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Stochastic orders for a multivariate Pareto distribution

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Abstract

In this article we give some theoretical results for equivalence between different stochastic orders of some kind multivariate Pareto distribution family. Weak multivariate orders are equivalent or imply different stochastic orders between extremal statistics order of two random variables sequences. The random variables in this article are not neccesary independent.

1 Introduction

Pareto distributions are popular models in finance, economics and related areas. Also, they are used to model random variables like risk, prices or income. Many generalized models was publised and the multivariate case is very interesting. Thus, Kotz et al. (2000) offered a continous multivariate distributions course. Weak stochastic orders (or weak hazard rate order) are equivalent (or imply) the stochastic (or hazard rate) order between extremal order statistics. Shaked and Shantikumar (2007) and Zbaganu (2004) offered results and

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a course in this field. In the multivariate case appear new stochastic orders inequalities with a weak form, due to $P(X \leq x)$ and $P(X \geq x, X \neq x)$. Catana (2016) proves some results which show us important difference between univariate distribution and survial functions and multivariate distribution and survial functions. Stochastic comparisons of two random vectors have been extensively studied and discussed in the literature. Recently, Balakrishnan et al. (2016) and Misra and Naqvi (2018) analized different multivariate stochastic orders of some family distributions.

In nature parallel and series systems are very common phenomena. Parallel and series system are 1-out-of-n and n-out-of-n system, respectively. A k-out-of-n systems is a system which functions if and only if at least k out of its n components function. Stochastic comparison of system lifetimes has always been a relevant topic in reliability optimization and life testing experiments. Several papers have dealt with comparisons among systems (largely on the parallel and the series) with heterogeneous independent components following a certain probability distribution with unbounded/bounded support, such as exponential, gamma, Weibull, generalized exponential, generalized Weibull, beta, Kumaraswamy, or log-Lindley. Ordering of the smallest Pareto order statistics has practical appeal. Balakrishnan et. al. (1998) offered an order statistics course. Samaniego (2007) also offered a system signatures course.

The multivariate Pareto distribution which we work in this article was proposed by Mardia (1962) with the joint density function

$$f_X(x_1, x_2, ..., x_n) = a(a+1)...(a+n-1) \cdot \left(\prod_{i=1}^{n} b_i \right)^{-1} \cdot$$
$$\left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1 \right)^{-(a+n)} \cdot 1_{(b_1,\infty)\times ... \times(b_n,\infty)} \left(x_1, x_2, ..., x_n\right),$$

$x_i > b_i > 0$, $a > 0$.

Arnold (1983) observed that the survial function of this distribution is

$$P\left(X \geq x\right) = \left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1 \right)^{-a}, \; x_i > b_i > 0, \; a > 0.$$

By the survial function continuity of this distribution it results $P(X \geq x, X \neq x) = \left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1 \right)^{-a}$.

Also, Arnold and Pourahmadi (1988) and Wesolowski and Ahsanullah (1995) presented some results for this distributions when $(b_1, ..., b_n) = (1, ..., 1)$.

Preda et. al. (2016) offered a new distributions family and studied its properties.

Catana (2019) analized the weak hazard rate function of a bivariate uniform distributions family. Bancescu (2018) presented the likelihood order of some classes of statistical distributions and Nadaralajah et al. (2017) analized different stochastic orders of smallest order statistics from some Pareto distributions.

We present the structure of this article. In the section 2 there are presented the preliminaries. In the section 3 we give results for different stochastic orders between multivariate Pareto distributions. In the section 4 we prove some parameters conditions imply stochastic and hazard rate orders of smallest Pareto statistics. In this case, the random variables are not neccesary independent. In the last section we discuss the conclusions.

2 Preliminaries

We consider $(\Omega, \mathcal{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}^n$ be a random vector with $n \geq 2$. For a random vector X we consider $\mu_X(B) = P(X \in B)$ be its distribution on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $F_X(x) = P(X \leq x)$ its distribution function and $F_X^*(x) = P(X \geq x, X \neq x)$.

If $X : \Omega \to \mathbb{R}$ is a random variable, we consider $\overline{F}_X(x) = 1 - F_X(x)$.

If X has density, we denote f_X its density function.

For $n \geq 1$, if F_X^* (respectively $\overline{F}_X$) is differentiable, we define the hazard rate function $r : Supp(F_X^*) \to \mathbb{R}^n$, $r(x) = \left(-\frac{\partial}{\partial x_i}\left(\ln F_X^*(x)\right)\right)_{i=\overline{1,n}}$, where for a function $g : \mathbb{R}^n \to \mathbb{R}$, $Supp(g) = \{x \in \mathbb{R}^n : g(x) \neq 0\}$.

We denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, with $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$, we denote $x_{(i)}$ is the i-th ordered number.

Definition. *(Shaked et al., 2007) Let $x, y \in \mathbb{R}^n$. We say x is smaller than y (and denote $x \leq y$) if $x_i \leq y_i$ $\forall i \in \{1, 2, ..., n\}$.*

Definition. *(Shaked et al., 2007) Let $x, y \in \mathbb{R}^n$. We denote* $\min(x, y) = (\min(x_1, y_1), ..., \min(x_n, y_n))$ and $\max(x, y) = (\max(x_1, y_1), ..., \max(x_n, y_n))$.

Also, for $x, y \in \mathbb{R}^n$, we denote $x < y$ if $x_i < y_i$ $\forall i \in \{1, 2, ..., n\}$.

Definition. *We say that the positive random variable X is distributed univariate Pareto(and denote $X \sim P(a, b)$, where $a, b \in \mathbb{R}_+$) if*

$$f_X(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{-(a+1)} \cdot 1_{(b,\infty)}(x).$$

For $x > b$

$$\overline{F}_X(x) = \left(\frac{x}{b}\right)^{-a}.$$

Definition. *We say that the positive n-dimensional random vector X is distributed multivariate Pareto (and denote $X \sim MP(a,b)$, where $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+^n$) if*

$$f_X(x_1, x_2, ..., x_n) = a(a+1)...(a+n-1) \cdot \left(\prod_{i=1}^{n} b_i\right)^{-1} \cdot$$
$$\left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1\right)^{-(a+n)} \cdot 1_{(b_1,\infty)\times...\times(b_n,\infty)}(x_1, x_2, ..., x_n)$$

For $x_1 > b_1, ..., x_n > b_n$

$$F_X^*(x_1, ..., x_n) = \left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1\right)^{-a}.$$

Remark. *If $X \sim MP(a,b)$ then $X_i \sim P(a, b_i)$.*

For completeness we present definition for different multivariate stochastic orders and important results.

Definition. *(Shaked et al., 2007) Let X, $Y : \Omega \to \mathbb{R}$ random variables. We say that X is said to be smaller than Y in the*

(i) usual stochastic order (written as $X \prec_{st} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ $\forall x \in \mathbb{R}$;

(ii) hazard rate order (written as $X \prec_{hr} Y$) if $r_X(x) \geq r_Y(x)$ $\forall x \in Supp(\overline{F}_X) \cap Supp(\overline{F}_Y)$.

Definition. *(Shaked et al., 2007) We say $C \subset \mathbb{R}^n$ is increasing if $\forall x \in C$ $\forall y \in \mathbb{R}^n$ then $x \leq y \Rightarrow y \in C$.*

Definition. *(Shaked et al., 2007) Let X, $Y : \Omega \to \mathbb{R}^d$ be two random variables, $d \geq 2$. We say that X is said to be smaller than Y in the*

(i) usual stochastic order (written as $X \prec_{st} Y$) if $P(X \in C) \leq P(Y \in C)$, for all increasing set $C \subset \mathbb{R}^d$;

(ii) weak stochastic order (written as $X \prec_{wst} Y$) if $F_X^*(x) \leq F_Y^*(x)$, for all $x \in \mathbb{R}^n$;

(iii) dual weak stochastic order (written as $X \prec_{dwst} Y$) if $F_X(x) \geq F_Y(x)$, for all $x \in \mathbb{R}^n$;

(iv) weak hazard rate order (written as $X \prec_{whr} Y$) if $r_X(x) \geq r_Y(x)$ $\forall x \in Supp(F_X^*) \cap Supp(F_Y^*)$;

(v) hazard rate order (written as $X \prec_{hr} Y$) if
$F_X^*(x) \cdot F_Y^*(y) \leq F_X^*(\min(x,y)) \cdot F_Y^*(\max(x,y))$ $\forall x, y \in \mathbb{R}^n$;

(vi) likelihood ratio order (written as $X \prec_{lr} Y$) if
$f_X(x) \cdot f_Y(y) \leq f_X(\min(x,y)) \cdot f_Y(\max(x,y))$ $\forall x, y \in \mathbb{R}^n$.

Theorem 2.1. *(Shaked et al., 2007, Theorem 6.G.15., p. 315) Let $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ be two nonnegative random vectors. Then* *(a)* $X \prec_{wst} Y$ *if, and only if,* $\min(a_1X_1,...,a_nX_n) \prec_{st} \min(a_1Y_1,...,a_nY_n)$ *whenever* $a_i > 0$, $i = 1, 2, ..., n$.

(b) $X \prec_{dwst} Y$ *if, and only if,* $\max(a_1X_1,...,a_nX_n) \prec_{st} \max(a_1Y_1,...,a_nY_n)$ *whenever* $a_i > 0$, $i = 1, 2, ..., n$.

Theorem 2.2. *(Shaked et al., 2007, Theorem 6.D.7., p. 294) Let* $X = (X_1,...,X_n)$ *and* $Y = (Y_1,...,Y_n)$ *be two nonnegative random vectors. If* $X \prec_{whr} Y$, *then* $\min(a_1X_1,...,a_nX_n) \prec_{hr} \min(a_1Y_1,...,a_nY_n)$ *whenever* $a_i > 0$, $i = 1, 2, ..., n$.

It is well known (Shaked et al., 2007):

$$\prec_{st} \Rightarrow \prec_{wst}, \prec_{dwst},$$
$$\prec_{hr} \Rightarrow \prec_{whr} \Rightarrow \prec_{wst}$$

and

$$(X_1,...,X_n) \prec_{st} (\prec_{wst}, \prec_{dwst})(Y_1,...,Y_n) \Rightarrow X_i \prec_{st} Y_i,$$
$$(X_1,...,X_n) \prec_{whr} (Y_1,...,Y_n) \Rightarrow X_i \prec_{hr} Y_i.$$

Theorem 2.3. *(Shaked et al., 2007, Theorem 6.E.6., p. 300) Let* $X = (X_1,...,X_n)$ *and* $Y = (Y_1,...,Y_n)$ *be two random vectors. If* $X \prec_{lr} Y$ *then* $X \prec_{hr} Y$.

Theorem 2.4. *(Shaked et al., 2007, Theorem 6.E.8., p. 301) Let* $X = (X_1,...,X_n)$ *and* $Y = (Y_1,...,Y_n)$ *be two random vectors. If* $X \prec_{lr} Y$ *then* $X \prec_{st} Y$.

Definition. *A function* $K : \mathbb{R}^n \to [0, \infty)$ *is said to be multivariate totally positive of order 2(*MTP_2*) if* $K(x) \cdot K(y) \leq K(\min(x, y)) \cdot K(\max(x, y))$ $\forall x, y \in \mathbb{R}^n$.

Theorem 2.5. *(Shaked et al., 2007, Theorem 6.D.1., p. 290) Let* $X = (X_1,...,X_n)$ *and* $Y = (Y_1,...,Y_n)$ *be two random vectors with common support* S *which is a lattice. If* F_X^* *or* F_Y^* *is* MTP_2, *then* $X \prec_{whr} Y \Rightarrow X \prec_{hr} Y$.

Remark. *For* $X = (X_1,...,X_n)$ *and* $Y = (Y_1,...,Y_n)$ *be two random vectors with common support* S *which is a lattice and* $f_X(x) \cdot f_Y(y) \leq f_X(y) \cdot f_Y(x)$ $\forall x, y \in \mathbb{R}^n$ $x \geq y$. *If* f_X *or* f_Y *is* MTP_2, *then* $X \prec_{lr} Y$.

(It is a same argument as the proof of theorem 2.5.)

3 Main results

Lemmas 3.1 and 3.2 give equvalence between usual stochastic order, hazard rate order and distributions parameters of univariate Pareto distributions.

Theorems 3.3 and 3.4 give implications between weak stochastic order and parameters of multivariate Pareto distributions.

Lemma 3.1. *Let* $X \sim P(a,b)$ *and* $Y \sim P(c,d)$*. Then* $X \prec_{st} Y \Leftrightarrow$*one of the conditions is hold:*

(i) $a = c$ *and* $b \leq d$*;*

(ii) $a > c$, $b \leq d$ *and* $d \geq \left(\frac{b^a}{d^c}\right)^{\frac{1}{a-c}}$.

Proof. Let us prove " $\Rightarrow$ " implication.

$\overline{F}_X(x) \leq \overline{F}_Y(x)\ \forall x > \max(b,d) \Leftrightarrow x^{c-a} \leq \frac{d^c}{b^a}\ \forall x > \max(b,d)$. Let $x \to \infty$ and it results $a \geq c$.

If $d < b$, then let $\varepsilon > 0$ such that $d + \varepsilon < b$.

$1 = \overline{F}_X(d+\varepsilon) \leq \overline{F}_Y(d+\varepsilon) = \left(\frac{d+\varepsilon}{d}\right)^{-c} < 1$, contradiction. Then $b \leq d$.

If $a = c$ then $\overline{F}_X(x) \leq \overline{F}_Y(x)\ \forall x > d \Leftrightarrow \frac{x}{b} \geq \frac{x}{d}\ \forall x > d \Leftrightarrow b \leq d$, which is true.

If $a > c$ then $x^{c-a} \leq \frac{d^c}{b^a}\ \forall x > d \Leftrightarrow x \geq \left(\frac{b^a}{d^c}\right)^{\frac{1}{a-c}}\ \forall x > d \Leftrightarrow d \geq \left(\frac{b^a}{d^c}\right)^{\frac{1}{a-c}}$.

Let us prove " $\Leftarrow$ " implication.

If $x \leq b$ then $\overline{F}_X(x) = \overline{F}_Y(x) = 1$.

If $x \in (b;d]$ then $\overline{F}_X(x) = \left(\frac{x}{b}\right)^{-a} \leq 1 = \overline{F}_Y(x)$.

If $x > d$ then $x > d \geq \left(\frac{b^a}{d^c}\right)^{\frac{1}{a-c}} \Leftrightarrow \left(\frac{x}{b}\right)^{-a} \leq \left(\frac{x}{d}\right)^{-c} \Leftrightarrow \overline{F}_X(x) \leq \overline{F}_Y(x)$. □

Lemma 3.2. *Let* $X \sim P(a,b)$ *and* $Y \sim P(c,d)$*. Then* $X \prec_{hr} Y \Leftrightarrow a \geq c$*.*

Proof. $r_X(x) = \frac{a}{x}$ and $r_Y(x) = \frac{c}{x}$, $x > 0$.

Then $X \prec_{hr} Y \Leftrightarrow r_X(x) \geq r_Y(x)\ \forall x > 0 \Leftrightarrow \frac{a}{x} \geq \frac{c}{x}\ \forall x > 0 \Leftrightarrow a \geq c$. □

Theorem 3.3. *Let* $X \sim MP(a,b)$ *and* $Y \sim MP(c,d)$*. If* $X \prec_{wst} Y$ *then one of the conditions is hold:*

(i) $a = c$ *and* $b_i \leq d_i\ \forall i \in \{1,...,n\}$*;*

(ii) $a > c$, $b_i \leq d_i$ *and* $d_i \geq \left(\frac{b_i^a}{d_i^c}\right)^{\frac{1}{a-c}}\ \forall i \in \{1,...,n\}$.

Proof. $X \prec_{wst} Y \Rightarrow X_i \prec_{st} Y_i \Rightarrow$one of the conditions is hold:

(i) $a = c$ and $b_i \leq d_i$;

(ii) $a > c$, $b_i \leq d_i$ and $d_i \geq \left(\frac{b_i^a}{d_i^c}\right)^{\frac{1}{a-c}}$. □

Theorem 3.4. *Let* $X \sim MP(a,b)$ *and* $Y \sim MP(c,d)$*. If* $a \geq c$ *and* $b \leq d$ *then* $X \prec_{wst} Y$*.*

Proof. Let $x \in \mathbb{R}^n, x > d$. Then

$$\left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1\right)^{-a} \leq \left(\sum_{i=1}^{n} \frac{x_i}{d_i} - n + 1\right)^{-a} \leq \left(\sum_{i=1}^{n} \frac{x_i}{d_i} - n + 1\right)^{-c}.$$

Thus $F_X^*(x_1,...,x_n) \leq F_Y^*(x_1,...,x_n)$.

If $x \ngtr d$ then

$$F_X^*(x_1,...,x_n) = \left(\sum_{i=1}^{n}\left(\frac{x_i}{b_i}-1\right)\cdot 1_{(b_i,\infty)}(x_i)+1\right)^{-a}$$

and

$$F_Y^*(x_1,...,x_n) = \left(\sum_{i=1}^{n}\left(\frac{x_i}{d_i}-1\right)\cdot 1_{(d_i,\infty)}(x_i)+1\right)^{-c}$$

But

$$\left(\frac{x_i}{b_i}-1\right)\cdot 1_{(b_i,\infty)}(x_i) \geq \left(\frac{x_i}{d_i}-1\right)\cdot 1_{(d_i,\infty)}(x_i) \Rightarrow$$

$$\sum_{i=1}^{n}\left(\frac{x_i}{b_i}-1\right)\cdot 1_{(b_i,\infty)}(x_i)+1 \geq \sum_{i=1}^{n}\left(\frac{x_i}{d_i}-1\right)\cdot 1_{(d_i,\infty)}(x_i)+1 \Rightarrow$$

$$\left(\sum_{i=1}^{n}\left(\frac{x_i}{b_i}-1\right)\cdot 1_{(b_i,\infty)}(x_i)+1\right)^{-a} \leq \left(\sum_{i=1}^{n}\left(\frac{x_i}{d_i}-1\right)\cdot 1_{(d_i,\infty)}(x_i)+1\right)^{-a} \leq$$

$$\left(\sum_{i=1}^{n}\left(\frac{x_i}{d_i}-1\right)\cdot 1_{(d_i,\infty)}(x_i)+1\right)^{-c}.$$

Thus $F_X^*(x_1,...,x_n) \leq F_Y^*(x_1,...,x_n)$. □

Theorem 3.5 gives equivalence between weak hazard rate order and parameters.

Theorem 3.5. *Let* $X \sim MP(a,b)$ *and* $Y \sim MP(c,d)$. *Then* $X \prec_{whr} Y \Leftrightarrow$

$b \leq d$, $\frac{b_i}{ab_j} \leq \frac{d_i}{cd_j}$, $d_{(n)} \geq \frac{(n-1)\left(\frac{b_i}{a}-\frac{d_i}{c}\right)}{\sum\limits_{j=1}^{n}\frac{d_i}{cd_j}-\sum\limits_{j=1}^{n}\frac{b_i}{ab_j}}$,

$\sum\limits_{j\in I\setminus\{i\}}^{n}\frac{b_i}{a} + \sum\limits_{j\in\{1,2,...,n\}\setminus(I\cup\{i\})}^{n}\left(\frac{b_i}{a}-\frac{d_i}{c}\right) \geq$

$\sum\limits_{j\in I\setminus\{i\}}^{n}\frac{b_i}{ab_j}d_j + \sum\limits_{j\in\{1,2,...,n\}\setminus(I\cup\{i\})}^{n}\left(\frac{b_i}{ab_j}-\frac{d_i}{cd_j}\right)b_j$

$\forall i,j \in \{1,...,n\}, I \subset \{1,2,...,n\}$.

Proof. For $x_1 > b_1, ..., x_n > b_n$ it results

$$-\ln F_X^*(x_1,...,x_n) = a\ln\left(\sum_{i=1}^{n}\frac{x_i}{b_i}-n+1\right)$$

and

$$(r_X)_i\,(x_1,...,x_n) = \frac{a}{b_i} \cdot \frac{1}{\sum\limits_{j=1}^{n} \frac{x_j}{b_j} - n + 1}.$$

For $x \in \mathbb{R}_+^n$ it results

$$(r_X)_i\,(x_1,...,x_n) = \frac{a}{x_i + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \frac{b_i}{b_j}\,(x_j - b_j) \cdot 1_{(b_j,\infty)}\,(x_j)}$$

and

$$(r_Y)_i\,(x_1,...,x_n) = \frac{c}{x_i + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \frac{d_i}{d_j}\,(x_j - d_j) \cdot 1_{(d_j,\infty)}\,(x_j)}.$$

Thus

$$X \prec_{whr} Y \Leftrightarrow (r_X)_i\,(x_1,...,x_n) \geq (r_Y)_i\,(x_1,...,x_n)\ \ \forall x \in \mathbb{R}_+^n \Leftrightarrow$$

$$\frac{a}{x_i + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \frac{b_i}{b_j}\,(x_j - b_j) \cdot 1_{(b_j,\infty)}\,(x_j)} \geq \frac{c}{x_i + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \frac{d_i}{d_j}\,(x_j - d_j) \cdot 1_{(d_j,\infty)}\,(x_j)}$$

$\forall x \in \mathbb{R}_+^n \Leftrightarrow$

$$\left(\frac{1}{a} - \frac{1}{c}\right) x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{b_i}{ab_j}\,(x_j - b_j) \cdot 1_{(b_j,\infty)}\,(x_j) - \frac{d_i}{cd_j}\,(x_j - d_j) \cdot 1_{(d_j,\infty)}\,(x_j)\right] \leq 0$$

$\forall x \in \mathbb{R}_+^n$.
Let us prove " $\Rightarrow$ " implication.
$X \prec_{whr} Y \Rightarrow X \prec_{wst} Y \Rightarrow a \geq c$ and $b \leq d$.
For $x > d$

$$\sum_{j=1}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j \leq (n-1) \left(\frac{b_i}{a} - \frac{d_i}{c}\right)$$

Let $x_j \to \infty$, $j \in \{1, ..., n\}$ and it results $\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \le 0 \Leftrightarrow \frac{b_i}{ab_j} \le \frac{d_i}{cd_j}$.
For $x_1 = x_2 = ... = x_n = x$ it results

$$\left(\sum_{j=1}^{n} \frac{b_i}{ab_j} - \sum_{j=1}^{n} \frac{d_i}{cd_j} \right) x \le (n-1) \left(\frac{b_i}{a} - \frac{d_i}{c} \right) \forall x \ge d_{(n)} \Leftrightarrow$$

$$d_{(n)} \ge \frac{(n-1)\left(\frac{b_i}{a} - \frac{d_i}{c}\right)}{\sum\limits_{j=1}^{n} \frac{d_i}{cd_j} - \sum\limits_{j=1}^{n} \frac{b_i}{ab_j}}.$$

For $x \not\geq d$ let $I = \{i : b_i < x_i \le d_i\}$ and $J = \{i : x_i > d_i\}$. Then

$$\left(\frac{1}{a} - \frac{1}{c} \right) x_i + \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} (x_j - b_j) + \sum_{j \in J \setminus \{i\}}^{n} \left[\frac{b_i}{ab_j} (x_j - b_j) - \frac{d_i}{cd_j} (x_j - d_j) \right] \le 0$$

$\forall x \not\geq d \Leftrightarrow$

$$\left(\frac{1}{a} - \frac{1}{c} \right) x_i + \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} x_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \right) x_j \le$$

$$\sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a} + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{a} - \frac{d_i}{c} \right) \ \forall x \not\geq d \Leftrightarrow$$

$$\sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a} + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{a} - \frac{d_i}{c} \right) \ge$$

$$\max_{\substack{x \in \mathbb{R}_+^n \\ x \not\geq d}} \left[\left(\frac{1}{a} - \frac{1}{c} \right) x_i + \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} x_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \right) x_j \right] =$$

$$\sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} d_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \right) b_j$$

Thus

$$\sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a} + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{a} - \frac{d_i}{c} \right) \ge \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} d_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \right) b_j.$$

Let us prove " $\Leftarrow$ " implication.
Let $i \in \{1, ..., n\}$, $x \in \mathbb{R}^n, x > d$.
$\frac{b_i}{ab_j} - \frac{d_i}{cd_j} \leq 0$ and $x_{(1)} \leq x_j$. Then
$\left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j \leq \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_{(1)}$. It results

$$\sum_{j=1}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j \leq \sum_{j=1}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_{(1)} =$$

$$\left(\sum_{j=1}^{n} \frac{b_i}{ab_j} - \sum_{j=1}^{n} \frac{d_i}{cd_j}\right) \cdot x_{(1)} \leq (n-1) \left(\frac{b_i}{a} - \frac{d_i}{c}\right).$$

Then $\sum\limits_{j=1}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j \leq (n-1) \left(\frac{b_i}{a} - \frac{d_i}{c}\right)$.
Thus $(r_X)_i (x_1, ..., x_n) \geq (r_Y)_i (x_1, ..., x_n)$.
If $x \not> d$ then

$$\left(\frac{1}{a} - \frac{1}{c}\right) x_i + \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} x_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j \leq$$

$$\max_{\substack{x \in \mathbb{R}_+^n \\ x \not> d}} \left[\left(\frac{1}{a} - \frac{1}{c}\right) x_i + \sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} x_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) x_j\right] \leq$$

$$\sum_{j \in I \setminus \{i\}}^{n} \frac{b_i}{ab_j} d_j + \sum_{j \in J \setminus \{i\}}^{n} \left(\frac{b_i}{ab_j} - \frac{d_i}{cd_j}\right) b_j.$$

Thus $(r_X)_i (x_1, ..., x_n) \geq (r_Y)_i (x_1, ..., x_n)$. $\square$

Remark. In this article $\sum\limits_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a} = \frac{b_i}{a} \cdot card(I \setminus \{i\})$ and
$\sum\limits_{j \in \{1,2,...,n\} \setminus (I \cup \{i\})}^{n} \left(\frac{b_i}{a} - \frac{d_i}{c}\right) = \left(\frac{b_i}{a} - \frac{d_i}{c}\right) \cdot card(\{1, 2, ..., n\} \setminus (I \cup \{i\}))$.
Theorem 3.6 shows that F_X^* and f_X are MTP$_2$.

Theorem 3.6. *Let $X \sim MP(a, b)$. Then F_X^* and f_X are MTP$_2$.*

Proof. It is obvoius

$$F_X^*(x) F_X^*(y) \leq F_X^*(\min(x, y)) F_X^*(\max(x, y)) \; \forall x, y > b \Leftrightarrow$$

$$\left(\sum_{i=1}^{n} \frac{x_i}{b_i} - n + 1\right) \left(\sum_{i=1}^{n} \frac{y_i}{b_i} - n + 1\right) \geq$$

$$\left(\sum_{i=1}^{n}\frac{\min(x_i,y_i)}{b_i}-n+1\right)\left(\sum_{i=1}^{n}\frac{\max(x_i,y_i)}{b_i}-n+1\right)\ \forall x,y>b$$

(it is similar the case $x \ngtr b$ or $y \ngtr b$ for the inequality $F_X^*(x)F_X^*(y) \leq F_X^*(\min(x,y))F_X^*(\max(x,y))$)
and

$$f_X(x)f_X(y) \leq f_X(\min(x,y))f_X(\max(x,y))\ \forall x,y\in\mathbb{R}^n \Leftrightarrow$$

$$\left(\sum_{i=1}^{n}\frac{x_i}{b_i}-n+1\right)\left(\sum_{i=1}^{n}\frac{y_i}{b_i}-n+1\right)\geq$$

$$\left(\sum_{i=1}^{n}\frac{\min(x_i,y_i)}{b_i}-n+1\right)\left(\sum_{i=1}^{n}\frac{\max(x_i,y_i)}{b_i}-n+1\right)\ \forall x>b,y>b.$$

But

$$\left(\sum_{i=1}^{n}\frac{x_i}{b_i}-n+1\right)\left(\sum_{i=1}^{n}\frac{y_i}{b_i}-n+1\right)\geq$$

$$\left(\sum_{i=1}^{n}\frac{\min(x_i,y_i)}{b_i}-n+1\right)\left(\sum_{i=1}^{n}\frac{\max(x_i,y_i)}{b_i}-n+1\right)\Leftrightarrow$$

$$\left(\sum_{i=1}^{n}\frac{x_i}{b_i}\right)\left(\sum_{i=1}^{n}\frac{y_i}{b_i}\right)\geq\left(\sum_{i=1}^{n}\frac{\min(x_i,y_i)}{b_i}\right)\left(\sum_{i=1}^{n}\frac{\max(x_i,y_i)}{b_i}\right)\Leftrightarrow$$

$$\sum_{i,j=1}^{n}\frac{x_iy_j}{b_ib_j}\geq\sum_{i,j=1}^{n}\frac{\min(x_i,y_i)\cdot\max(x_j,y_j)}{b_ib_j}.$$

Let us prove $x_iy_j+x_jy_i \geq \min(x_i,y_i)\cdot\max(x_j,y_j)+\min(x_j,y_j)\cdot\max(x_i,y_i)$ $\forall x,y\in\mathbb{R}^n$.

If $x_i \leq y_i, x_j \leq y_j$ then

$$\min(x_i,y_i)\cdot\max(x_j,y_j)+\min(x_j,y_j)\cdot\max(x_i,y_i)=x_iy_j+x_jy_i.$$

If $x_i \geq y_i, x_j \geq y_j$ then

$$\min(x_i,y_i)\cdot\max(x_j,y_j)+\min(x_j,y_j)\cdot\max(x_i,y_i)=y_ix_j+y_jx_i=x_iy_j+x_jy_i.$$

If $x_i \le y_i, x_j \ge y_j$ then

$$x_i y_j + x_j y_i - \min(x_i, y_i) \cdot \max(x_j, y_j) - \min(x_j, y_j) \cdot \max(x_i, y_i) =$$

$$x_i y_j + x_j y_i - x_i x_j - y_j y_i = x_j (y_i - x_i) + y_j (x_i - y_i) =$$

$$(y_i - x_i)(x_j - y_j) \ge 0$$

If $x_i \ge y_i, x_j \le y_j$ then

$$x_i y_j + x_j y_i - \min(x_i, y_i) \cdot \max(x_j, y_j) - \min(x_j, y_j) \cdot \max(x_i, y_i) =$$

$$x_i y_j + x_j y_i - y_i y_j - x_j x_i = x_j (y_i - x_i) + y_j (x_i - y_i) =$$

$$(y_i - x_i)(x_j - y_j) \ge 0$$

Thus $x_i y_j + x_j y_i \ge \min(x_i, y_i) \cdot \max(x_j, y_j) + \min(x_j, y_j) \cdot \max(x_i, y_i)\ \forall x, y \in \mathbb{R}^n$.

$\Rightarrow \sum\limits_{i,j=1}^{n} \frac{x_i y_j}{b_i b_j} \ge \sum\limits_{i,j=1}^{n} \frac{\min(x_i, y_i) \cdot \max(x_j, y_j)}{b_i b_j}\ \forall x, y \in \mathbb{R}^n$.

It results $f_X(x) f_X(y) \le f_X(\min(x, y)) f_X(\max(x, y))$ and

$F_X^*(x) F_X^*(y) \le F_X^*(\min(x, y)) F_X^*(\max(x, y))\ \forall x, y \in \mathbb{R}^n$.

Therefore F_X^* and f_X are MTP_2. □

Theorem 3.7 gives equivalence between hazard rate order and parameters.

Theorem 3.7. *Let* $X \sim MP(a, b)$ *and* $Y \sim MP(c, d)$. *Then* $X \prec_{hr} Y \Leftrightarrow b \le d$, $\frac{b_i}{a b_j} \le \frac{d_i}{c d_j}$,

$d_{(n)} \ge \frac{(n-1)\left(\frac{b_i}{a} - \frac{d_i}{c}\right)}{\sum\limits_{j=1}^{n} \frac{d_i}{c d_j} - \sum\limits_{j=1}^{n} \frac{b_i}{a b_j}}$,

$\sum\limits_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a} + \sum\limits_{j \in \{1,2,\dots,n\} \setminus (I \cup \{i\})}^{n} \left(\frac{b_i}{a} - \frac{d_i}{c}\right) \ge$

$\sum\limits_{j \in I \setminus \{i\}}^{n} \frac{b_i}{a b_j} d_j + \sum\limits_{j \in \{1,2,\dots,n\} \setminus (I \cup \{i\})}^{n} \left(\frac{b_i}{a b_j} - \frac{d_i}{c d_j}\right) b_j$

$\forall i, j \in \{1, \dots, n\}, I \subset \{1, 2, \dots, n\}$.

Proof. It results from theorems 3.5, 3.6 and 2.5. □

Theorem 3.8 gives equivalence between likelihood order and parameters.

Theorem 3.8. *Let* $X \sim MP(a, b)$ *and* $Y \sim MP(c, d)$. *Then* $X \prec_{lr} Y \Leftrightarrow b \le d$, $\frac{b_i}{b_j(a+n)} \le \frac{d_i}{d_j(c+n)}$,

$d_{(n)} \geq \frac{(n-1)\left(\frac{d_i}{c+n}-\frac{b_i}{a+n}\right)}{\sum\limits_{j=1}^{n}\frac{d_i}{d_j(c+n)}-\sum\limits_{j=1}^{n}\frac{b_i}{b_j(a+n)}}\ \forall i,j \in \{1,...,n\}.$

Proof. Let us prove " $\Rightarrow$ " implication.
$X \prec_{lr} Y \Rightarrow X \prec_{st} Y \Rightarrow X \prec_{wst} Y \Rightarrow b \leq d$ and
$X \prec_{lr} Y \Rightarrow f_X(x)\, f_Y(y) \leq f_X(y)\, f_Y(x)\ \ \forall x,y \in \mathbb{R}^n\ x \geq y.$
But

$$f_X(x)\, f_Y(y) \leq f_X(y)\, f_Y(x)\ \ \forall x,y \in \mathbb{R}^n\ x \geq y \Leftrightarrow$$

$$\forall i \in \{1,...,n\}\ x_i \mapsto \tfrac{f_Y(x)}{f_X(x)} \text{ is increasing on } \{x \in \mathbb{R}^n : x > d\} \Leftrightarrow$$

$$\frac{\partial}{\partial x_i}\frac{f_Y(x)}{f_X(x)} \geq 0\ \forall i \in \{1,...,n\}\ \forall x > d.$$

For $x > d$

$$\frac{\partial}{\partial x_i}\frac{f_Y(x)}{f_X(x)} =$$

$$\frac{\partial}{\partial x_i}\left(\frac{c(c+1)...(c+n-1)\cdot\left(\prod\limits_{j=1}^{n} d_j\right)^{-1}\cdot\left(\sum\limits_{j=1}^{n}\frac{x_j}{d_j}-n+1\right)^{-(c+n)}}{a(a+1)...(a+n-1)\cdot\left(\prod\limits_{j=1}^{n} b_j\right)^{-1}\cdot\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)^{-(a+n)}}\right) =$$

$$\frac{c(c+1)...(c+n-1)\cdot\left(\prod\limits_{j=1}^{n} d_j\right)^{-1}}{a(a+1)...(a+n-1)\cdot\left(\prod\limits_{j=1}^{n} b_j\right)^{-1}}\cdot\frac{\partial}{\partial x_i}\frac{\left(\sum\limits_{j=1}^{n}\frac{x_j}{d_j}-n+1\right)^{-(c+n)}}{\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)^{-(a+n)}} =$$

$$\frac{c(c+1)...(c+n-1)\cdot\left(\prod\limits_{j=1}^{n} d_j\right)^{-1}}{a(a+1)...(a+n-1)\cdot\left(\prod\limits_{j=1}^{n} b_j\right)^{-1}}\cdot$$

$$\frac{\left(\sum\limits_{j=1}^{n}\frac{x_j}{d_j}-n+1\right)^{-(c+n)}\cdot\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)^{-(a+n)}}{\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)^{-2(a+n)}}\cdot$$

$$\left(\frac{a+n}{b_i\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)}-\frac{c+n}{d_i\left(\sum\limits_{j=1}^{n}\frac{x_j}{d_j}-n+1\right)}\right).$$

Thus

$$\frac{\partial}{\partial x_i}\frac{f_Y(x)}{f_X(x)}\geq 0\ \forall x>d\Leftrightarrow$$

$$\frac{a+n}{b_i\left(\sum\limits_{j=1}^{n}\frac{x_j}{b_j}-n+1\right)}\geq\frac{c+n}{d_i\left(\sum\limits_{j=1}^{n}\frac{x_j}{d_j}-n+1\right)}\ \forall x>d\Leftrightarrow$$

$$\sum_{j=1}^{n}\left(\frac{b_i}{b_j\left(a+n\right)}-\frac{d_i}{d_j\left(c+n\right)}\right)x_j\leq(n-1)\left(\frac{b_i}{a+n}-\frac{d_i}{c+n}\right)\ \forall x>d.$$

Let $x_j\to\infty$, $j\in\{1,...,n\}$ and it results $\frac{b_i}{b_j(a+n)}\leq\frac{d_i}{d_j(c+n)}$.
For $x_1=x_2=...=x_n=x$ it results

$$\left(\sum_{j=1}^{n}\frac{b_i}{b_j\left(a+n\right)}-\sum_{j=1}^{n}\frac{d_i}{d_j\left(c+n\right)}\right)x\leq(n-1)\left(\frac{b_i}{a+n}-\frac{d_i}{c+n}\right)\ \forall x>d_{(n)}\Leftrightarrow$$

$$d_{(n)}\geq\frac{(n-1)\left(\frac{d_i}{c+n}-\frac{b_i}{a+n}\right)}{\sum\limits_{j=1}^{n}\frac{d_i}{d_j(c+n)}-\sum\limits_{j=1}^{n}\frac{b_i}{b_j(a+n)}}.$$

Let us prove " $\Leftarrow$ " implication.
Let $i\in\{1,...,n\}$, $x\in\mathbb{R}^n,x>d$.
$\frac{b_i}{b_j(a+n)}-\frac{d_i}{d_j(c+n)}\leq 0$ and $x_{(1)}\leq x_j$. Then
$\left(\frac{b_i}{b_j(a+n)}-\frac{d_i}{d_j(c+n)}\right)x_j\leq\left(\frac{b_i}{b_j(a+n)}-\frac{d_i}{d_j(c+n)}\right)x_{(1)}$. It results

$$\sum_{j=1}^{n}\left(\frac{b_i}{b_j\,(a+n)}-\frac{d_i}{d_j\,(c+n)}\right)x_j \le \sum_{j=1}^{n}\left(\frac{b_i}{b_j\,(a+n)}-\frac{d_i}{d_j\,(c+n)}\right)x_{(1)} =$$

$$\left(\sum_{j=1}^{n}\frac{b_i}{b_j\,(a+n)}-\sum_{j=1}^{n}\frac{d_i}{d_j\,(c+n)}\right)x_{(1)} \le (n-1)\left(\frac{b_i}{a+n}-\frac{d_i}{c+n}\right).$$

Thus $\sum\limits_{j=1}^{n}\left(\frac{b_i}{b_j(a+n)}-\frac{d_i}{d_j(c+n)}\right)x_j \le (n-1)\left(\frac{b_i}{a+n}-\frac{d_i}{c+n}\right)$.

Then $\frac{\partial}{\partial x_i}\frac{f_Y(x)}{f_X(x)} \ge 0$.

But f_X and f_Y are MTP$_2$. It results $X \prec_{lr} Y$. □

4 Corollaries

Corollaries 4.1 and 4.2 show that some conditions for multivariate Pareto distributions parameters imply univariate stochastic order or hazard rate order between extremal statistic orders of two random variables sequences. Corollary 4.3 gives sufficient conditions to have usual multivariate stochastic order.

Corollary 4.1. *Let* $X_1,...,X_n$ *random variables with* $(X_1,...,X_n) \sim MP(a,b)$ *and* $Y_1,...,Y_n$ *random variables with* $(Y_1,...,Y_n) \sim MP(c,d)$*. If* $a \ge c$ *and* $b \le d$ *then* $\min(a_1X_1,...,a_nX_n) \prec_{st} \min(a_1Y_1,...,a_nY_n)$ *whenever* $a_i > 0$, $i = 1,2,...,n$.

Proof. We apply theorem 3.4 and theorem 2.1. □

Corollary 4.2. *Let* $X_1,...,X_n$ *random variables with* $(X_1,...,X_n) \sim MP(a,b)$ *and* $Y_1,...,Y_n$ *random variables with* $(Y_1,...,Y_n) \sim MP(c,d)$*. If* $b \le d$, $\frac{b_i}{ab_j} \le \frac{d_i}{cd_j}$, $d_{(n)} \ge \frac{(n-1)\left(\frac{b_i}{a}-\frac{d_i}{c}\right)}{\sum\limits_{j=1}^{n}\frac{d_i}{cd_j}-\sum\limits_{j=1}^{n}\frac{b_i}{ab_j}}$,

$$\sum_{j\in I\setminus\{i\}}^{n}\frac{b_i}{a}+\sum_{j\in\{1,2,...,n\}\setminus(I\cup\{i\})}^{n}\left(\frac{b_i}{a}-\frac{d_i}{c}\right)\ge$$
$$\sum_{j\in I\setminus\{i\}}^{n}\frac{b_i}{ab_j}d_j+\sum_{j\in\{1,2,...,n\}\setminus(I\cup\{i\})}^{n}\left(\frac{b_i}{ab_j}-\frac{d_i}{cd_j}\right)b_j$$

$\forall i,j \in \{1,...,n\}, I \subset \{1,2,...,n\}$,

then $\min(a_1X_1,...,a_nX_n) \prec_{hr} \min(a_1Y_1,...,a_nY_n)$ *whenever* $a_i > 0$, $i = 1,2,...,n$.

Proof. We apply theorem 3.5 and theorem 2.2. □

Corollary 4.3. *Let* $X \sim MP(a,b)$ *and* $Y \sim MP(c,d)$*. If* $b \leq d$, $\frac{b_i}{b_j(a+n)} \leq \frac{d_i}{d_j(c+n)}$,
$d_{(n)} \geq \frac{(n-1)\left(\frac{d_i}{c+n}-\frac{b_i}{a+n}\right)}{\sum\limits_{j=1}^{n}\frac{d_i}{d_j(c+n)}-\sum\limits_{j=1}^{n}\frac{b_i}{b_j(a+n)}}$ $\forall i,j \in \{1,...,n\}$ *then* $X \prec_{st} Y$.

Proof. It results from theorems 3.8 and 2.4. □

5 Conclusions and future research

In this article we proved that the weak stochastic order implies some conditions between parameters and the simply order between parameters implies weak stochastic order. Also, we proved that the weak hazard rate order and likelihood order are equivalent with some conditions of parameters. In a future research we will analyze stochastic orders for other multivariate distributions.

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