 10, 11, 14]) for hypergroups associated to binary or n-ary relations and extended also to the fuzzy case [10].

Recently, Kankaras and Cristea [20] have investigated the (fuzzy) reducibility of complete hypergroups, i.p.s hypergroups and a particular non-complete $1-$ hypergroups. The fuzzy reducibility was considered with respect to the grade fuzzy set $\widetilde{\mu}$, introduced by Corsini [5] and studied by Corsini and Cristea for the definition of the fuzzy grade of a hypergroup [6]. For an element u in hypergroup, grade fuzzy set $\widetilde{\mu}(u)$ is the average value of the reciprocals of the sizes of all hyperproducts which contain the element u. Fuzzy grade of the hypergroup represents the length of the sequence of join spaces and fuzzy sets associated with the given hypergroup.

Motivated by the above mentioned studies, in this note we aim to study the reducibility and the fuzzy reducibility property of Corsini hypergroups. First, we will recall some definitions and properties related with Corsini hypergroups, then we will establish conditions under which these hypergroups are reduced or fuzzy reduced. Finally, we will focus on the study of the product of Corsini hypergroups and its reducibility. Conclusions and new ideas for further research are covered by the last section.

2 Preliminaries

In this section we briefly recall the main definitions concerning the reducibility and the fuzzy reducibility of hypergroups, as well as the concept of Corsini hypergroup. For more details and a solid background of the theory of algebraic hypergroups the readers can consult the monographs [4, 7, 15].

Definition 2.1. [19] Two elements x, y in a hypergroup $(H, \circ)$ are called:

1. *operationally equivalent* or by short *o-equivalent*, and write $x \sim_o y$, if $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any $a \in H$;

2. *inseparable* or by short *i-equivalent*, and write $x \sim_i y$, if, for all $a, b \in H$, $x \in a \circ b \Longleftrightarrow y \in a \circ b$;

3. *essentially indistinguishable* or by short *e-equivalent*, and write $x \sim_e y$, if they are operationally equivalent and inseparable.

Definition 2.2. [19] A hypergroup is called *reduced* if the equivalence class of each element with respect to the essentially indistinguishable relation is a singleton.

If now we consider a fuzzy set $\mu : H \longrightarrow [0, 1]$ defined on a hypergroup H, then we can extend the reducibility property to the fuzzy case. As in the classical aproach, first we define three equivalences relations that keep the same model as the inseparability and operationally equivalence.

Definition 2.3. [20] In a crisp hypergroup $(H, \circ)$ endowed with a fuzzy set μ, for two arbitrary elements $x, y \in H$, we say that:

1. x and y are *fuzzy operationally equivalent* and write $x \sim_{fo} y$ if, for any $a \in H$, $\mu(x \circ a) = \mu(y \circ a)$ and $\mu(a \circ x) = \mu(a \circ y)$;
2. x and y are *fuzzy inseparable* and write $x \sim_{fi} y$ if $\mu(x) \in \mu(a \circ b) \Longleftrightarrow \mu(y) \in \mu(a \circ b)$, for $a, b \in H$;
3. x and y are *fuzzy essentially indistinguishable* and write $x \sim_{fe} y$, if they are fuzzy operationally equivalent and fuzzy inseparable.

Definition 2.4. [20] The crisp hypergroup $(H, \circ)$ is a *fuzzy reduced hypergroup* if the equivalence class of each element in H with respect to the fuzzy essentially indistinguishable relation is a singleton, i.e for all $x \in H, \hat{x}_{fe} = \{x\}$.

As it was already explained in [20], the fuzzy reducibility depends on the considered fuzzy set, so it can change when we consider different fuzzy sets. For any hypergroupoid $(H, \circ)$, the grade fuzzy set $\widetilde{\mu}$ is defined as follows:

$$\widetilde{\mu}(u) = \frac{A(u)}{q(u)},$$

where $A(u) = \sum\limits_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}, Q(u) = \{(x, y) \in H^2 : u \in x \circ y\}, q(u) = |Q(u)|$. For $Q(u) = \emptyset$, by default we take $\widetilde{\mu}(u) = 0$.

In the first studies concerning the relationship between hypergroups and hypergraphs, Corsini defined the following hypergroupoid.

Definition 2.5. [2] On a non empty set H, define the hyperoperation $\circ$ as follows. For all $(x, y) \in H^2$,

1. $x \circ y = x \circ x \cup y \circ y$,
2. $x \in x \circ x$,
3. $y \in x \circ x \iff x \in y \circ y$.

Theorem 2.6. [2] *A hypergrupoid $(H, \circ)$ satisfying the conditions in Definition 2.5 is a hypergroup if and only if also the following condition is valid:*

$$\forall (a, c) \in H^2 \quad c \circ c \circ c \setminus c \circ c \subseteq a \circ a \circ a.$$

This hypergroup was studied also in [1], where the authors named it "Corsini hypergroup" and investigated also its properties connected with the Cartesian product. Here we recall one result, that we will need in our research.

Theorem 2.7. [1] *Let $(H, \circ_1)$ and $(H, \circ_2)$ be two Corsini hypergroups. Then the direct product of hypergroups $(H \times H, \circ_1 \times \circ_2)$ is a Corsini hypergroup if and only if $(H, \circ_1)$ or $(H, \circ_2)$ (or both) is a total hypergroup.*

Note that, for two given hypergroups defined on the same support set H, the hyperoperation $\bigotimes = \circ_1 \times \circ_2$ is defined as $(x_1, x_2) \bigotimes (y_1, y_2) = (x_1 \circ_1 y_1, x_2 \circ_2 y_2), x_1, x_2, y_1, y_2 \in H$. The structure $(H \times H, \bigotimes)$ is called the *direct product of hypergroups*.

We end this preliminary section with one particular type of Corsini hypergroup, studied for its important properties in the theory of automata and languages [22], and called *B-hypergroup* by G. Massouros, after the binary result that the hyperoperation gives. It was also investigated in connection with fortified join spaces [21] or breakable semihypergroups [18].

Definition 2.8. [22] Let H be any non-empty set. For any $(x, y) \in H^2$, define $\star$ as follows

$$x \star y = \{x, y\}.$$

Then the hypergroup $(H, \star)$ is called a B-hypergroup.

Proposition 2.9. *[1] Any B-hypergroup $(H, \star)$ is a Corsini hypergroup.*

3 The reducibility in Corsini hypergroups

In this section we determine necessary and sufficient condition for the Corsini hypergroup to be reduced. Secondly, we prove that any B-hypergroup is always reduced. Also, we give an example of a reduced hypergroup which is not a B-hypergroup.

Proposition 3.1. *Let $(H, \circ)$ be a Corsini hypergroup. If there exist some different elements x, y in H such that $x \circ x = y \circ y$, then the hypergroup $(H, \circ)$ is not reduced.*

Proof. Let x, y be arbitrary elements in H such that $x \neq y$ and $x \circ x = y \circ y$. It is easy to see that $x \circ a = y \circ a$, for any $a \in H$, since $x \circ a = x \circ x \cup a \circ a = y \circ y \cup a \circ a = y \circ a$. Using the commutativity, we obtain that $a \circ x = a \circ y$, for any $a \in H$. Hence, $x \sim_o y$. Let $x \in c \circ d$, with $x, c, d \in H$. Then $x \in c \circ c \cup d \circ d$, which implies that $x \in c \circ c$ or $x \in d \circ d$. Since $(H, \circ)$ is a Corsini hypergroup, the previous implication gives $c \in x \circ x$ or $d \in x \circ x$ and $c \in y \circ y$ or $d \in y \circ y$. Using the same property, we conclude that $y \in c \circ d$. Similarly, one proves the converse implication. Therefore, $x \sim_i y$. Hence, the hypergroup $(H, \circ)$ is not reduced. □

As a consequence of Proposition 3.1, we obtain the following results.

Proposition 3.2. *A Corsini hypergroup* $(H, \circ)$ *with at least two different elements is reduced if and only if* $x \circ x \neq y \circ y$, *for all* $x, y \in H$.

Proof. The contraposition of Proposition 3.1 directly gives the first direction. Suppose now that $x \circ x \neq y \circ y$, for all $x, y \in H$. Take two arbitrary elements $x \neq y$ from H. We will prove that $x \circ a = y \circ a$, for all $a \in H$, just in case when $x = y$. Assume that $x \circ a = y \circ a$, for all $a \in H$. From here, we have $x \circ x = y \circ x$, which gives $x \circ x = y \circ y \cup x \circ x$. The last equality is possible only if $y \circ y \subseteq x \circ x$. Similarly, since $x \circ y = y \circ y$, it follows the other inclusion $x \circ x \subseteq y \circ y$. Therefore, $x \circ a = y \circ a$ is equivalent with $x \circ x = y \circ y$, which contradicts the hypothesis. Hence, two arbitrary elements x and $y, x \neq y$ are not operationally equivalent, thus $\hat{x}_e = \{x\}$ for all $x \in H$, meaning that H is a reduced hypergroup. □

Proposition 3.3. *Any B-hypergroup is reduced.*

Proof. This immediately follows from Proposition 3.2, since in a B-hypergroup there is $x \circ x = \{x\}$, for all elements x. □

In the following example we present a reduced Corsini hypergroup, which is not a B-hypergroup.

Example 3.4. On the set $H = \{a, b, c\}$ define the hyperoperation " $\circ$ " by the following table:

$\circ$	a	b	c
a	H	H	H
b	H	a, b	H
c	H	H	a, c

Since all the rows in the table are different, it follows that $\hat{x}_o = \{x\}$ for any $x \in H$, which clearly implies the reducibility of the hypergroup.

4 Fuzzy reducibility in Corsini hypergroups

The aim of this section is to prove that a Corsini hypergroup $(H, \circ)$ is not fuzzy reduced with respect to the grade fuzzy set $\tilde{\mu}$. For doing this, first we present some properties regarding the hyperproducts $x_i \circ x_i$, with $x_i \in H$. For a finite hypergroup H with n elements, we will denote its cardinality by $|H| = n$. Recall also that, for any $u \in H$, $\widetilde{\mu}(u) = \frac{A(u)}{q(u)}$, where $A(u) = \sum\limits_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}, Q(u) = \{(x, y) \in H^2 : u \in x \circ y\}, q(u) = |Q(u)|$.

Proposition 4.1. *Let* $(H, \circ)$ *be a Corsini hypergroup with* n *elements. If an element* x_i *appears in exactly* k *hyperproducts* $x_j \circ x_j, j = 1, 2, \ldots, n$, *then* $q(x_i) = 2nk - k^2$.

Proof. Let x_i be an arbitrary element from $H = \{x_1, x_2, x_3, \ldots, x_n\}$ which appears in k hyperproducts $x_j \circ x_j$, for some $j = 1, \ldots, n$. By the definition of the hyperoperation of a Corsini hypergroup, it follows that x_i appears in every hyperproduct $x_j \circ x_k$, with $k \in \{1, \ldots, n\}$. For one fixed k, because the commutativity, x_i appears in $n + n - 1$ hyperproducts. The sum of all such cases is:

$$(2n-1) + (2n-1) - 2 \cdot 1 + (2n-1) - 2 \cdot 2 + \ldots + (2n-1) - 2 \cdot (k-1) =$$
$$(2n-1) \cdot k - 2(1 + 2 + \ldots + k) = (2n-1) \cdot k - (k-1)k = 2nk - k^2.$$

□

Proposition 4.2. *The sum of all cardinalities of $x_i \circ x_i, x_i \in H$ when $|H|$ is odd (even) is an odd (even) number.*

Proof. Let $|H| = n$ be an even number. If $|x_i \circ x_i| = 1$, for every $x_i \in H$, then $\sum_i^n |x_i \circ x_i| = 1 \cdot n = n$ which is an even number. Let add k elements to a hyperproduct $x_i \circ x_i, k \leq n-1$. In that case, by the property 3 of the definition of the hyperoperation " $\circ$ ", we have to add the element x_i to $k-$ hyperproducts $x_j \circ x_j$. All together, we add $k + k = 2k$ elements, which is again an even number. Continuing this process, so adding an arbitrary number of elements to any hyperproduct $x_i \circ x_i$, we always get an even number. Summing arbitrary even numbers, we obtain at the end an even number. The proof is analogous in the case when n is an odd number. □

Proposition 4.3. *Let $(H, \circ)$ be a Corsini hypergroupoid of cardinality n. The number of all possible different sums of the cardinalities of the hyperproducts $x_i \circ x_i, x_i \in H$, is $\frac{n^2-n}{2} + 1$.*

Proof. The proof will be performed using the mathematical induction. For hypergroups of cardinality 2, the property is easily satisfied, because if H contains two elements, we have exactly two possibilities. The hyperproducts $x \circ x$ are singleton, or equal to H. In the first case the sum of the cardinalities of $x_i \circ x_i$ is 2, while in the second case the sum is 4. Thus, the number of the different sums is 2, i.e. $\frac{2^2-2}{2} + 1$. Assume that for $|H| = n$ the number of the different sums is equal to $\frac{n^2-n}{2} + 1$. Let us prove that the claim is valid for $|H| = n + 1$. In this case we have to analyse only the hyperproduct $x_{n+1} \circ x_{n+1}$. If $x_{n+1} \circ x_{n+1} = \{x_{n+1}\}$, then we have $\frac{n^2-n}{2} + 1$ possible sums, i.e. the number of sums is the same as in the inductive case. The other cases are: $x_{n+1} \circ x_{n+1} = \{x_{n+1}, x_i\}, x_{n+1} \circ x_{n+1} = \{x_{n+1}, x_i, x_j\}, \ldots, x_{n+1} \circ x_{n+1} = H$. It gives n sums more, which is finally $\frac{n^2-n}{2} + 1 + n$, i.e. number of possible sums when $|H| = n + 1$ is equal to $\frac{(n+1)^2-(n+1)}{2} + 1$, which proves the proposition.

□

Remark 4.4. Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. There are at least $\frac{n^2-n}{2}+1$ Corsini hypergroups of order n up to isomorphism. Since the hyperproducts $x \circ x, x \in H$ completely determine the hypergroup, it folows that $\frac{n^2-n}{2}+1$ different sums define at least as many different hypergroups. One sum can form more different tables, and in case when $n \geq 3$ the number of hypergroups is greater.

Proposition 4.5. *Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. If an element x_i appears in k hyperproducts $x_j \circ x_j$, and if we assume that the cardinalities of those sets are, respectively $m_1, m_2, \ldots, m_k$, then*

$$\widetilde{\mu}(x_i) = \frac{\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_k} + 2 \cdot \sum\limits_{\substack{i \neq j \\ i=1,\ldots,k}} \frac{1}{|x_i \circ x_j|}}{2nk - k^2}$$

Proof. According to definition of the fuzzy grade set $\widetilde{\mu}$ and Proposition 4.1, the result is clearly satisfied. □

Remark 4.6. If two elements of a Corsini hypergroup have the same number of appearances in some hyperproducts $x_j \circ x_j$, and the cardinalities of those hyperproducts are the same for both elements, based on Proposition 4.5, then their values under the grade fuzzy set $\widetilde{\mu}$ are the same. Hereinafter, we will say that elements with this property are in the *same formation.*

Proposition 4.7. *In any Corsini hypergroup $(H, \circ)$, the fuzzy operational equivalence implies the fuzzy inseparability.*

Proof. Let $x, y \in H$ be two arbitrary elements in H such that $x \sim_{fo} y$, i.e. $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(y \circ a)$, for $\forall a \in H$. It means that:

$$\widetilde{\mu}(x \circ x \cup a \circ a) = \widetilde{\mu}(y \circ y \cup a \circ a)$$

$\Longleftrightarrow$

$$\widetilde{\mu}(x \circ x) \cup \widetilde{\mu}(a \circ a) = \widetilde{\mu}(y \circ y) \cup \widetilde{\mu}(a \circ a).$$

Since this equality is satisfied for every set $\widetilde{\mu}(a \circ a), a \in H$, it follows that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$ and contains both $\widetilde{\mu}(x)$ and $\widetilde{\mu}(y)$ by property 3. of Definition 2.5. If $\widetilde{\mu}(x) = \widetilde{\mu}(y)$, then clearly $x \sim_{fi} y$. Let us consider now the case when $\widetilde{\mu}(x) \neq \widetilde{\mu}(y)$. Suppose that $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ d) = \widetilde{\mu}(c \circ c) \cup \widetilde{\mu}(d \circ d)$. Let us take $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ c)$. It follows that, for some $z \in c \circ c, \widetilde{\mu}(x) = \widetilde{\mu}(z)$. The equality $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$ means that $\{\widetilde{\mu}(l) \mid l \in x \circ x\} = \{\widetilde{\mu}(k) \mid k \in y \circ y\}$, i.e. for every $l \in x \circ x$ there exists $k \in y \circ y$ such that $\widetilde{\mu}(l) = \widetilde{\mu}(k)$. Now, since $\widetilde{\mu}(x) = \widetilde{\mu}(z) \in \widetilde{\mu}(x \circ x)$, $\widetilde{\mu}(x) \in \widetilde{\mu}(x \circ x)$, and $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$, we conclude that $\widetilde{\mu}(z) \in \widetilde{\mu}(y \circ y)$. Thus there exists $l \in y \circ y$ such that $\widetilde{\mu}(z) = \widetilde{\mu}(l)$. But $\widetilde{\mu}(z) \in \widetilde{\mu}(c \circ c)$, so $\widetilde{\mu}(l) \in \widetilde{\mu}(c \circ c)$, with $l \in y \circ y$, which finally gives $\widetilde{\mu}(y) \in \widetilde{\mu}(c \circ c)$. The converse implication can be proved taking $\widetilde{\mu}(y) \in \widetilde{\mu}(c \circ c)$ and proving that $\widetilde{\mu}(x) \in \widetilde{\mu}(c \circ c)$. This shows that $\widetilde{\mu}(x)$ and $\widetilde{\mu}(y)$ appear in the same $\widetilde{\mu}(c \circ c)$. Finally, according to the definition of $\widetilde{\mu}(x \circ y)$, it is easy to prove that the previous equivalence implies the fuzzy inseparability. □

Proposition 4.8. *Let $(H, \circ)$ be a Corsini hypergroup of cardinality n. If x is an element such that $x \circ x$ is a singleton, i.e. $x \circ x = \{x\}$, then* $\widetilde{\mu}(x) = \dfrac{1+2\cdot \sum\limits_{a \neq x} \dfrac{1}{|x \circ a|}}{2n-1}$, *with $a \in H$.*

Proof. Using Proposition 4.1 we easily get that $q(x) = 2n-1$. Since x appears in every product $x \circ a, a \in H$, and the commutativity holds, then $A(x) = 1 + 2 \cdot \sum\limits_{a \neq x} \dfrac{1}{|x \circ a|}$, which clearly gives the formula. □

Based on this result, we can state sufficient conditions such that two elements in a Corsini hypergroup are fuzzy essentially indistinguishable.

Proposition 4.9. *If there exist two elements x, y in a Corsini hypergroup $(H, \circ)$ such that $x \circ x = x$ and $y \circ y = y$, then $x \sim_{fe} y$.*

Proof. Using Proposition 4.8 this obviously holds, because $\widetilde{\mu}(x) = \widetilde{\mu}(y)$. □

Proposition 4.10. *If there exist two elements x, y in Corsini hypergroup $(H, \circ)$ such that $x \circ x = y \circ y = H$, then $x \sim_{fe} y$.*

Proof. Since $x \circ x = H$, based on condition 3 of Definition 2.5 it follows that x appears in all hyperproducts $z \circ z$, with $z \in H$, and similarly holds for y. So x and y are in the same formation. According to Proposition 4.5, we have $\widetilde{\mu}(x) = \widetilde{\mu}(y)$, so x and y are fuzzy inseparable. Besides, $\mu(x \circ a) = \mu(y \circ a) = \mu(\{x \mid x \in H\})$, which implies the fuzzy operational equivalence. Therefore, $x \sim_{fe} y$. □

Theorem 4.11. *Any B-hypergroup is not fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. Regarding to the definition of a B-hypergroup, we have $|x \circ x| = 1$ and $|x \circ a| = 2$ for every $x \neq a$, so $A(x) = 1 + 2 \cdot (n-1) \cdot \frac{1}{2} = n$. Using Proposition 4.1, we know that $q(x) = 2n-1$, which clearly gives that, for any $x \in H, \widetilde{\mu}(x) = \frac{n}{2n-1}$. Hence, two arbitrary elements in a B-hypergroup are fuzzy inseparable. Besides, $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(y \circ a)$, for any $a \in H$ since $\widetilde{\mu}(x) = \widetilde{\mu}(y)$ for two arbitrary elements from H, and $\widetilde{\mu}(x \circ a) = \widetilde{\mu}(\{x, a\}) = \{\widetilde{\mu}(x), \widetilde{\mu}(a)\}$. □

Proposition 4.12. *Let $(H, \circ)$ be a Corsini hypergroup with $|H| \geq 2$. There always exist two elements $x, y \in H$ such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$.*

Proof. We will split the proof in some cases. Using Propositions 4.9 and 4.10 we can eliminate the cases when there exist $x, y \in H$ such that $x \circ x$ and $y \circ y$ are singleton or equal to H. It remains then to consider other three cases.

1. There exists $x \in H$ such that $x \circ x = H$.

2. There exists $x \in H$ such that $x \circ x = x$,

3. The hypergroup doesn't contain any element x such that $x \circ x$ is equal to x or H.

Case 1. Without losing the generality, assume that $H = \{x_1, x_2, \ldots, x_n\}$ and $x_n \circ x_n = H$. This means that any $x_i \in H$ belongs to $x_n \circ x_n$, that implies $x_n \in x_i \circ x_i$, for any $i = 1, 2, \ldots, n$.
Subcase 1.1. If $x_i \circ x_i = \{x_i, x_n\}, i = 1, 2, \ldots, n-1$ and $x_n \circ x_n = H$, then by Proposition 4.5, we know that $\widetilde{\mu}(x_i)$ is the same, for all $i = 1, 2, \ldots, n-1$. This also implies that $\widetilde{\mu}(x_1 \circ x_1) = \widetilde{\mu}(x_2 \circ x_2) = \ldots = \widetilde{\mu}(x_{n-1} \circ x_{n-1})$, which concludes the result.
Subcase 1.2. Extending the previous subcase, that can be considered as a "*base case*", we can analyze now the situation when we add another element x_k, $k \neq n \neq i$, to the hyperproduct $x_i \circ x_i$. This leads to have $x_k \circ x_k = x_i \circ x_i = \{x_k, x_i, x_n\}$, which clearly gives $\widetilde{\mu}(x_i \circ x_i) = \widetilde{\mu}(x_k \circ x_k)$, which proves the proposition. Continuing the process, we can extend now this subcase into two ways:

- by adding another element to a hyperproduct $x \circ x$, with $x \in H \setminus \{x_i, x_k, x_n\}$ and again we obtain the conclusion of the result, or

- by adding a different element x_l to one of the hyperproducts $x_i \circ x_i$ or $x_k \circ x_k$. Suppose that we add it to $x_i \circ x_i$. Thus we get $x_i \circ x_i = \{x_i, x_k, x_l, x_n\}, x_l \circ x_l = \{x_l, x_i, x_n\}, x_k \circ x_k = \{x_k, x_i, x_n\}$, meaning that x_l and x_k are in the same formations, so $\widetilde{\mu}(x_k) = \widetilde{\mu}(x_l)$ and thereby $\widetilde{\mu}(x_k \circ x_k) = \widetilde{\mu}(x_l \circ x_l)$.

Continuing this process by the above described procedure, we will always get two distinct elements such that $\widetilde{\mu}(x \circ x)$=$\widetilde{\mu}(y \circ y)$. The process is finite, since we stop when we get two hyperproducts $x \circ x = H$.

Case 2. There exists $x_i \in H$ such that $x_i \circ x_i = x_i$. First, the "*base case*" is when all the other hyperproducts $x \circ x$, with $x \in H \setminus \{x_i\}$, contain two elements. This is possible only if the cardinality of H is odd. If the cardinality of H is an even number, the "*base case*" is when one hyperproduct $x_j \circ x_j$, with $j \neq i$, has three elements, and all the other hyperproducts $x \circ x$ have exactly two elements. The value $\widetilde{\mu}(x_i)$ of all elements x_i such that $|x_i \circ x_i| = 2$ is the same. Repeating the same procedure as in Case 1, we will always obtain two elements x and y which satisfy the result.

Case 3. There doesn't exist x_i such that $x_i \circ x_i = H$ nor $x_i \circ x_i = x_i$. The "*base cases*" are exactly the same as in the second case and they depend on the parity of the cardinality of H. For example, in the case when cardinality is an even number, we can set hyperproducts as: $x_1 \circ x_1 = x_2 \circ x_2 = \{x_1, x_2\}$, $x_3 \circ x_3 = x_4 \circ x_4 = \{x_3, x_4\}, \ldots, x_{n-1} \circ x_{n-1} = x_n \circ x_n = \{x_{n-1}, x_n\}$. The values of all $\widetilde{\mu}(x_i)$ are the same for all $i \in \{1, 2, \ldots, n\}$, so $\widetilde{\mu}(x_i \circ x_i)$ are also the same for $i \in \{1, 2, \ldots, n\}$. In the case when the cardinality is an odd number, we can form hyperproducts $x_i \circ x_i$ as in the previous case for $i = 2, \ldots, n-1$, but take $x_1 \circ x_1 = \{x_1, x_2, x_n\}, x_n \circ x_n = \{x_n, x_1\}$. This case reduces to the first case, too. Using already mentioned procedure of constructing other Corsini hypergroups, we will always get two elements x, y such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$. □

It is worth noticing that the procedure described above permits us to construct all finite Corsini hypergroups.

Theorem 4.13. *Any Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set* $\widetilde{\mu}$.

Proof. According to Proposition 4.12 we can always find two elements x and y such that $\widetilde{\mu}(x \circ x) = \widetilde{\mu}(y \circ y)$. This implies the fuzzy operational equivalence of these two elements. From here, according to Proposition 4.7, we conclude that they are also fuzzy inseparable. Hence, in any Corsini hypergroup there always exist two elements in the same equivalence class with respect to the fuzzy essential indistinguishability, which gives that the hypergroup is not fuzzy reduced, with respect to the grade fuzzy set $\widetilde{\mu}$. □

Remark 4.14. Do to a manner of construction of Corsini hypergroups, showed in the Proposition 4.12, it is easy to conclude that the infinite Corsini hypergroup is also not fuzzy reduced with respect to the $\widetilde{\mu}$.

Example 4.15. On the set $H = \{1, 2, 3, \ldots, n\}$ let define the hyperoperation $\circ_\rho$ by $x \circ_\rho y = x \circ_\rho x \cup y \circ_\rho y$, where $x \circ_\rho x = \{z \mid x\rho z\}$ and the relation ρ is defined as $x\rho y \iff x \leq y$. Then $(H, \circ_\rho)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Indeed, note that $i \circ n = \{1, 2, 3, \ldots, max\{i, n\}\}$. Since 1 is the smallest element in the set H, then $1 \circ i = i \circ 1 = \{1, 2, \ldots, i\}$, for any $i \in H$. Here, 1 appears in any hyperproduct, so $q(1) = n^2$, and the cardinalities of the sets where 1 appears are: $1, 2, \ldots, n$, respectively. Similarly, $2 \circ i = i \circ 2 = \{1, 2, 3, \ldots, i\}$, and $q(2) = n^2 - 1$, because 2 doesn't appear only in the hyperproduct $1 \circ 1$. The element 2 appears in the sets of cardinalities $2, 3, 4, \ldots, n-1$ respectively. For an arbitrary element k, we can conclude that it doesn't appear in hyperproducts $j \circ i$ and $i \circ j$ where $i, j \leq k$. Cardinalities of the sets where k appears are $k, k+1, \ldots n$, because k appear in every $i \circ j$, where i or j are greater than or equal to k. The set of cardinality n where k appears is every set $i \circ n$, for any $i \leq n$. Using the commutativity we conclude that we have a $2n - 1$ such sets. Similarly, the set of cardinality $n-1$ where k appears is every set $i \circ (n-1), i \leq n-1$ and the number of them is $2(n-1)+1$. Continuing the procedure, we get that the set of cardinality k where k appears is $i \circ k, i \leq k$ and the number of them is $(2k-1)$. Calculating $A(k)$, we get that k appears in $(2k-1) + (2(k+1)-1) + \ldots + (2n-1)$ hyperproducts, which finally gives:

$$\widetilde{\mu}(k) = \frac{\frac{1}{k} \cdot (k+k-1) + \frac{1}{k+1}(k+1+k-1-1) + \ldots + \frac{1}{n}(n+n-1)}{(2k-1) + (2k+1) + (2k+3) + ... + (2n-1)}.$$

By summing and arranging members we get $\widetilde{\mu}(k) = \frac{2(n-k+1) - (\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n})}{(n+k-1)(n-k+1)}$. By simple calculations it can be proved that $\widetilde{\mu}(k+1) \leq \widetilde{\mu}(k)$, hence k and $k+1$ are not fuzzy essentially indistinguishable. From the previous inequality we have $\widetilde{\mu}(1) \geq \widetilde{\mu}(2) \geq \ldots \geq \widetilde{\mu}(n)$ so the equivalence class of any element in H is a singleton. Hence, $(H, \circ)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$.

Remark 4.16. Notice that the previous hypergroup is not a Corsini one, but it satisfies the first two conditions of Definition 2.5.

5 Reducibility of the direct product of Corsini hypergroups

We start this section by stating one known result about the reducibility of the product of hypergroups. After that, we study the fuzzy reducibility of the product of two non-fuzzy reduced hypergroups, that will be used for the examination of the fuzzy reducibility of the direct product of Corsini hypergroups.

Theorem 5.1 ([12])**.** *The hypergroup $(H \times H, \bigotimes)$ is reduced if and only if the hypergroups $(H, \circ_1)$ and $(H, \circ_2)$ are reduced.*

Proposition 5.2 ([13])**.** *If $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are the grade fuzzy sets of H_1 and H_2, and $\widetilde{\mu}$ is the grade fuzzy set of the direct product $H_1 \times H_2$ then $\widetilde{\mu}(x, y) = \widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y), x, y \in H$.*

Proposition 5.3. *Let $(H, \circ_1)$ and $(H, \circ_2)$ be non-fuzzy reduced hypergroups constructed on the support set H with at least two elements. Then the direct product $(H \times H, \circ_1 \times \circ_2)$ is a non-fuzzy reduced hypergroup with respect to the grade fuzzy set $\widetilde{\mu}$.*

Proof. For two elements a and b, we know that $\mu(a \circ b) = \{\mu(x) \mid x \in a \circ b\}$. Since $(H, \circ_1)$ is not fuzzy reduced, assume that x_1, x_2 are two elements such that $x_1 \sim_{f.e} x_2$, i.e. $\widetilde{\mu}_1(x_1 \circ_1 a) = \widetilde{\mu}_1(x_2 \circ_1 a)$, for all $a \in H$. Also, $\widetilde{\mu}_1(x_1)$ and $\widetilde{\mu}_1(x_2)$ appear in the same $\widetilde{\mu}_1(a \circ b), a, b \in H$. Similarly, since $(H, \circ_2)$ is not fuzzy reduced, let y_1 and y_2 be elements in H such that they are fuzzy essential indistinguishable. Our goal is to prove that the ordered pairs (x_1, y_1) and (x_2, y_2) are fuzzy essential indistinguishable. Since $(x_1, y_1) \circ_1 \times \circ_2 (a, b) = (x_1 \circ_1 a, y_1 \circ_2 b)$, it follows that $\widetilde{\mu}((x_1, y_1) \circ_1 \times \circ_2 (a, b)) = \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) | x \in x_1 \circ_1 a, y \in y_1 \circ_2 b\}$. Denote the last set with A and the set $\mu((x_2, y_2) \circ_1 \times \circ_2 (a, b))$ with B. Since $x_1 \sim_{f.o} x_2$, we have $\{\widetilde{\mu}_1(x) \mid x \in x_1 \circ_1 a\} = \{\widetilde{\mu}_1(y) : y \in x_2 \circ_1 a\}$, and $y_1 \sim_{f.o} y_2$ implies $\{\widetilde{\mu}_2(x) \mid x \in y_1 \circ_2 b\} = \{\widetilde{\mu}_2(y) \mid y \in y_2 \circ_2 b\}$, meaning that $A = B$. This proves the fuzzy operational equivalence of the corresponding elements. For the proof of the fuzzy inseparability, let a, c be elements from H such that $\widetilde{\mu}_1(x_1) \in \widetilde{\mu}_1(a \circ_1 c)$. From here, due to the fuzzy inseparability in $(H, \circ_1)$, $\widetilde{\mu}_1(x_2)$ belongs to the same set. On the other side, let b, d be elements from H such that $\widetilde{\mu}_2(y_1) \in \widetilde{\mu}_2(b \circ_2 d)$, from where we conclude that $\widetilde{\mu}_2(y_2) \in \widetilde{\mu}(b \circ_2 d)$. Using the last two implications, we get:

$$\widetilde{\mu}_1(x_1) \cdot \widetilde{\mu}_2(y_1) \in \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) : x \in a \circ_1 c, y \in b \circ_2 d\} =$$

$$\{\widetilde{\mu}(x, y) : x \in a \circ_1 c, y \in b \circ_2 d\} = \widetilde{\mu}(a \circ_1 c, b \circ_2 d)$$

This means that $\widetilde{\mu}(x_1, y_1) \in \widetilde{\mu}(a \circ_1 c, b \circ_2 d)$. The above mentioned implications show that $\widetilde{\mu}(x_2, y_2)$ belongs to the same set. Similarly, one proves the converse implication. Hence, (x_1, y_1) and (x_2, y_2) are fuzzy inseparable and therefore, $(H, \circ_1)$ and $(H, \circ_2)$ are not fuzzy reduced.

□

Proposition 5.4. *The direct product of B-hypergoups is reduced.*

Proof. Since any B-hypergroup is reduced, this is a direct corollary of Theorem 5.1. □

The converse of Proposition 5.3 doesn't hold, as we can see in Examples 5.5 and 5.6.

Example 5.5. Let $(H, \circ_1)$ and $(H, \circ_2)$ be hypergroups, where the hyperoperations " $\circ_1$ " and " $\circ_2$ " are defined by the following tables.

$\circ_1$	a	b	c	d
a	a	a	a,b,c	a,b,d
b	a	a	a,b,c	a,b,d
c	a,b,c	a,b,c	a,b,c	c,d
d	a,b,d	a,b,d	c,d	a,b,d

$\circ_2$	a	b	c	d
a	b	b	a,b,c	a,b,d
b	b	b	a,b,c	a,b,d
c	a,b,c	a,b,c	a,b,c	c,d
d	a,b,d	a,b,d	c,d	a,b,d

Here, we will consider fuzzy reducibility with respect to the grade fuzzy set $\widetilde{\mu}$.

By easy calculations, we get: $\widetilde{\mu}_1(a) = \frac{11}{21}, \widetilde{\mu}_1(b) = \frac{1}{3}, \widetilde{\mu}_1(c) = \frac{8}{21}, \widetilde{\mu}_1(d) = \frac{8}{21}$. We can notice that the only rows which are the same are those corresponding to a and b. This implies $a \sim_o b$, which easily gives $a \sim_{fo} b$, but here, $\widetilde{\mu}(a)$ belongs to $\widetilde{\mu}(a \circ a)$, while $\widetilde{\mu}(b)$ does not belong to it, so $a \nsim_{fi} b$. Hence, $a \nsim_{fe} b$. It is easy to see that except a and b all other pairs of elements are not fuzzy operational equivalent, which, together with $a \nsim_{fe} b$ implies that $\hat{x}_{fe} = \{x\}$, for all $x \in H$. Hence, $(H, \circ_1)$ is fuzzy reduced.

Regarding $(H, \circ_2)$, due to the isomorphism of hypergroups, we get the same values of the elements under the fuzzy grade $\widetilde{\mu}_2$. At the same way as for the previous hypergroup, we can conclude that $(H, \circ_2)$ is fuzzy reduced.

Here, $(a,a) \sim_{f.o} (b,b)$, because $\widetilde{\mu}((a,a)\circ_1 \times \circ_2(m,n)) = \{\widetilde{\mu}_1(x)\cdot\widetilde{\mu}_2(y) \mid x \in a \circ_1 m, y \in a \circ_2 n\} = \{\widetilde{\mu}_1(x) \cdot \widetilde{\mu}_2(y) \mid \widetilde{\mu}_1(x) \in \{\frac{11}{21}, \frac{1}{3}, \frac{8}{21}\}, \widetilde{\mu}_2(y) \in \{\frac{1}{3}, \frac{11}{21}, \frac{8}{21}\}\}$, where $m, n \in \{\ a, b, c, d\}$. This set is equal to $\widetilde{\mu}((b,b)\circ_1 \times \circ_2(m,n))$.

Further more, $\widetilde{\mu}(a,a) = \widetilde{\mu}_1(a) \cdot \widetilde{\mu}_2(a) = \frac{11}{21} \cdot \frac{1}{3} = \widetilde{\mu}(b,b)$, which ensures that $(a,a) \sim_{f.i} (b,b)$. Hence, we got non-fuzzy reduced hypergroup as a direct product of two fuzzy reduced hypergroups.

Example 5.6. Let $(H, \circ_1)$ and $(H, \circ_2)$ be hypergroups, where the hyperoperations " $\circ_1$ " and " $\circ_2$ " are defined by the following tables:

$\circ_1$	a	b	c
a	a,b	a,b	H
b	a,b	a,b	H
c	H	H	c

$\circ_2$	a	b	c
a	a	a	H
b	a	a	H
c	H	H	H

Easy calculations of the fuzzy grade sets $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ show that the first hypergroup $(H, \circ_1)$ is not fuzzy reduced, while $(H, \circ_2)$ is fuzzy reduced with respect to the grade fuzzy set $\widetilde{\mu}$. As in the previous example, it can be shown that $(b,a) \sim_{f.e} (a,a)$, which proves the non-fuzzy reducibility of $(H \times H, \circ_1 \times \circ_2)$.

Proposition 5.7. *The direct product of two Corsini hypergroups is non-fuzzy reduced with respect to the grade fuzzy set* $\widetilde{\mu}$.

Proof. Since an arbitrary Corsini hypergroup is not fuzzy reduced according to Theorem 4.13, using Proposition 5.3 it follows that the direct product of two Corsini hypergroups is not fuzzy reduced. □

Corollary 5.8. *The direct product of a Corsini hypergroup and a total hypergroup is non-fuzzy reduced with respect to the grade fuzzy set* $\widetilde{\mu}$.

Proof. This is a direct consequence of Theorem 2.7. □

6 Conclusions and Open Problems

In this paper, we have investigated different types of Corsini hypergroups with the aim to study their reducibility and fuzzy reducibility with respect to the grade fuzzy set $\widetilde{\mu}$. In the second part of the paper, we have presented some conditions which give the reducibility and fuzzy reducibility of the direct product of hypergroups of Corsini hypergroups. In a future work we will extend our study to the reducibility and fuzzy reducibility of the direct product of arbitrary hypergroups. Besides, it would be interesting to construct hyperrings composed of Corsini's hypergroups and study their reducibility.

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