DOI: 10.2478/auom-2021-0006
An. Şt. Univ. Ovidius Constanţa **Vol. 29(1),2021, 83–92**

Invariance property of a five matrix product involving two generalized inverses

Bo Jiang, Yongge Tian

Abstract

Matrix expressions composed by generalized inverses can generally be written as $f(A_1^-, A_2^-, \ldots, A_k^-)$, where $A_1, A_2, \ldots, A_k$ are a family of given matrices of appropriate sizes, and $(\cdot)^-$ denotes a generalized inverse of matrix. Once such an expression is given, people are primarily interested in its uniqueness (invariance property) with respect to the choice of the generalized inverses. As such an example, this article describes a general method for deriving necessary and sufficient conditions for the matrix equality $A_1A_2^-A_3A_4^-A_5 = A$ to always hold for all generalized inverses A_2^- and A_4^- of A_2 and A_4 through use of the block matrix representation method and the matrix rank method, and discusses some special cases of the equality for different choices of the five matrices.

1 Introduction

Throughout this note, $\mathbb{C}^{m\times n}$ denotes the collection of all $m \times n$ complex matrices, $r(A)$, $\mathscr{R}(A)$, and $\mathscr{N}(A)$ denote the rank, the range, and the null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m, $[\,A,\, B]$ denotes a row block matrix consisting of A and B. We next introduce the definition and notation of generalized inverses of matrix. The

Key Words: Generalized inverse; matrix product; matrix equation; rank formula; invariance
2010 Mathematics Subject Classification: Primary 15A09; 15A24; Secondary 47A05; 47A50.
Received: 01.11.2019
Accepted: 04.05.2020

Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3)\ (AX)^{*} = AX,\quad (4)\ (XA)^{*} = XA. \tag{1}$$

A matrix X is called a $\{i,\ldots,j\}$-generalized inverse of A, denoted by $A^{(i,\ldots,j)}$, if it satisfies the ith,..., jth equations in (1). The collection of all $\{i,\ldots,j\}$-generalized inverses of A is denoted by $A\{i,\ldots,j\}$. A matrix X is called a generalized inverse of A if it satisfies $AXA = A$, and is denoted by A^{-}.

One of the principal problems in the theory of matrix algebra is to study various matrix-valued functions from theoretical and applied point of view, including the establishment of matrix equations and identities that involve unknown matrices. In [6], we presented some standard algebraic procedures for deriving the invariance properties of various multilinear matrix-valued functions (MMVFs). As examples, we recently have concentrated our efforts on understanding the uniqueness (invariance property) of the following MMVFs

$$f(X_1,\ldots,X_k) = (A_1 + B_1X_1C_1)(A_2 + B_2X_2C_2)\cdots(A_k + B_kX_kC_k), \tag{2}$$

$$g(X_1,Y_1,\ldots,X_k,Y_k) = (A_1 + B_1X_1C_1 + D_1Y_1E_1)\cdots(A_k + B_kX_kC_k + D_kY_kE_k), \tag{3}$$

and have constructed necessary and sufficient conditions in block matrix forms for some special cases of $f(X_1,\ldots,X_k) = A$ and $g(X_1, Y_1,\ldots,X_k, Y_k) = A$ to hold respectively for all variable matrices $X_1, \ldots, X_k, Y_1, \ldots, Y_k$, which we denote by $f(X_1,\ldots,X_k) \equiv A$ and $g(X_1, Y_1,\ldots,X_k, Y_k) \equiv A$ associated with (2) and (3), respectively. The results obtained can be used to establish various specified matrix identities that involve multiple variable matrices, including the characterizations of various complicated matrix equalities that involve matrices and their generalized inverses.

Recall that matrix expression that involves generalized inverses can generally be written as $f(A_1^{(i_1,\ldots,j_1)}, A_2^{(i_2,\ldots,j_2)},\ldots,A_k^{(i_k,\ldots,j_k)})$, where $A_1, A_2,\ldots,A_k$ are a family of given matrices of appropriate sizes. The invariance property of this matrix expression can be formulated as the following matrix identity problem $f(A_1^{(i_1,\ldots,j_1)}, A_2^{(i_2,\ldots,j_2)},\ldots,A_k^{(i_k,\ldots,j_k)}) \equiv A$ for the generalized inverses. Especially, people are interested in the uniqueness of matrix products composed by matrices and generalized inverses (cf. [1, 4, 6, 14, 16]). One concrete example of matrix identities that involves a product of five matrices and their generalized inverses is given by

$$A_1A_2^{-}A_3A_4^{-}A_5 \equiv A, \tag{4}$$

where $A_1 \in \mathbb{C}^{m_1\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_2}$, $A_3 \in \mathbb{C}^{m_3\times m_4}$, $A_4 \in \mathbb{C}^{m_5\times m_4}$, and $A_5 \in \mathbb{C}^{m_5\times m_6}$, and $A \in \mathbb{C}^{m_1\times m_6}$ are given matrices. The matrix identity problem in (4) and some of its special cases occur in reverse order law problems for generalized inverses, such as, $(AB)^- = B^-A^-$, $(ABC)^- = (BC)^-B(AB)^-$, etc. Thus the work is indeed meaningful and inclusive. It is easy to figure out that we cannot directly give any simple and explicit necessary and sufficient conditions for (4) to hold under general assumptions. In this note, we shall solve this problem using analytical formulas of generalized inverses, multilinear matrix identities, as well as matrix rank formulas. Some special cases of (4) in applications will also be discussed.

Note from the definitions of generalized inverses of a matrix that they are in fact defined to be (common) solutions of some matrix equations. Thus analytical expressions of generalized inverses of matrices can be written as certain matrix-valued functions with one or more variable matrices. In fact, analytical formulas of generalized inverses of matrices and their functions are important issues and tools in matrix analysis. For instance, the basic formula in the following lemma can be found, e.g., in [2, 3, 9].

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times n}$. Then the general expression of A^- of A can be written as*

$$A^- = A^\dagger + F_AU + VE_A, \tag{5}$$

where $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$, and U, $V \in \mathbb{C}^{n\times m}$ are arbitrary.

In order to simplify matrix expressions and matrix equalities associated with (4), we need to use the following rank formulas and their consequences.

Lemma 1.2 ([8]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then*

$$r[A,\, B] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{6}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{7}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{8}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & D - CA^\dagger B \end{bmatrix}. \tag{9}$$

In particular, the following results hold.

(a) $r[A,\, B] = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow AA^\dagger B = B \Leftrightarrow E_AB = 0$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*) \Leftrightarrow CA^\dagger A = C \Leftrightarrow CF_A = 0$.

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) \Leftrightarrow E_B A F_C = 0.$

(d) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A),\ \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*),$ *and* $CA^\dagger B = D.$

Lemma 1.3 ([10])**.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$r(D - CA^\dagger B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{10}$$

In particular,

$$r(D - CA^\dagger B) = r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \quad if \quad \mathscr{R}(B) \subseteq \mathscr{R}(A), \tag{11}$$

$$r(D - CA^\dagger B) = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A) \quad if \quad \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*). \tag{12}$$

We shall use the following two simple facts:

$$\mathscr{R}(A_1) = \mathscr{R}(A_2) \text{ and } \mathscr{R}(B_1) = \mathscr{R}(B_2) \Rightarrow r[A_1,\, B_1] = r[A_2,\, B_2], \tag{13}$$

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \text{ and } r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{14}$$

and use the following known result to approach the matrix equality problem in (4).

Lemma 1.4 ([6])**.** *The bilinear matrix equation*

$$M_1(A_1 + B_1X_1 + Y_1C_1)M_2(A_2 + B_2X_2 + Y_2C_2)M_3 = M \tag{15}$$

holds for all matrices X_1, X_2, Y_1, *and* Y_2, *namely,* (15) *is a matrix identity, if and only if one of the following six block matrix equalities holds*:

(i) $[M,\, M_1] = 0,$ (ii) $\begin{bmatrix} M \\ M_3 \end{bmatrix} = 0,$ (iii) $\begin{bmatrix} M & 0 \\ 0 & M_2 \end{bmatrix} = 0,$

(iv) $\begin{bmatrix} M & M_1A_1M_2 & M_1B_1 \\ 0 & C_1M_2 & 0 \end{bmatrix} = 0,$ (v) $\begin{bmatrix} M & 0 \\ M_2A_2M_3 & M_2B_2 \\ C_2M_3 & 0 \end{bmatrix} = 0,$

(vi) $\begin{bmatrix} M_1A_1M_2A_2M_3 - M & M_1A_1M_2B_2 & M_1B_1 \\ C_1M_2A_2M_3 & C_1M_2B_2 & 0 \\ C_2M_3 & 0 & 0 \end{bmatrix} = 0.$

2 Main results

To understand the matrix identity in (4), we need first to convert it into the following bilinear matrix equation

$$A_1A_2^-A_3A_4^-A_5 = A_1(A_2^\dagger + F_{A_2}U_1 + U_2E_{A_2})A_3(A_4^\dagger + F_{A_4}V_1 + V_2E_{A_4})A_5 = A, \tag{16}$$

where U_1, $U_2 \in \mathbb{C}^{m_2\times m_3}$ and V_1, $V_2 \in \mathbb{C}^{m_4\times m_5}$ are unknown matrices. Apparently (16) is a special case of (15). Hence, we can deduce necessary and sufficient conditions for (16) to always hold from Lemma 1.4 and a variety of algebraic matrix calculations.

Theorem 2.1. *Let A_i and A be as given in* (4), $i = 1, \dots, 5$. *Then the following four statements are equivalent*:

(a) $A_1A_2^-A_3A_4^-A_5 = A$ *holds for all* A_2^- *and* A_4^-, *i.e.*, $A_1A_2^-A_3A_4^-A_5 \equiv A$.

(b) $A_1A_2^-A_3A_4^-A_5$ *is invariant with respect to the choice of* A_2^- *and* A_4^-, *and* $A = A_1A_2^\dagger A_3A_4^\dagger A_5$.

(c) *One of the following six assertions holds*:

  (i) $A_1 = 0$ *and* $A = 0$.

  (ii) $A_3 = 0$ *and* $A = 0$.

  (iii) $A_5 = 0$ *and* $A = 0$.

  (iv) $A = 0$, $A_1A_2^\dagger A_3 = 0$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*)$, *and* $\mathscr{R}(A_3) \subseteq \mathscr{R}(A_2)$.

  (v) $A = 0$, $A_3A_4^\dagger A_5 = 0$, $\mathscr{R}(A_3^*) \subseteq \mathscr{R}(A_4^*)$, *and* $\mathscr{R}(A_5) \subseteq \mathscr{R}(A_4)$.

  (vi) $A = A_1A_2^\dagger A_3A_4^\dagger A_5$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*)$, $\mathscr{R}(A_5) \subseteq \mathscr{R}(A_4)$, $\mathscr{R}[(A_1A_2^\dagger A_3)^*] \subseteq \mathscr{R}(A_4^*)$, $\mathscr{R}(A_3A_4^\dagger A_5) \subseteq \mathscr{R}(A_2)$, *and* $E_{A_2}A_3F_{A_4} = 0$.

(d) *One of the following six assertions holds*:

  (i) $A_1 = 0$ *and* $A = 0$.

  (ii) $A_3 = 0$ *and* $A = 0$.

  (iii) $A_5 = 0$ *and* $A = 0$.

  (iv) $A = 0$ *and* $r\begin{bmatrix} A_2 & A_3 \\ A_1 & 0 \end{bmatrix} = r(A_2)$.

  (v) $A = 0$ *and* $r\begin{bmatrix} A_4 & A_5 \\ A_3 & 0 \end{bmatrix} = r(A_4)$.

(vi) $A = A_1A_2^{\dagger}A_3A_4^{\dagger}A_5$, $\mathscr{R}([\,0,\,A_1]^*) \subseteq \mathscr{R}\left(\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}\right)^*$, $\mathscr{R}\begin{bmatrix} 0 \\ A_5 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}$, *and* $r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

Proof. The equivalence of (a) and (b) follows from setting $A_2^- = A_2^{\dagger}$ and $A_4^- = A_4^{\dagger}$ in (16).

Applying Lemma 1.4 to (16), we see that (16) always holds for all U_1, U_2, V_1, and V_2 if and only if one of the following six equalities holds

$$[A,\, A_1] = 0, \qquad \begin{bmatrix} A \\ A_5 \end{bmatrix} = 0, \qquad \begin{bmatrix} A & 0 \\ 0 & A_3 \end{bmatrix} = 0, \tag{17}$$

$$\begin{bmatrix} A & A_1A_2^{\dagger}A_3 & A_1F_{A_2} \\ 0 & E_{A_2}A_3 & 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} A & 0 \\ A_3A_4^{\dagger}A_5 & A_3F_{A_4} \\ E_{A_4}A_5 & 0 \end{bmatrix} = 0, \tag{18}$$

$$\begin{bmatrix} A_1A_2^{\dagger}A_3A_4^{\dagger}A_5 - A & A_1A_2^{\dagger}A_3F_{A_4} & A_1F_{A_2} \\ E_{A_2}A_3A_4^{\dagger}A_5 & E_{A_2}A_3F_{A_4} & 0 \\ E_{A_4}A_5 & 0 & 0 \end{bmatrix} = 0. \tag{19}$$

Expanding the three equalities in (18) and (19) and applying Lemma 1.2(a) and (b) lead to the six statements in (c).

The equivalence of (iv) of (c) and (iv) of (d), as well as the equivalence of (v) of (c) and (v) of (d) follow from Lemma 1.2(d). By Lemma 1.2(c),

$$r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4) \Leftrightarrow E_{A_2}A_3F_{A_4} = 0, \tag{20}$$

establishing the equivalence of the last term in (vi) of (c) and the last rank equality in (vi) of (d). By (20),

$$\begin{aligned} r\left(\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}\begin{bmatrix} A_4^* & 0 \\ 0 & I_{m_2} \end{bmatrix}\right) &= r\begin{bmatrix} A_3A_4^* & A_2 \\ A_4A_4^* & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_2 \\ A_4A_4^* & 0 \end{bmatrix} \\ &= r(A_2) + r(A_4) = r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}, \end{aligned} \tag{21}$$

$$\begin{aligned} r\left(\begin{bmatrix} A_3^* & A_4^* \\ A_2^* & 0 \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & I_{m_5} \end{bmatrix}\right) &= r\begin{bmatrix} A_3^*A_2 & A_4^* \\ A_2^*A_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_4^* \\ A_2^*A_2 & 0 \end{bmatrix} \\ &= r(A_2) + r(A_4) = r\begin{bmatrix} A_3^* & A_4^* \\ A_2^* & 0 \end{bmatrix}. \end{aligned} \tag{22}$$

Applying (14) to (21) and (22) yields

$$\mathscr{R}\begin{bmatrix} A_3A_4^* & A_2 \\ A_4A_4^* & 0 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}, \quad \mathscr{R}\begin{bmatrix} A_3^*A_2 & A_4^* \\ A_2^*A_2 & 0 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_3^* & A_4^* \\ A_2^* & 0 \end{bmatrix}. \tag{23}$$

Applying (6) and (7) to the {1,2}-block $A_1A_2^{\dagger}A_3F_{A_4}$ and {2,1}-block $E_{A_2}A_3A_4^{\dagger}A_5$ in (19) and simplifying, respectively, we obtain

$$\begin{aligned}
r(A_1A_2^{\dagger}A_3F_{A_4}) &= r\begin{bmatrix} A_4 \\ A_1A_2^{\dagger}A_3 \end{bmatrix} - r(A_4) = r\left(\begin{bmatrix} 0 \\ A_1 \end{bmatrix} A_2^{\dagger}A_3 + \begin{bmatrix} A_4 \\ 0 \end{bmatrix}\right) - r(A_4) \\
&= r\begin{bmatrix} A_2^*A_2 & A_2^*A_3 \\ 0 & A_4 \\ A_1 & 0 \end{bmatrix} - r(A_2) - r(A_4) \quad (\text{by } \mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*) \text{ and (12)}) \\
&= r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \\ 0 & A_1 \end{bmatrix} - r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix} \quad (\text{by (13), (20), and (23)}), \qquad (24)
\end{aligned}$$

$$\begin{aligned}
r(E_{A_2}A_3A_4^{\dagger}A_5) &= r[\, A_2,\ A_3A_4^{\dagger}A_5] - r(A_2) = r([\, A_2,\ 0] + A_3A_4^{\dagger}[\,0,\ A_5]) - r(A_2) \\
&= r\begin{bmatrix} A_4A_4^* & 0 & A_5 \\ A_3A_4^* & A_2 & 0 \end{bmatrix} - r(A_2) - r(A_4) \quad (\text{by } \mathscr{R}(A_5) \subseteq \mathscr{R}(A_4) \text{ and (11)}) \\
&= r\begin{bmatrix} A_3 & A_2 & 0 \\ A_4 & 0 & A_5 \end{bmatrix} - r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix} \quad (\text{by (13), (20), and (23)}). \qquad (25)
\end{aligned}$$

Setting both sides of (24) and (25) equal to zero and applying Lemma 1.2(a) and (b) lead to the second and third range inclusions in (vi) of (d), respectively. □

Various consequences can be derived from Theorem 2.1 on invariance properties of quintuple matrix products involving two generalized inverses. In particular, the rank and range formulas can be simplified further when the five matrices are given in certain specified forms. We next give some direct consequences of Theorem 2.1.

Corollary 2.1. *Let $A_1 \in \mathbb{C}^{m_1\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_2}$, $A_3 \in \mathbb{C}^{m_4\times m_3}$, $A_4 \in \mathbb{C}^{m_4\times m_5}$, and $A \in \mathbb{C}^{m_1\times m_5}$. Then the following four statements are equivalent*:

(a) $A_1A_2^-A_3^-A_4 \equiv A$.

(b) *$A_1A_2^-A_3^-A_4$ is invariant with respect to the choice of A_2^- and A_3^- and $A = A_1A_2^{\dagger}A_3^{\dagger}A_4$.*

(c) *One of the following three assertions holds*:

  (i) $A_1 = 0$ *and* $A = 0$.

  (ii) $A_4 = 0$ *and* $A = 0$.

  (iii) $A = A_1A_2^{\dagger}A_3^{\dagger}A_4$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*)$, $\mathscr{R}(A_4) \subseteq \mathscr{R}(A_3)$, $\mathscr{R}[(A_1A_2^{\dagger})^*] \subseteq \mathscr{R}(A_3^*)$, $\mathscr{R}(A_3^{\dagger}A_4) \subseteq \mathscr{R}(A_2)$, *and* $E_{A_2}F_{A_3} = 0$.

(d) *One of the following three assertions holds*:

(i) $A_1 = 0$ *and* $A = 0$.

(ii) $A_4 = 0$ *and* $A = 0$.

(iii) $A = A_1 A_2^{\dagger} A_3^{\dagger} A_4$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}[(A_3 A_2)^*]$, $\mathscr{R}(A_4) \subseteq \mathscr{R}(A_3 A_2)$, *and* $\mathscr{N}(A_3) \subseteq \mathscr{R}(A_2)$.

The equivalence of (a) and (c) in Corollary 2.1 was established in [1, Theorem]; the equivalence of (b) and (c) in Corollary 2.1 was established in [7, Theorem 5].

Corollary 2.2. *Let* $A_1 \in \mathbb{C}^{m_1 \times m_2}$, $A_2 \in \mathbb{C}^{m_1 \times m_3}$, *and* $A_3 \in \mathbb{C}^{m_4 \times m_3}$ *be three nonzero matrices. Then* $A_1^- A_2 A_3^-$ *is invariant with respect to the choice of* A_1^- *and* A_3^- *if and only if* $r(A_1) = m_2$, $r(A_3) = m_4$, $\mathscr{R}(A_2) \subseteq \mathscr{R}(A_1)$, *and* $\mathscr{R}(A_2^*) \subseteq \mathscr{R}(A_3^*)$.

Two well-known mixed reverse-order laws for the matrix products AB and ABC are given by

$$(AB)^- = B^- A^-, \quad (ABC)^- = (BC)^- B (AB)^-. \tag{26}$$

Some special forms of (26) were approached in [5, 10, 13, 15]. Concerning the invariance properties of the products ABB^-A^-AB, $(BC)^-B(AB)^-$ and $ABC(BC)^-B(AB)^-ABC$, we are able to derive from Theorem 2.1 the following consequences.

Corollary 2.3 ([6])**.** *Let* $A \in \mathbb{C}^{m \times n}$ *and* $B \in \mathbb{C}^{n \times p}$. *Then* $ABB^-A^-AB \equiv AB \Leftrightarrow ABB^-A^-AB \equiv ABB^{\dagger}A^{\dagger}AB \Leftrightarrow AB = 0$ *or* $r(AB) = r(A) + r(B) - n$.

Corollary 2.4. *Let* A, B, *and* C *be three nonzero matrices of the appropriate sizes. Then the* AA^-BC^-C *is invariant with respect to the choice of* A^- *and* C^- *if and only if both* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(B^*) \subseteq \mathscr{R}(C^*)$.

Corollary 2.5. *Let* $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$, *and* $C \in \mathbb{C}^{p \times q}$ *be three nonzero matrices. Then the following two results hold.*

(a) $(BC)^-B(AB)^- \equiv (BC)^{\dagger}B(AB)^{\dagger} \Leftrightarrow r(ABC) = r(B) = m = q$.

(b) $ABC(BC)^-B(AB)^-ABC \equiv ABC(BC)^{\dagger}B(AB)^{\dagger}ABC$
$\Leftrightarrow ABC(BC)^-B(AB)^-ABC \equiv ABC$
$\Leftrightarrow ABC = 0$ *or* $r(ABC) = r(AB) + r(BC) - r(B)$.

We have characterized a matrix product identity that involve the mixed products of five matrices and their generalized inverses using the block matrix representation method and the matrix rank method, and have featured several examples that shed new perspectives on the invariance problems of matrix products involving generalized inverses. In addition, it would be of interest to approach the matrix product equalities

$$A_1A_2^{(i,\dots,j)}A_3A_4^{(k,\dots,l)}A_5 = A$$

for different generalized inverses of $A_2^{(i,\dots,j)}$ and $A_4^{(k,\dots,l)}$, as well as, other matrix equalities

$$\begin{aligned}
&A^{(i,\dots,j)} = B^{(k,\dots,l)}, \quad A^{(i,\dots,j)} = B^{(k,\dots,l)}C^{(s,\dots,t)},\\
&A^{(i,\dots,j)} = B^{(k,\dots,l)} + C^{(s,\dots,t)},\\
&A_1A_2^{(i_2,\dots,j_2)}\cdots A_{p-1}A_p^{(i_p,\dots,j_p)}A_{p+1} = A,\\
&A(A_1^{(i_1,\dots,j_1)} + A_2^{(i_2,\dots,j_2)} + \cdots + A_p^{(i_p,\dots,j_p)})A = A,
\end{aligned}$$

(cf. [11,12]). Also recall that generalized inverses of elements can symbolically be defined in many other algebraic structures. Thus it would be of interest to consider invariance properties of algebraic expressions involving generalized inverses of elements in other algebraic structures by means of analogous methods developed in [6] and this note.

Acknowledgments: The authors are grateful to the reviewer for helpful comments and suggestions which helped to improve the presentation of the results in the note. This work is supported in part by the Shandong Provincial Natural Science Foundation #ZR2019MA065.

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Bo Jiang,
Shandong Institute of Business and Technology,
Yantai, China.
Email: jiangbo@email.cufe.edu.cn

Yongge Tian,
Shanghai Business School,
Shanghai, China.
Email: yongge.tian@gmail.com